Mathematics I

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Chapters

1) Functions and Limits
2) Differentiation
3) Integration
4) Partial Differentiation
5) Maclaurin Series (Taylor series)
6) Series, Infinite Sums

All chapters are inter-related and are essentially an introduction to Calculus

Books

Both M.L. Boas and
K.F. Riley, M.P. Hobson, S.J. Bence

v 00000
Chapter 1

Functions and Limits

1.1 Definition, Range, Domain

1.2 Common functions

1.3 Log (ln), Exponential and Hyperbolic functions

1.4 Limits of functions

1.5 Examples of non-trivial limits
Chapter 1: Functions + Limits

1.1 Definition

- If two variables, $x$ and $y$, follow a rule:
  "when $x$ is given, then $y$ is determined as..." then $y$ is said to be a function of $x$.

- We write $y = f(x)$

  $x$ - independent variable
  $y$ - dependent variable

Example

Circle of radius $r$

Area $A = \pi r^2$

dependent variable

independent variable
Range and Domain

For a set of values of \( x \) ("domain")
there is a corresponding set of \( y \) values ("range")

\[ A = \pi r^2 \]

Given domain \( 0 \leq r \leq 2 \)

\[ \Rightarrow \text{range of } A \text{ is} \]
\[ 0 \leq A \leq 4\pi \]

Common Notation

usually write \( y = f(x) \) but
sometimes simply \( y = y(x) \).

Thus

\[ f(x) = x + x^2 \quad \iff \quad y = x + x^2 \]
1.2 Common Functions

a) Polynomials

\[ y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \]

\[ = \sum_{n=0}^{N} a_n x^n \quad \text{integer} \]

N = degree of polynomial

b) Linear functions (poly of degree 1)

\[ y = a_0 + a_1 x \]

intercept slope

\[ a_0 \]

\[ x \]

c) Quadratic functions

\[ y = a_0 + a_1 x + a_2 x^2 \]

(poly of degree 2, parabola)
Sketch

\[ y \]
\[ \begin{array}{c}
  a_2 > 0 \\
  \text{a}_2 < 0 \\
\end{array} \]

A useful tip: Completing the square

\[ a_2 x^2 + a_1 x + a_0 = \left( \frac{\sqrt{a_2} x + a_1}{2 \sqrt{a_2}} \right)^2 - \frac{a_1^2}{4a_2} + a_0 \]

Check:

\[ = a_2 x^2 + 2 \cdot \frac{\sqrt{a_2} x \cdot a_1}{2 \sqrt{a_2}} + \frac{a_1^2 - a_2}{4a_2} + a_0 \]

\[ \checkmark \]

1) Power laws

\[ y = a \cdot x^\lambda \]

\[ \lambda \text{ = power, index or exponent} \]

Take a positive, also \( x > 0 \).
Differetiate \( \frac{dy}{dx} = ax^{x-1} \),

thus

\[ \begin{align*}
\alpha > 1 & \quad \text{\( \infty \) loop} \\
\alpha = 1 & \quad \text{\"o\" slope} \\
0 < \alpha < 1
\end{align*} \]

Many laws of Physics expressed this way

e.g.

In two dimensions

\[ M \sim (T_c - T)^{\frac{1}{8}} \quad ; \quad T < T_c \]

(Lars Onsager, Nobel prize 1968)
e) Trigonometric functions: \( \sin x, \cos x \ldots \)

f) Heaviside Step Function \( H(x) \)

\[
H(x) = \begin{cases} 
0 & \text{if } x < 0 \\
1 & \text{if } x > 0 
\end{cases}
\]

The function is discontinuous at \( x = 0 \).

E.g., \( y(x) = H(x-1) \cdot \sin \frac{\pi x}{2} \)
9) **Modulus Function**: \( |x| \)

\[ |x| = \begin{cases} 
  x & \text{if } x > 0 \\
  -x & \text{if } x < 0 
\end{cases} \]

![Graph of |x| function]

b) **Even and Odd Functions**

\( f(x) \) is even if \( f(x) = f(-x) \)

![Graph of even function example]

Thus, \( f(x) = x^2 \) is even since replacing \( x \) with \( -x \):

\[ (-x)^2 = (-1)^2 x^2 = x^2 \]

Similarly, \( f(x) = \cos x \) is even since

\[ f(-x) = \cos(-x) = \cos x = f(x) \]
1. \( f(x) \) is odd if \( f(x) = -f(-x) \)

   e.g.

   ![Graph showing odd function]

   \( y \) odd

   \( x \)

   \[ f(x) = x^3 \] odd since

   \[ f(-x) = (-x)^3 = -x^3 = -f(x) \]

   Similarly \( \sin x, \tan x, x^5, x^7 \) odd

---

**Example**

Is \( y = \sin(x^5) \) odd, even or neither?

\[ f(x) = \sin(x^5) \]

\[ f(-x) = \sin((-x)^5) = \sin(-x^5) = -\sin(x^5) = -f(x) \]

**Odd**

---

**Notes**

Not all functions odd or even

E.g. \( f(x) = x + x^2 \)

Even function \( x \) even function = even

Odd \( x \) odd product = even
1) Inverse Functions

A function \( y = f(x) \) can sometimes be inverted to get \( x \) in terms of \( y \)

\[ x = g(y). \]

\( g = \) the inverse of \( f \)

Example

\[ y = x^2 \quad \text{with} \quad x > 0 \]

\[ g(y) = \sqrt{y} \]

\( f(x) = x^2 \)

\[ x^2 = y \]

\[ x = \pm \sqrt{y} \]

\[ x = \sqrt{y} \]

Inverse \( g(y) = \sqrt{y} \).

Does not matter what variable we do

\( g(x) = \sqrt{x} \)

\( \text{flip axes of } (f(x) \text{ vs } x \text{ graph) } \)
Notation: the inverse function $g(x)$ is often written $f^{-1}(x)$.

The inverse of $f(x) = x^2$ ($x$ positive) is $f^{-1}(x) = \sqrt{x}$.

Question: what is $f(f^{-1}(x))$?

In above example $f(x) = x^2$ thus

$f(f^{-1}(x)) = (\sqrt{x})^2 = x$

\[ f(f^{-1}(x)) = x \]  

\[ \text{general result} \]

(j) Functions of a function

Given two functions $f(x)$, $g(x)$ can calculate functions of a function

$\text{e.g. } f(x) = x^2, \ g(x) = \sin x$
Then

\[ f(g(x)) = (\sin x)^2 = \sin^2 x \]

\[ \sin \]

\[ g(f(x)) = \sin(x^2) \]

k) Many-valued functions

Consider trig. \( f^6 \quad y = \sin x \)

For each value of \( x \), \( \exists \) only one \( y \)
Now consider inverse

\[ y = \sin^{-1} x \quad \text{or} \quad (y = \csc^{-1} x) \]

For each value of \( x \) (between -1 and +1) there are many values of \( y \).
Hence (to clarify) we define principle value of \( \sin^{-1} x \) to be
in (domain) \( -\pi/2 < \sin^{-1} x < \pi/2 \)

(similarly for \( y = \cos^{-1} x \) define \( \text{PV} : 0 < \cos^{-1} x < \pi \))
3. **Logarithm**

The natural logarithm, \( \ln x \) or \( \log_e x \), is defined as

\[
\ln x = \int_{1}^{x} \frac{dt}{t} \quad \text{if } x > 0
\]

**Notes:** It follows that

1. \( \frac{d}{dx} \ln x = \frac{1}{x} \)
2. \( \ln 1 = 0 \)
3. \( \ln x, x_2 = \ln x_1 + \ln x_2 \)

**Proof:**

Consider \( \ln x_1 = \int_{1}^{x_1} \frac{dt}{t} \)

Now let \( s = t x_2 \) be new variable (\( x_2 \) fixed)

\[
\frac{ds}{s} = \frac{dt}{t}
\]

\[
\Rightarrow \ln x_1 = \int_{x_2}^{x_1 x_2} \frac{ds}{s}
\]
\[
\begin{align*}
\ln x_i &= \ln x_1 x_2 - \ln x_2 \\
\text{Q.E.D.}
\end{align*}
\]

(liv) \( \ln \frac{1}{x} = -\ln x \)

(follows from (iii) putting \( x_2 = x, x_1 = \frac{1}{x} \))

(v) \( \ln x^2 = 2 \ln x \) \quad (x_1 = x_2 = x)

\( \ln x^3 = 3 \ln x \) \quad (x_1 = x_2 = x)

\( \ln x^n = n \ln x \)

\[ y = \ln x \]

\( \ln x \) tends to \(+\infty\) (with vanishing slope) as \( x \) increases.

\( \ln x \) tends to \(-\infty\) as \( x \) tends to \( 0 \).
Exponential function

Consider \( x = \ln y \). What is function \( f(x) \) ? i.e. what is inverse of natural log?

Let \( x_1 = \ln y_1 \) so \( y_1 = f(x_1) \)

\[ x_2 = \ln y_2 \quad \text{so} \quad y_2 = f(x_2) \]

Then

\[ x_1 + x_2 = \ln y_1 + \ln y_2 = \ln (y_1 y_2) \]

\[ \Rightarrow y_1 y_2 = f(x_1 + x_2) \]

or \( f \) must satisfy

\[ f(x_1 + x_2) = f(x_1) f(x_2) \]

This implies \( f(x) \) has to be of form \( f(x) = a^x \) since only this satisfies \( a^{x_1} a^{x_2} = a^{x_1 + x_2} \).

What is value of \( "a" \)?

If \( x = \ln y \Rightarrow \frac{dx}{dy} = \frac{1}{y} \)

or \( \{dy/dx = y\} \)
hence final question is what value of \( a \) gives \( \frac{d}{dx} a^x = a^x \)?

The number which satisfies this is found to be \( e = 2.71828 \ldots \) (an example of an irrational number).

\[
y = e^x
\]

is the inverse of \( y = \ln x \).

\( y = e^x \) goes to \( 0 \) as \( x \) goes to \( 0 \) and goes to \( +\infty \) as \( x \) goes to \( +\infty \).

Note similarly the function \( y = e^{-x} \) looks like

\[
y = e^{-x}
\]
Hyperbolic functions

\[ \cosh x = \frac{1}{2} \left( e^x + e^{-x} \right) \]

\[ \sinh x = \frac{1}{2} \left( e^x - e^{-x} \right) \]

\[ \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \]
(i) Behave similar to trig functions:

\[ \cosh^2 x - \sinh^2 x = 1 \]

\[ \sinh (x_1 + x_2) = \sinh x_1 \cosh x_2 + \sinh x_2 \cosh x_1 \]
\[ \cosh (x_1 + x_2) = \cosh x_1 \cosh x_2 + \sinh x_1 \sinh x_2 \]

Also

\[ \frac{d}{dx} \sinh x = \cosh x, \quad \frac{d}{dx} \cosh x = \sinh x \]

\[ \frac{d}{dx} \tanh x = \text{sech}^2 x \]

where

\[ \text{sech} x = \frac{1}{\cosh x}, \quad \text{cosech} x = \frac{1}{\sinh x}, \quad \text{coth} y = \frac{1}{\tanh y} \]

(ii) Not coincidental! e.g.

\[ \cosh ix = \cos x \]

\[ \uparrow \]

\[ \frac{1}{i} \]

(iii) Inverse function: related to logs...
e.g. \( y = \sinh^{-1} x \) means \( x = \sinh y \)

This simplifies
\[
x = \sinh y
\]
\[
x = \frac{1}{2} (e^y - e^{-y})
\]
\[
\Rightarrow e^{2y} - 2xe^y - 1 = 0
\]

Quadratic eqn for \( e^y \), thus
\[
e^y = 2x \pm \sqrt{4x^2 + 4}
\]
\[
\Rightarrow e^y = x \pm (x^2 + 1)^{1/2}
\]
\[
\Rightarrow y = \ln\left( x \pm \sqrt{x^2 + 1} \right)
\]
\[
\text{e.g. } \sinh^{-1} x \equiv \ln\left( x + \sqrt{x^2 + 1} \right)
\]

Similarly
\[
cosh^{-1} x \equiv \ln\left( x + \sqrt{x^2 - 1} \right)
\]
1.4 Limit of functions

Example. Consider

\[ f(x) = \frac{\sin x}{x} \quad ; \quad x \neq 0 \]

\[ f(x) \text{ not defined at } x = 0 \text{ since } \frac{0}{0}. \]

But plotting this function (using radians) numerically shows \( f(x) \) gets closer and closer to 1 as \( x \) gets closer to 0.

\[ f(0.1) = \]
\[ f(0.01) = \]
\[ f(0.001) = \]

\[ \begin{array}{c}
\text{1} \\
\end{array} \]

Question:
What happens to \( \frac{\sin x}{x} \) as \( x \) tends to 0 (as \( x \to 0 \))?
Proof

Consider section of unit circle

\[ O \left\langle \cos x \rightarrow B \quad 1 \quad \tan x \quad \sin x \right\rangle \]

By inspection

Area of \( \Delta ODB \) > Area sector > Area \( \Delta OAB \)

\[ \frac{\tan x}{2} > \frac{x}{2\pi}, \pi > \frac{\sin x}{2} \]

Divide by \( \sin x / 2 \)

\[ \frac{1}{\cos x} > \frac{x}{\sin x} > 1 \]

\[ \cos x < \frac{\sin x}{x} < 1 \]

\[ \text{As } x \to 0, \cos x \to 1 \Rightarrow \lim_{x \to 0} \frac{\sin x}{x} \to 1 \]
Notation. We write

\[ \frac{\sin x}{x} \rightarrow 1 \quad \text{as} \quad x \rightarrow 0 \]

or \[ \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \]

More generally, we write the limit \( F \) of a function \( f(x) \) at a point \( x_0 \) as

\[ f \rightarrow F \quad \text{as} \quad x \rightarrow x_0 \]

or \[ \lim_{x \rightarrow x_0} f(x) = F \]

Notes

1) Limits trivial (and unnecessary) if function is "well-behaved" at \( x_0 \).

E.g., \[ f(x) = x^2 + 3 \]

\[ \lim_{x \rightarrow 4} (x^2 + 3) = 19 \]
1) Some simple rules for sums, products:

If \( f(x) \to F \) as \( x \to x_0 \)
and \( g(x) \to G \) as \( x \to x_0 \)

then

\[
af + bg \to aF + bG
\]

\[
f \cdot g \to FG
\]

\[
\frac{f}{g} \to \frac{F}{G} \quad \text{(provided \( g \neq 0 \))}
\]

2) \[
\lim_{x \to 2} (x^2 + 2) \cdot \cos \frac{\pi x}{2}
\]

\[
= \lim_{x \to 2} (x^2 + 2) \cdot \lim_{x \to 2} \cos \frac{\pi x}{2}
\]

\[
= 6 \cdot \cos \pi
\]

\[
= -6
\]
Non-trivial limits

Concept of limit essential when we encounter combinations like
\[
\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0 \cdot \infty, \quad \infty - \infty
\]

1. Limits of type "0/0"

For this we use L'Hopital's rule \( \text{proof later} \)

For \( \lim_{x \to x_0} \frac{f(x)}{g(x)} \) where \( f(x_0) = g(x_0) = 0 \), then

\[
\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}
\]

If both \( f'(x_0) \) and \( g'(x_0) \) are still zero then differentiate again

\[
\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f''(x)}{g''(x)} \quad \text{etc}
\]
Examples of $L'H$ rule:

1) \( \lim_{x \to 0} \frac{\sin x}{x} \) "$0/0$" type.

By $L'H$, \( \lim_{x \to 0} \frac{\cos x}{1} = 1 \)

2) \( \lim_{x \to 1} \frac{x^3 - x^2 + 2x - 2}{x^3 + x^2 - 2} \)

"$0/0$" type: by $L'H$

\[
\lim_{x \to 1} \frac{3x^2 - 2x + 2}{3x^2 + 2x} = \frac{3}{5}
\]

Check using independent method

Write \( x = 1 + h \)

\[
\lim_{h \to 0} \frac{(1+h)^3 - (1+h)^2 + 2(1+h) - 2}{(1+h)^3 + (1+h)^2 - 2}
\]

\[
= \lim_{h \to 0} \frac{(1+3h+3h^2+h^3)-(1+2h+h^2)+2(1+h)-2}{(1+3h+3h^2+h^3)+(1+2h+h^2)-2}
\]

\[
= \lim_{h \to 0} \frac{3h + 2h^2 + h^3}{5h + 4h^2 + h^3}
\]

\[
= \lim_{h \to 0} \frac{3 + 2h + h^2}{5 + 4h + h^2} = \frac{3}{5}
\]

\( \checkmark \)
\( \lim_{x \to 0} \frac{(1+x)^{\frac{y_2}{2}} - (1+2x)^{\frac{y_2}{2}}}{x} \)

"0/0" ... by L'Hopital's Rule

\[ \lim_{x \to 0} \frac{\frac{1}{2} (1+x)^{\frac{-y_2}{2}} - \frac{1}{2} (1+2x)^{\frac{-y_2}{2}} \cdot 2}{1} \]

\[ = -1/2 \]

//

2) Limits of type "\( \infty / \infty \)"

\[ \text{e.g.} \quad \lim_{x \to \infty} \frac{2x^5 + 2x^2 - 1}{x^5 - x^3 + 1} \]

Take dominant term on top and bottom outside

\[ = \lim_{x \to \infty} \frac{2x^5 + 2x^2 - 1}{x^5 - x^3 + 1} \frac{x^5}{x^5} \]

\[ = \lim_{x \to \infty} \frac{2 + 2x^3 - x^5}{1 - x^{-2} + x^{-5}} = 2 \]

Note: can see get finite number as numerator and denominator both have biggest power in same. Whereas,

\[ \lim_{x \to \infty} \frac{x^7 + 6}{x^6 - 5} = \infty \quad \text{and} \quad \lim_{x \to \infty} \frac{x^4 + 3}{x^3 + 2} = 0 \]
(3) Limits of type "∞ - 0"

(i) \[ \lim_{x \to \infty} x^{\sqrt{2}} \left\{ \left( x + \frac{1}{x} \right)^{\sqrt{2}} - \left( x - \frac{1}{x} \right)^{\sqrt{2}} \right\} \]

\[ = \lim_{x \to \infty} x^{\sqrt{2}} \left\{ x^{\sqrt{2}} \left( 1 + \frac{1}{x} \right)^{\sqrt{2}} - x^{\sqrt{2}} \left( 1 - \frac{1}{x} \right)^{\sqrt{2}} \right\} \]

\[ = \lim_{x \to \infty} x \left\{ \left( 1 + \frac{1}{x} \right)^{\frac{1}{2}} - \left( 1 - \frac{1}{x} \right)^{\frac{1}{2}} \right\} \]

Now use binomial expansion

\[ (1 + t)^4 = 1 + 2t + \frac{1}{2} t^2 \cdot \frac{2!}{2!} \cdot t^2 + \cdots \quad \text{if} \quad |t| < 1 \]

thus

\[ \left( 1 + \frac{1}{x} \right)^{\sqrt{2}} = 1 + \frac{1}{2x} - \frac{1}{8x^2} + \cdots \]

\[ \left( 1 - \frac{1}{x} \right)^{\sqrt{2}} = 1 - \frac{1}{2x} - \frac{1}{8x^2} + \cdots \]

Then

\[ \text{Limit} = \lim_{x \to \infty} x \left\{ 1 + \frac{1}{2x} + O\left( \frac{1}{x^2} \right) \right. \]

\[ - \left( 1 - \frac{1}{2x} + O\left( \frac{1}{x^2} \right) \right) \]

Digression \( O\left( \frac{1}{x^2} \right) \) short-hand for

\[ \frac{\text{Constant}}{x^2} \quad \frac{\text{Constant}}{x^3} + \cdots \]
\[
\begin{align*}
\lim_{x \to \infty} x \left\{ \frac{1}{x} + O\left(\frac{1}{x^2}\right) \right\} &= \lim_{x \to \infty} 1 + O\left(\frac{1}{x}\right) = 1 \\
\lim_{x \to \infty} x^2 \left\{ \left(1 + \frac{2}{x^2}\right)^{\frac{1}{2}} - \left(1 - \frac{3}{x^2}\right)^{\frac{1}{2}} \right\} &= \lim_{x \to \infty} x^2 \left\{ 1 + \frac{1}{x^2} + O\left(\frac{1}{x^4}\right) - \left(1 - \frac{3}{2x^2} + O\left(\frac{1}{x^4}\right)\right) \right\} \\
&= \lim_{x \to \infty} x^2 \left\{ \frac{5}{2x^2} + O\left(\frac{1}{x^4}\right) \right\} \\
&= \frac{5}{2}
\end{align*}
\]
Chapter 2

Differentiation

2.1 First Principles

2.2 Product, Quotient, chain-rules etc etc

2.3 Curve sketching

2.4 Parametric representation of curves

2.5 Polar co-ordinates
Chapter 2: Differentiation

1. Differentiation from First Principles

What is tangent to curve at point $P$?

\[ y = f(x) \]

\[ \Delta y \]

\[ \Delta x \]

\[ P \]

\[ Q \]

Gradient of chord $PQ$ is given by:

\[
\text{Gradient of chord } PQ = \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{\Delta y}{\Delta x}
\]

Now, at fixed $x$, let $\Delta x \to 0$. If limit exists, then

\[
\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{dy}{dx}
\]

is derivative of $f(x)$ at $x$. \[ \star \]
Examples

i) \( y = x^2 \)

\[
\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{(x+\delta x)^2 - x^2}{\delta x}
\]

\[
= \lim_{\delta x \to 0} \frac{x^2 + 2x\delta x + \delta x^2 - x^2}{\delta x}
\]

\[
= \lim_{\delta x \to 0} \frac{2x + \delta x}{\delta x} = 2x
\]

\[
\delta x \to 0
\]

ii) \( y = \sin x \)

\[
\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\sin (x+\delta x) - \sin x}{\delta x}
\]

Recall \( \sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2} \)

\[
\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{2 \cos \left( x + \frac{\delta x}{2} \right) \sin \frac{\delta x}{2}}{\delta x}
\]

\[
= \lim_{\delta x \to 0} \cos \left( x + \frac{\delta x}{2} \right) \cdot \lim_{\delta x \to 0} \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}}
\]

\[
= \cos x \cdot 1 = \cos x
\]
2.2 Useful results/techniques

1) **Product rule**
\[
\frac{d}{dx} (fg) = \frac{df}{dx}g + f\frac{dg}{dx}
\]

2) **Quotient rule**
\[
\frac{d}{dx} \frac{f}{g} = \frac{f'g - fg'}{g^2}
\]

Shw he, \( f' = \frac{df}{dx}, \ f'' = \frac{d^2f}{dx^2} \) etc.

3) **Function of a function**

E.g., \( y = \ln \cos x \)

Write \( y = \ln g \) and \( g = \cos x \)

\[
\frac{dy}{dx} = \frac{dy}{dg} \cdot \frac{dg}{dx} = \frac{1}{g} (-\sin x) = -\tan x
\]

\[\text{chain-rule} \]
Parametric differentiation

Suppose \( y = y(t) \) and \( x = x(t) \) e.g. coordinates of moving body.

\[
\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{dy}{dt} \frac{dx}{dt} = \frac{y'}{x'}
\]

\( y' = \frac{dy}{dt}, \quad y'' = \frac{d^2y}{dt^2} \) etc.

Similarly

\[
\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{dt}{dx} \frac{d}{dt} \left( \frac{y'}{x'} \right)
\]

\[
\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{y''x - y'x'}{x'^3}
\]
Example

\[ y = \sin t \]
\[ x = \cos t \]

\[ \frac{dy}{dx} = \frac{\cos t}{-\sin t} = -\cot t \]

Similarly,\[ \frac{dy}{dx} = \frac{\sin^2 t + \cos^2 t}{(-\sin t)^3} = -\frac{1}{y^3} \]

1) Differentiation of inverse (useful \( \ast \))

\( \text{e.g. } 1) \) \[ y = \sin^{-1} x \]

\[ \sin y = x \Rightarrow \cos y \cdot \frac{dy}{dx} = 1 \]
\[ \frac{dy}{dx} = \frac{1}{\cos y} \]
\[ \frac{dy}{dx} = \frac{1}{\sqrt{1 - \sin^2 y}} \Rightarrow y' = \frac{1}{\sqrt{1 - x^2}} \]

\( \text{e.g. } 2) \) \[ y = \tan^{-1} x \]

\[ \tan y = x \Rightarrow \sec^2 y \cdot y' = 1 \Rightarrow y' = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} \]
\[ y' = \frac{1}{1 + x^2} \]

Similarly, for \( y = \cos^{-1} x, y = \sec^{-1} x, \ldots \)
e) Implicit Functions

Sometimes relationship between $x$ and $y$ is implicit

$$F(x, y) = 0$$

rather than explicitly $y = f(x)$.

E.g.

$$x^2 \sin y + xy = 1$$

Differentiate w.r.t. $x$ (using product rule)

$$2x \sin y + x^2 \cos y \frac{dy}{dx} + y + x \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{(y + 2x \sin y)}{x^2 \cos y + x}$$
1) Stationary Points

At a Stationary pt

Two basic types

Maximum

\[ y = 0 \text{ and } y'' < 0 \]

Minimum

\[ y' = 0, \quad y'' > 0 \]

At a point of inflexion \( \frac{d^2y}{dx^2} = 0 \)

but \( \frac{dy}{dx} \) may or not be zero.

If \( y' = 0 \) \( \implies \) stationary pt or saddle-point of inflexion.

If \( y' \neq 0 \) \( \implies \) non-stationary pt of inflexion.

Corresponds to a change of curvature.
Examples of points of inflexion

\[ y = x^2 (x-1) \]

Taking derivatives,

\[ y' = 3x^2 - 2x = x(3x-2) \]
\[ y'' = 6x - 2 \]

Thus stationary points \((y' = 0)\) at \(x = 0, \frac{2}{3}\)

- at \(x = 0\), \(y'' = -2 \Rightarrow \text{Max} \)
- at \(x = \frac{2}{3}\), \(y'' = +2 \Rightarrow \text{Min} \)
- \(y'' = 0\) at \(x = \frac{1}{3}\) \(\Rightarrow\) point of inflexion
1.3 Curve Sketching

Get main features by

1. Examining behaviour for \( x \to 0, +\infty, -\infty \)

2. Look for symmetries (even/odd)

If \( y = \frac{P(x)}{Q(x)} \) with \( P, Q \) polynomials

(i) Zeros of \( P \) give intersections with \( Ox \)

(ii) Zeros of \( Q \) give infinite discontinuities or asymptotes

Stationary pts for finer detail

Example

\[
y = \frac{x(x-2)}{x-3}
\]

1. \( y = 0 \) when \( x = 0, 2 \)

2. \( y \sim \frac{2}{3}x \) for \( b \) small

near \( x = 3 \), \( y \sim \frac{3}{x-3} \)
Thus

\[ y \to +\infty \quad \text{as} \quad x \to 3^+ \]

\[ y \to -\infty \quad \text{as} \quad x \to 3^- \]

---

**Digression**

\( x \to 3^+ \) means "\( x \) tends to 3 from above". Think \( x = 3.1, 3.01, 3.001 \ldots \)

\( x \to 3^- \) means "\( x \) tends to 3 from below". Think \( x = 2.9, 2.99, 2.999 \ldots \)

---

@ behaviour at large \( x \)

\[
y = \frac{x(x-2)}{x-3}
\]

\[
= \frac{x^2(1 - \frac{2}{x})}{x(1 - \frac{3}{x})}
\]

\[
= x \left(1 - \frac{2}{x}\right) \left(1 + \frac{3}{x}\right)^{-1}
\]

\[
= x \left(1 + \frac{1}{x} + O\left(\frac{1}{x^2}\right)\right)
\]

\[
= x + 1 + O\left(\frac{1}{x}\right)
\]
\[ y' = \frac{x^2 - 6x + 6}{(x-3)^2} \implies \text{stationary points at } x = 3 \pm \sqrt{3} \]

(can already tell if max/min)

2.4 Parametric representation of curves

Given \( x = x(t) \) and \( y = y(t) \)
- \( t \) = time or some abstract variable

\[ \text{g. a)} \quad x = acost, \quad y = bsint \]
In this case can eliminate $t$

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]

**ellipse**

$\theta$

\[t = \pi, \quad t = \pi/2, \quad t = 0, \quad t = 3\pi/2\]

Parametric representation often useful when cannot (easily) eliminate $t$

\[x = a(t - \sin t)\]
\[y = a(1 - \cos t)\]

**Cycloid**

\[\theta\]

\[2a\]

\[0, 2\pi a, 4\pi a, x\]
2.5 Polar Co-ordinates

In 2D often best to use plane polar co-ordinates \((r, \theta)\) instead of Cartesian \((x, y)\)

\[
x = r \cos \theta, \quad y = r \sin \theta
\]

\((x, y)\) in terms of \((r, \theta)\)

- \(r = \sqrt{x^2 + y^2}\)
- \(\theta = \arctan \frac{y}{x}\)

\((-\pi < \theta < 2\pi)\)

\((x, y)\) in terms of \((r, \theta)\)

\(r^2 = 2a^2 \cos 2\theta\)

Since \(r\) real, no solutions when \(\cos 2\theta < 0\) e.g. \(\frac{3\pi}{2} < 2\theta < \frac{5\pi}{2}\)
Mathematical name for infinity symbol $\infty$. 

"lemniscate"
Chapter 3

Integration

3.1 Riemann's Definition

3.2 Fundamental Theorem of Calculus

3.3 Infinite and Improper Integrals

3.4 Useful tricks (intg' by parts etc)

3.5 Applications: Area, Volume, Path length, Surface area

3.6 Centre-of-Mass
Chapter 5: Integration

3.1 Riemann's definition

Integrals evolved from intuitive ideas about area (or volume) in geometry.

Consider area $A$ under curve $f(x)$ between $x = a$ and $x = b$.

To calculate $A$, we imagine $n$ non-overlapping rectangular strips located at $x_0, x_1, \ldots, x_n$ with end points $x_0 = a, x_n = b$.

Area of all strips $S_n = \sum_{i=0}^{n-1} f(x_i) \delta x_i$.
where \( f(x_i) \) is strip height and \( \Delta x_i = x_{i+1} - x_i \) is strip width.

Our intuition is that \( S_n \rightarrow A \) as number of strips, \( n \rightarrow \infty \), since the error (due to shaded bits) vanishes in this limit.

This is indeed the case! \textbf{Riemann} generalised above approach so that the height \( f \) of the strip can be evaluated at any point \( \xi_i \) between \( x_i \) and \( x_{i+1} \).

Riemann's \textit{definite integral} is the limit of

\[
S_n^* = \sum_{i=0}^{n-1} f(\xi_i) \Delta x_i
\]

with \( x_i < \xi_i < x_{i+1} \). He showed that as \( n \rightarrow \infty \) we have \( S_n^* \rightarrow S_n \rightarrow A \). In this limit we define
\[ \int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=0}^{n} f(x_i) \Delta x_i \]

as the integral of \( f \) between \( a \) and \( b \).

**Notes**

a) \( f(x) \) = integrand
   - \( a \) = lower limit of integral
   - \( b \) = upper limit of integral

\( x \) is a dummy variable. Any symbol will do!

\[ \int_a^b f(x) \, dx = \int_a^b f(s) \, ds = \int_a^b f(t) \, dt \quad \text{etc} \]

For example

\[ \int \frac{d \text{cabin}}{\text{cabin}} = \left[ \log_2 \text{cabin} \right]^2 = \log_2 2 \]
b) We also define

$$\int_{a}^{b} f(x) \, dx = -\int_{b}^{a} f(x) \, dx$$

Hence

$$\int_{a}^{a} f(x) \, dx = 0$$

No area under curve.

And

$$\int_{a}^{c} f(x) \, dx = \int_{a}^{b} f(x) \, dx + \int_{b}^{c} f(x) \, dx$$

valid for any $b$. Just means areas can be added (and subtracted).

\[ \text{Diagram showing areas of integrals.} \]
3.2 Fundamental theorem of Calculus

Integration is inverse of differentiation. That is, if we define a function

\[ F(x) = \int_\alpha^x f(u) \, du \]

then

\[ \frac{dF}{dx} = f(x) \]

Proof

\[ \frac{dF}{dx} = \lim_{\delta x \to 0} \frac{F(x+\delta x) - F(x)}{\delta x} \]

\[ = \lim_{\delta x \to 0} \left( \int_0^{x+\delta x} f(u) \, du - \int_0^x f(u) \, du \right) \]

\[ = \lim_{\delta x \to 0} \left( \int_0^x f(u) \, du + \int_x^{x+\delta x} f(u) \, du \right) \]

\[ = \lim_{\delta x \to 0} \left( \frac{f(x) \delta x}{\delta x} = f(x) \right) \]
Two comments

- In above def of $F(x)$, lower limit is arbitrary $\Rightarrow$ can be added to $F(x)$

- The definite integral

$$\int_{a}^{b} f(u) \, du = F(b) - F(a)$$

3.3 Infinite and Improper Integrals

Infinite integrals have a $+\infty$ (or $-\infty$) in the upper (lower) limit.

What is meaning of $\int_{a}^{\infty} f(x) \, dx$?

To decide if meaningful, integrate to finite value $N$

$$I(N) = \int_{a}^{N} f(x) \, dx$$

If $I(N)$ has finite limit as $N \to \infty$ then infinite integral exists.
Example 1)
\[
\int_{\alpha}^{\infty} e^{-x} \, dx = \lim_{N \to \infty} \int_{\alpha}^{N} e^{-x} \, dx
\]
\[
= \lim_{N \to \infty} \left( -e^{-a} - e^{-N} \right)
\]
\[
= -a \quad \text{Integral exist}
\]

Example 2)
\[
\int_{\alpha}^{\infty} \frac{dx}{x} = \lim_{N \to \infty} \int_{\alpha}^{N} \frac{dx}{x}
\]
\[
= \lim_{N \to \infty} \ln N - \ln \alpha \to \infty
\]
\[
i.e. \text{ Integral does not exist}
\]

Similarly \underline{Improper} integrals involve a singularity of the integrand in range of integration.

Example 3)
\[
\int_{0}^{1} x^{-\frac{1}{2}} \, dx
\]
Potentially problematic because $x^{-\frac{1}{2}}$ is $\infty$ at $x = 0$.

To decide, integrate from $E \ (> 0)$ to 1, then take limit $E \to 0$.

$$I(E) = \int_{E}^{1} x^{-\frac{1}{2}} \, dx$$

No infinity anywhere

$$= \left[ 2x^{\frac{1}{2}} \right]_{E}^{1} = 2 - 2\sqrt{E}$$

Then

$$\int_{0}^{1} x^{-\frac{1}{2}} \, dx = \lim_{E \to 0} I(E)$$

$$= \lim_{E \to 0} 2 - 2\sqrt{E} = 2 \quad \checkmark$$

All O.K.

Example 4)

$$\int_{0}^{1} \frac{dx}{x^2} = \lim_{E \to 0} \int_{E}^{1} \frac{dx}{x^2}$$

$$= \lim_{E \to 0} \left[ -x^{-1} \right]_{E}^{1}$$

Integral does not

$$\to +\infty$$
This is blaw in the calculation.

\[
\int_{-1}^{+1} \frac{dx}{x^2} = \left[ -x^{-1} \right]_{-1}^{+1} = -2
\]

The area under the curve is infinite.

3.4 Useful tricks for evaluating integrals

(a) Partial fractions. E.g.,

\[
\int \frac{dx}{x(x+1)} = \int \left( \frac{1}{x} - \frac{1}{x+1} \right) dx
\]

\[
= \ln|x| - \ln|x+1| + C
\]

\[
= \ln \left| \frac{x}{x+1} \right| + C
\]
(b) Change of variable

(i) \[ \int x e^{-x^2} \, dx \]

let \( u = x^2 \) \( \Rightarrow du = 2x \, dx \)

\[ \Rightarrow \frac{1}{2} \int e^{-u} \, du = -\frac{e^{-u}}{2} + C \]

\[ = -\frac{e^{-x^2}}{2} + C \]

(ii) For trig. integrals often useful to define

\[ t = \tan \frac{x}{2} \]

Then

\[ \cos x = \frac{1-t^2}{1+t^2}, \quad \sin x = \frac{2t}{1+t^2} \]

\[ \frac{dx}{dt} = \frac{2}{1+t^2} \]
Digression

If \( t = \tan \frac{x}{2} \) then double the formula \( \tan 2A = \frac{2\tan A}{1-\tan^2 A} \) identifies

\[ \tan x = \frac{2t}{1-t^2} \]

Draw \( \Delta \)

![Diagram](image)

The hypotenuse is \( 1+t^2 \) by Pythagoras

Then

\[ \sin x = \frac{2t}{1+t^2} \quad \cos x = \frac{1-t^2}{1+t^2} \]

Finally differentiate (1)

\[ \frac{dx}{dt} = \sec^2 x = \frac{2(1+t^2)}{(1-t^2)^2} \]

\[ \frac{dx}{dt} = \frac{2(1+t^2)}{(1-t^2)^2} \cdot \frac{1}{1+t^2} = \frac{2(1+t^2)}{(1-t^2)^2} \cdot \frac{1}{1+t^2} \]

\[ = \frac{2(1+t^2)}{(1-t^2)^2(1+4t^2)} = \frac{2(1+t^2)}{(1-t^2)^2 + 4t^2} \]

\[ = \frac{2}{1+t^2} \]
Example

\[ \int \frac{dx}{2 + \cos x} \]

let \( t = \tan \frac{x}{2} \)

\[ = \int \frac{dx}{dt} \frac{1}{2 + 1 - t^2} \cdot dt \]

\[ = \int \frac{2}{1 + t^2} \cdot \frac{1}{2 + 1 - t^2} \cdot dt \]

\[ = 2 \int \frac{dt}{3 + t^2} \]

standard integral (remember or look up)

\[ = 2 \tan^{-1} \frac{t}{\sqrt{3}} + C \]

\[ = 2 \tan^{-1} \left[ \frac{\tan \frac{x}{2}}{\sqrt{3}} \right] + C \]
(c) Integration by parts

From product rule

\[ \int \frac{u}{dx} \, dv \, dx = [uv] - \int \frac{v}{dx} \, du \, dx \]

Example 1)

\[ \int x \, e^x \, dx = xe^x - \int e^x \, dx = (x-1)e^x + C \]

\[ \uparrow \quad \uparrow \quad \Rightarrow \quad \frac{du}{dx} = 1, \quad v = e^x \]

Example 2)

\[ \int \ln x \, dx \]

Write \[ \int \frac{1}{x} \cdot \ln x \, dx = x \ln x - \int \frac{dx}{x} = x(\ln x - 1) + C \]

\[ \frac{dv}{dx} \quad \Rightarrow \quad v = x, \quad \frac{du}{dx} = \frac{1}{x} \]

Example 3)

\[ \int \tan^{-1} x \, dx = \int (1 + \tan^{-1} x) \, dx = \tan^{-1} x + C \]

\[ \uparrow \quad \uparrow \quad \Rightarrow \quad u = x \]

\[ \frac{dv}{dx} \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{1 + x^2} \]
\[ x + \tan^{-1} x - \frac{1}{2} \ln (1 + x^2) + C \]

3.5 Applications of Integration

1) Mean-Value

Consider a function \( f(x) \) and a specified interval \([a, b]\).

![Graph](image)

The mean (average) value of \( f \) over interval \( [a, b] \) is

\[
\bar{f} = \frac{1}{b-a} \int_a^b f(x) \, dx
\]

more modern notation, inspired by Dirac's formulation of A.M. is \( \langle f \rangle \).

Geometrically, \( \bar{f} \) is height of rectangle whose area is same as under curve
Similarly, the root mean square value of \( f \) over \([a, b]\) is given by:

\[
\text{r.m.s.} = \frac{1}{\sqrt{b-a}} \left[ \int_a^b \frac{f(x) \, dx}{b-a} \right]^{1/2}
\]

(in Dirac's notation \( \langle f^2 \rangle^{1/2} \))

Both \( \bar{f} \) and \( \text{r.m.s.} \) characterise function \( f \) over the interval.

\[\text{r.m.s.} - \bar{f} \quad \text{small}\]

\[\text{r.m.s.} - \bar{f} \quad \text{large}\]
b) **Area in Polars**

Also convenient to measure area using plane polars \((r, \theta)\).

2. What is area of wedge defined by polar curve \(r(\theta)\)?

Look at infinitesimal wedge section

\[ \delta A \approx \frac{1}{2} r \delta \theta \cdot r \]

\[ \text{Area} = \int_{A}^{B} \frac{1}{2} r^2 \cos \theta \, d\theta \]

*required to remember*
Example

Area of semi-circle \( x^2 + y^2 = a^2 \)

![Graph of a semi-circle](image)

- **Cartesian**
  \[
  A = \int_{-a}^{a} y \, dx = \int_{-a}^{a} \sqrt{a^2 - x^2} \, dx = 2 \int_{0}^{a} \sqrt{a^2 - x^2} \, dx
  \]

  Let \( x = a \sin \theta \Rightarrow dx = a \cos \theta \, d\theta 

  \[
  A = 2 \int_{0}^{\pi/2} \sqrt{a^2 - a^2 \sin^2 \theta} \cdot a \cos \theta \, d\theta 
  \]

  \[
  = 2a^2 \int_{0}^{\pi/2} \cos^2 \theta \, d\theta 
  \]

  \[
  = 2a^2 \int_{0}^{\pi/2} \left(1 + \cos 2\theta \right) \, d\theta 
  = \frac{2a^2}{2} \left[ \theta + \frac{\sin 2\theta}{2} \right]_{0}^{\pi/2} 
  \]

  \[
  = \frac{\pi a^2}{2} \quad \checkmark
  \]

- **Polar**:
  \( r = a \) for \( 0 < \theta < \pi \)

  \[
  A = \int_{0}^{\pi} \frac{1}{2} r^2 \, d\theta = \frac{1}{2} a^2 \int_{0}^{\pi} d\theta = \frac{\pi a^2}{2} 
  \]
c) Path length

Cartesians

\[ \delta s = \delta x^2 + \delta y^2 \]

\[ \Rightarrow \quad \delta s = \delta x \sqrt{1 + \left( \frac{\delta y}{\delta x} \right)^2} \]

\[ L = \int_{a}^{b} ds = \int_{a}^{b} dx \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \]

Parametrically

As above but \( x = x(t), \ y = y(t) \) (so \( L \) is total distance travelled)

\[ \delta s^2 = \delta x^2 + \delta y^2 \]

\[ = \left( \frac{\delta x}{\delta t} \right)^2 + \left( \frac{\delta y}{\delta t} \right)^2 \delta t^2 \]

\[ \Rightarrow \quad ds = \sqrt{\dot{x}^2 + \dot{y}^2} \, dt \]

\[ L = \int_{t_a}^{t_b} \sqrt{\dot{x}^2 + \dot{y}^2} \, dt \]

\[ \text{speed} = \left| \dot{v} \right| \]
Look at "infinitesimal" contribution

\[ s_s^2 = \delta r^2 + r^2 \delta \theta^2 \]
\[ = \left[ \left( \frac{\delta r}{\delta \theta} \right)^2 + r^2 \right] \delta \theta^2 \]

\[ L = \int_A^B ds = \int_A^B \sqrt{r^2 + \left( \frac{dr}{d\theta} \right)^2} \, d\theta \]
Example 1) Arc length of quarter circle

\[ x^2 + y^2 = a^2 \]

\[ L = \int_0^a dx \sqrt{1 + y'^2} \]

Now \( y = \sqrt{a^2 - x^2} \) \( \Rightarrow \) \( y' = \frac{-x}{\sqrt{a^2 - x^2}} \)

\[ L = \int_0^a dx \sqrt{1 + \frac{x^2}{a^2 - x^2}} = \int_0^a dx \frac{a}{\sqrt{a^2 - x^2}} \]

\[ = a \left[ \sin^{-1} \frac{x}{a} \right]_0^a = \frac{\pi a}{2} \checkmark \]

Polar: \( r = a, \ 0 < \theta < \frac{\pi}{2} \)

\[ L = \int_0^{\pi/2} \frac{dr}{\sqrt{r^2 - a^2}} \ \theta = \int_0^{\pi/2} \frac{a \ d\theta}{\sqrt{2}} = \frac{\pi a}{2} \]
Example 2)

An infinite spiral

\[ r(\theta) = e^{-\theta/2\pi} \]

\[ r(0) = 1 \]
\[ r(2\pi) = e^{-1} \]
\[ r(4\pi) = e^{-2} \]

\[ L = \int_{0}^{8} d\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \]

\[ r = e^{-\theta/2\pi} \implies \frac{dr}{d\theta} = -\frac{1}{2\pi} e^{-\theta/2\pi} \]

\[ L = \int_{0}^{8} d\theta \sqrt{1 + \frac{1}{4\pi^2}} e^{-\theta/2\pi} \]

\[ L = \int_{0}^{8} \sqrt{1 + \frac{1}{4\pi^2}} \left[-2\pi e^{-\theta/2\pi}\right] \]

\[ L = \sqrt{1 + 4\pi^2} \]
3.6 Centre of Mass

Imagine we have \( N \) masses \( m_i \) \( (i = 1, 2, \ldots, N) \) at positions \( (x_i, y_i) \).

The Centre of Mass \( \overline{x}, \overline{y} \) is defined to be at

\[
\overline{x} = \frac{\sum_{i=1}^{N} m_i x_i}{M}, \quad \overline{y} = \frac{\sum_{i=1}^{N} m_i y_i}{M}
\]

This concept is useful in Mechanics.

Wish to generalise this to a continuous system (a 2D plate or lamina) bounded by curve \( y(x) \).

- Mass density \( \rho \) (uniform)
- Total mass \( M = \rho \int_a^b y \, dx \)
Centre of Mass at \((x, y)\) w.r.t

\[
\overline{x} = \frac{\int_a^b xy(x) \, dx}{\int_a^b y(x) \, dx}
\]

\[
\overline{y} = \frac{\int_a^b \frac{1}{2} y^2(x) \, dx}{\int_a^b y(x) \, dx}
\]

**Proof**

Split into strips

Mass of strip = \(Ey(x) \, dx\)

Total mass \(M = \int \text{Area}\)

Centre of Mass of each strip is at \((x, y)\)

\[
\overline{x} = \frac{\int x \, \text{Area}}{M} = \frac{\int xy \, dx}{\int y \, dx} = \frac{\int xy \, dx}{\int y \, dx}
\]

\[
\overline{y} = \frac{\int \frac{y^2}{2} \, \text{Area}}{M} = \frac{\int \frac{1}{2} y^2 \, dx}{\int y \, dx} = \frac{1}{2} \frac{\int y^2 \, dx}{\int y \, dx}
\]
Example 1) Uniform semi-circle

\[ x^2 + y^2 = a^2 \]

\[ \overline{x} = \frac{\int_{-a}^{a} xy \, dx}{\pi a^2/2} = -\frac{\int_{-a}^{a} x\sqrt{a^2-x^2} \, dx}{\pi a^2/2} = 0 \]

(\( \overline{x} \) must be 0 by symmetry)

\[ \overline{y} = \frac{1}{2} \frac{\pi a^2}{2} \int_{-a}^{a} y^2 \, dx \]

\[ = \frac{2}{\pi a^2} \int_{0}^{a} (a^2-x^2) \, dx \]

\[ = \frac{2}{\pi a^2} \left[ a^2x - \frac{x^3}{3} \right]_{0}^{a} \]

\[ = \frac{4a}{3\pi} \]
Example 2) Uniform semi-circular wire

radius \( a \)

Mass \( M \)

density \( \rho = \frac{M}{\pi a^2} \)

\( \bar{y} = 0 \) by symmetry

Split into arc segments

\( ds = a \theta \)

\( dm = \rho ds = a \rho \theta \)

\[ x = a \cos \theta \]

\[ \bar{x} = \frac{1}{M} \int x \, dm \]

\[ = \frac{1}{M} \int_{-\pi/2}^{\pi/2} a^2 \rho \theta \cos \theta \, d\theta \]

\[ = \frac{a^2 \rho}{M} \left[ \sin \theta \right]_{-\pi/2}^{\pi/2} = \frac{2 a^2 \rho}{M} \]

\[ = \frac{2a}{\pi} \]
Volume and Surface of Revolution

Take function $y(x)$ in 2D plane and rotate about $x$-axis to get 3D shape

To get volume $V$, surface area $S$ divide to circular cross-sections width $\delta x$

**Volume**

$$\delta V = \pi y^2 \delta x$$

$$V = \pi \int_{a}^{b} y^2 \, dx$$

**Surface area**

$$\delta S = 2\pi y \delta s$$

$$S = 2\pi \int_{a}^{b} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$
Example: Paraboloid

Take \( y = \sqrt{x} \) and rotate. Have \( 0 \leq x \leq 1 \) say.

\[
V = \pi \int_0^1 y^2 \, dx = \pi \int_0^1 x \, dx = \frac{\pi}{2}
\]

\[
S = 2\pi \int_0^1 y \sqrt{1 + y'^2} \, dx
\]

\[
y = \sqrt{x} \implies y' = \frac{1}{2} x^{-1/2}
\]

\[
S = 2\pi \int_0^1 x^{1/2} \left(1 + \frac{1}{4x}\right)^{1/2} \, dx
\]

\[
= 2\pi \int_0^1 (x + \frac{1}{4})^{3/2} \, dx
\]

\[
= \frac{4\pi}{3} \left[ \left( x + \frac{1}{4} \right)^{3/2} \right]_0^1 = \frac{4\pi}{3} \left[ \left( \frac{5}{4} \right)^{3/2} - \left( \frac{1}{4} \right)^{3/2} \right]
\]

\[
= \frac{\pi}{6} \left( 5^{3/2} - 1 \right)
\]
Chapter 4

Partial Differentiation

4.1 Definition

4.2 The total derivative

4.3 Changing variables - the chain-rule

4.4 From Cartesian to Polar

4.5 Laplace's equation

4.6 Implicit functions

4.7 Exact differentials

4.8 Stationary points
Consider a function \( u = u(x, y) \) of two independent variables \( x, y \).

E.g., local height \( u \) of a surface above point \( (x, y) \).

\[ u(x, y) = x^2 + y^2 \]

Can also picture a contour map (à la mythical geography) showing lines of equal height \( u \) (\( u = 1 \) and \( u = 4 \) say).
In general, the value of $u$ changes if we move in any direction. To cope with this we define \textbf{Partial Derivatives} as follows.

4.1 Definition

\[ \frac{\partial u}{\partial x} \] is the rate of change of $u$ as we move in the $x$ direction at fixed (constant) $y$.

\[ \frac{\partial u}{\partial x} = \lim_{\delta x \to 0} \frac{u(x+\delta x, y) - u(x, y)}{\delta x} \]

This is the \underline{first partial derivative} of $u$ w.r.t. $x$. Notation is

\[ \left( \frac{\partial u}{\partial x} \right)_y = \frac{\partial u}{\partial x} \equiv u_x \]

Same as ordinary derivative treating $y$ variable as a constant.
Similarly, the 1st derivative w.r.t. \( y \) is

\[
\frac{du}{dy} = \lim_{\delta y \to 0} \frac{u(x, y+\delta y) - u(x, y)}{\delta y}
\]

\[
= \left( \frac{du}{dy} \right)_x \equiv uy
\]

Same as ordinary derivative treating \( x \) as a constant.

**Examples**

1. \( u = x^2 y^3 \Rightarrow \frac{du}{dx} = 2xy^3, \frac{du}{dy} = 3x^2y^2 \)

2. \( u = \sin \frac{x}{y} \Rightarrow \frac{du}{dx} = \frac{1}{y} \cos \frac{x}{y} \)
\[
\frac{du}{dy} = -\frac{x}{y^2} \cos \frac{x}{y}
\]

This definition trivially generalizes to three (and more) variables \( u = u(x, y, z) \). Thus, \( \frac{\partial u}{\partial x} \) keeps both \( y \) and \( z \) constant, etc.
Can also define higher derivatives

\[ \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) = u_{xx} \]

\[ \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right) = u_{yy} \]

And mixed derivatives

\[ u_{xy} = \frac{\partial u}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial y} \right) \]

\[ u_{yx} = \frac{\partial u}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} \right) \]

However, for almost all functions:

\[ u_{xy} = u_{yx} \]

\[ \{ u_{xy} = u_{yx} \} \] can assume this.

Example

1) \[ u = x^2 y^3 \Rightarrow u_x = 2xy^3, \quad u_y = 3x^2 y^2 \]

\[ \frac{\partial^2 u}{\partial x^2} = 2y^3, \quad \frac{\partial^2 u}{\partial y^2} = 6x^2 y, \quad \frac{\partial^2 u}{\partial x \partial y} = 6xy^2 = \frac{\partial^2 u}{\partial y \partial x} \]
1) For \( u = \frac{\sin x}{y} \)

\[
\frac{du}{dx} = \frac{1}{y} \frac{\cos x}{y} \quad , \quad \frac{du}{dy} = -\frac{x}{y^2} \frac{\cos x}{y}
\]

Then

\[
\frac{d^2u}{dx^2} = -\frac{1}{y^2} \frac{\sin x}{y}
\]

\[
\frac{d^2u}{dy^2} = \frac{2x}{y^3} \frac{\cos x}{y} - \frac{x^2}{y^4} \frac{\sin x}{y}
\]

\[
\frac{d^3u}{dx^3} = \frac{1}{y^2} \frac{\cos x}{y} + \frac{x}{y^3} \frac{\sin x}{y}
\]

\[
\frac{d^3u}{dy^3} = -\frac{1}{y^2} \frac{\cos x}{y} + \frac{x}{y^3} \frac{\sin x}{y}
\]
4.2 The total derivative

Consider a function of one variable \( y(x) \).
If we make a small change \( x \to x + \delta x \),
the corresponding change in \( y \to y + \delta y \) is
\[
\delta y = \frac{dy}{dx} \delta x
\]
(from definition of derivative).

Similarly for a function of two variables:
\( u = u(x, y) \) defined \( (x, y) \to (x + \delta x, y + \delta y) \) the total change \( (u \to u + \delta u) \) is
\[
\delta u = \frac{\partial u}{\partial x} \delta x + \frac{\partial u}{\partial y} \delta y
\]
and similarly for more variables. In the infinitesimal limit, we define the total derivative
\[
\delta u = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy
\]

Very simple but with many important consequences.
Example (how to make good estimates of % changes).

Suppose \( f(x, y, z) = 8x^2y^3z^4 \) and we are told \( x \) increases by \( 1\% \), \( y \) by \( 2\% \) and \( z \) decreases by \( 1/2 \% \). What is % change in \( f \)?

\[
\delta f \approx \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial z} \delta z
\]

\[
= 16xy^3z^4 \delta x + 24xy^2z^4 \delta y + 32xyz^3 \delta z
\]

Divide by \( f = 8x^2y^3z^4 \)

\[
\frac{\delta f}{f} \approx 2 \frac{\delta x}{x} + 3 \frac{\delta y}{y} + 4 \frac{\delta z}{z}
\]

Now \( \delta x/x = 0.01 \), \( \delta y/y = 0.01 \), \( \delta z/z = -0.005 \)

\[
\frac{\delta f}{f} \approx 0.02 + 0.03 - 0.02 = 0.03
\]

\[
= 0.03 \Rightarrow 3\% \text{ increase}
\]
Simple Case:

Suppose \( u = u(x, y) \) and also \( x = x(t) \) and \( y = y(t) \). E.g. \( u = u(x, y) \) is local temperature at position \((x, y)\) of a heated 2D plate. Then \( x(t), y(t) \) could be coordinates of a body moving with time \( t \).

![Diagram of path of moving body](image)

Local temp \( u = u(x, y) \)

What is \( \frac{du}{dt} \)?

i.e. variation of temp. with time

Suppose in small time interval \( \delta t \), body moves \( x \to x + \delta x \), \( y \to y + \delta y \).

Then

\[
\delta u = \frac{\partial u}{\partial x} \delta x + \frac{\partial u}{\partial y} \delta y + \ldots
\]

Divide by \( \delta t \) and take infinitesimal limit
\[ \frac{du}{dt} = \frac{du}{dx} \frac{dx}{dt} + \frac{du}{dy} \frac{dy}{dt} \]  \hspace{1cm} \text{chain rule}

Notice "straight" $\frac{dx}{dt}$ since $x = x(t)$ is a function of the variable only, and similarly for $\frac{dy}{dt}$.

**Example**

Circular motion $x = \cos t$, $y = \sin t$.

Then for any temperature distribution $u = u(x,y)$

\[ \frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} \]

\[ u = -\sin t \, u_x + \cos t \, u_y \]

Suppose, specifically $u(x,y) = xy$

\[ u = -y \sin t + x \cos t \]

\[ = -\sin^2 t + \cos^2 t \]

\[ u = \cos 2t \]

**Check** $u(t) = x(t) \, y(t) = \cos t \, \sin t = \frac{1}{2} \sin 2t$

\[ \dot{u} = \cos 2t \quad \checkmark \]
Chain rule for two sets of independent variables

Suppose $u = u(x, y)$ and also

\[ x = x(s, t) \]
\[ y = y(s, t) \]

Best to regard $(s, t)$ as a new choice of coordinate system. Equally we could write

\[ s = s(x, y) \]
\[ t = t(x, y) \]

A very important question is: How is \( \frac{\partial u}{\partial x} \) (and \( \frac{\partial u}{\partial y} \)) related to \( \frac{\partial u}{\partial t} \) and \( \frac{\partial u}{\partial s} \)?

Suppose that small changes

\[ x \rightarrow x + \delta x, \quad y \rightarrow y + \delta y \]

correspond to changes

\[ s \rightarrow s + \delta s, \quad t \rightarrow t + \delta t. \]

Can express relation between partial derivatives in two equivalent ways:
(1) can write
\[ \delta u = \frac{\partial u}{\partial x} \delta x + \frac{\partial u}{\partial y} \delta y \]

Then dividing by \( \delta t \) or taking limit
\[ \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t} \]

dividing by \( \delta s \)
\[ \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} \]

Note: all derivatives are partial

(2) conversely, can write
\[ \delta u = \frac{\partial u}{\partial s} \delta s + \frac{\partial u}{\partial t} \delta t \]

which implies (divide by \( \delta x \) or \( \delta y \))
\[ \frac{\partial u}{\partial x} = \frac{\partial u}{\partial s} \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial x} \]
\[ \frac{\partial u}{\partial y} = \frac{\partial u}{\partial s} \frac{\partial s}{\partial y} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial y} \]
(i) A very (very) common mistake is to think
\[ \frac{dx}{ds} = \frac{1}{\frac{ds}{dx}} \quad \text{WRONG!!} \]
why? \( \frac{dx}{ds} \) means \( x = x(s, t) \) and the partial derivative is at constant \( t \) : \( \left( \frac{\partial x}{\partial s} \right)_t \)
But \( \frac{ds}{dx} \) means \( \left( \frac{\partial s}{\partial x} \right)_y \) i.e. \( s = s(x, y) \) + derivative at constant \( y \) : \( \left( \frac{\partial y}{\partial s} \right)_t \neq \left( \frac{\partial s}{\partial x} \right)_y \)

(ii) Changing variables in this manner is crucial for solving partial differential eqn (Maxwell's eqns, Heat eqn, Diffusion eqn, Schrödinger eqn).

Also understanding the invariance of laws of physics under change of co-ordinates \((x, y, z, t) \rightarrow (x', y', z', t')\) is basis of General Relativity (Einstein's theory of gravity).
4.4 Example: From Cartesian to Polar

We want to transform from Cartesian co-ordinates in the two independent variables \((x, y)\) to two new independent variables \((r, \theta)\) which are polar co-ordinates. The pair \((r, \theta)\) therefore play the role of \((a, \xi)\) in (4), (5), (7) and (8). The relation between these two sets of variables with \(x\) and \(y\) expressed in terms of \(r\) and \(\theta\) is

\[
x = r \cos \theta, \quad y = r \sin \theta
\]

whereas the other way round we have

\[
r^2 = x^2 + y^2, \quad \theta = \tan^{-1} \frac{y}{x}.
\]

From (9) we have

\[
\frac{\partial x}{\partial r} = \cos \theta, \quad \frac{\partial y}{\partial r} = \sin \theta, \quad \frac{\partial x}{\partial \theta} = -r \sin \theta, \quad \frac{\partial y}{\partial \theta} = r \cos \theta.
\]

From (10) we have

\[
\frac{\partial r}{\partial x} = \frac{x}{r} = \cos \theta, \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin \theta,
\]

and

\[
\frac{\partial \theta}{\partial x} = -\frac{y}{x^2 + y^2} = -\frac{\sin \theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2} = \frac{\cos \theta}{r}.
\]

Now we are ready to use the chain rule as in (3) and (4):

\[
\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} = \frac{\partial u}{\partial x} \cos \theta + \frac{\partial u}{\partial y} \sin \theta
\]

and

\[
\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta} = -\frac{\partial u}{\partial x} (r \sin \theta) + \frac{\partial u}{\partial y} (r \cos \theta).
\]

Conversely

\[
\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \frac{\partial \theta}{\partial x} = \frac{\partial u}{\partial r} \cos \theta - \frac{\partial u}{\partial \theta} \left( \frac{\sin \theta}{r} \right)
\]

and

\[
\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta} \frac{\partial \theta}{\partial y} = \frac{\partial u}{\partial r} \sin \theta + \frac{\partial u}{\partial \theta} \left( \frac{\cos \theta}{r} \right).
\]

From (16) and (17) we can write the derivative operations \(\frac{\partial}{\partial x}\) and \(\frac{\partial}{\partial y}\) as

\[
\frac{\partial}{\partial x} = \cos \theta \frac{\partial}{\partial r} - \left( \frac{\sin \theta}{r} \right) \frac{\partial}{\partial \theta}, \quad \frac{\partial}{\partial y} = \sin \theta \frac{\partial}{\partial r} + \left( \frac{\cos \theta}{r} \right) \frac{\partial}{\partial \theta}
\]

These operator relations are best way to think of the relationship between the partial derivatives.

Note in eqns (11) and (12) that \(\frac{\partial x}{\partial r} \neq \frac{1}{\partial \theta} \frac{\partial r}{\partial x}\)
Laplace's equation: changing from Cartesian to polar co-ordinates

Laplace's equation (a partial differential equation or PDE) in Cartesian co-ordinates is

$$u_{xx} + u_{yy} = 0.$$  \hfill (20)

We would like to transform to polar co-ordinates.

We found that the $x$ and $y$-derivatives of $u$ transform into polar co-ordinates in the following way:

$$u_x = (\cos \theta) u_r - \left( \frac{\sin \theta}{r} \right) u_\theta \quad u_y = (\sin \theta) u_r + \left( \frac{\cos \theta}{r} \right) u_\theta.$$ \hfill (21)

Likewise the operation $\frac{\partial}{\partial x}$ becomes

$$\frac{\partial}{\partial x} = (\cos \theta) \frac{\partial}{\partial r} - \left( \frac{\sin \theta}{r} \right) \frac{\partial}{\partial \theta}.$$ \hfill (22)

and the operation $\frac{\partial}{\partial y}$ becomes

$$\frac{\partial}{\partial y} = (\sin \theta) \frac{\partial}{\partial r} + \left( \frac{\cos \theta}{r} \right) \frac{\partial}{\partial \theta}.$$ \hfill (23)

Hence

$$u_{xx} = \frac{\partial u_x}{\partial x} = \left[ (\cos \theta) \frac{\partial}{\partial r} - \left( \frac{\sin \theta}{r} \right) \frac{\partial}{\partial \theta} \right] \left[ (\cos \theta) u_r - \left( \frac{\sin \theta}{r} \right) u_\theta \right].$$ \hfill (24)

Now we work this out using the product rule. Remember that $u_r$ and $u_\theta$ are functions of both $r$ and $\theta$. We get

$$u_{xx} = (\cos^2 \theta) u_{rr} + \left( \frac{\sin^2 \theta}{r^2} \right) u_r + 2 \left( \frac{\cos \theta \sin \theta}{r^2} \right) u_\theta - 2 \left( \frac{\cos \theta \sin \theta}{r} \right) u_{r\theta} + \left( \frac{\sin^2 \theta}{r^2} \right) u_{\theta\theta}.$$ \hfill (25)

Now we do the same for $u_{yy}$ to get

$$u_{yy} = \frac{\partial u_y}{\partial y} = \left[ (\sin \theta) \frac{\partial}{\partial r} + \left( \frac{\cos \theta}{r} \right) \frac{\partial}{\partial \theta} \right] \left[ (\sin \theta) u_r + \left( \frac{\cos \theta}{r} \right) u_\theta \right]$$ \hfill (26)

and therefore

$$u_{yy} = (\sin^2 \theta) u_{rr} + \left( \frac{\cos^2 \theta}{r^2} \right) u_r - 2 \left( \frac{\cos \theta \sin \theta}{r^2} \right) u_\theta + 2 \left( \frac{\cos \theta \sin \theta}{r} \right) u_{r\theta} + \left( \frac{\cos^2 \theta}{r^2} \right) u_{\theta\theta}.$$ \hfill (27)

Summing (25) and (27) and remembering that $\cos^2 \theta + \sin^2 \theta = 1$, we find that

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}.$$ \hfill (28)

and so Laplace's equation converts to

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0.$$ \hfill (29)
1) We abbreviate the Laplacian

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \nabla^2 u \]

Similarly in 3D the Laplacian of

\[ u = u(x, y, z) \]

is

\[ \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \]

pronounced "del squared". E.g. Schrödinger eq.

or wavefunction \( \Psi(x, y, z, t) \)

\[ -\frac{\hbar^2}{2m} \nabla^2 \Psi + V(x, y, z) \Psi = i\hbar \frac{\partial \Psi}{\partial t} \]

Changing from Cartesian \((x, y, z)\) to 3D polar \((r, \theta, \phi)\) required to get Hydrogen spectrum.

Extra z axis

- very similar to previous calculation.
4.6 Implicit Functions

Recall that the relation

\[ F(x, y) = 0 \]

implicitly defines \( y \) as a function of \( x \). Since \( F \) is always zero, varying \( x \) causes no change. Hence the total derivative

\[ \frac{dF}{dx} = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = 0 \]

vanishes. This means we can write the derivative \( \frac{dy}{dx} \) as

\[
\frac{dy}{dx} = - \frac{F_x}{F_y}
\]

Example. (taken from p. 35 section 2.2)

\[ F(x, y) = x^2 \sin y + xy - 1 = 0 \]

\[
\frac{dy}{dx} = - \frac{F_x}{F_y} = - \left( \frac{2x \sin y + y}{x^2 \cos y + x} \right)
\]

\( \checkmark \)
Similarly if we have
\[ F(x, y, z) = 0 \]
this defines
\[ x = x(y, z) \] or \[ y = y(x, z) \] or \[ z = z(x, y) \]
is above, no change in \( F \) implies
\[
\frac{\partial F}{\partial x} \frac{dx}{dx} + \frac{\partial F}{\partial y} \frac{dy}{dy} + \frac{\partial F}{\partial z} \frac{dz}{dz} = 0
\]
Thus at constant \( y \):
\[
\left( \frac{\partial z}{\partial x} \right)_y = - \frac{F_x}{F_z}
\]
at constant \( x \):
\[
\left( \frac{\partial z}{\partial y} \right)_x = - \frac{F_y}{F_z}
\]
at constant \( z \):
\[
\left( \frac{\partial y}{\partial x} \right)_z = - \frac{F_x}{F_y}
\]

Example

In thermodynamics the equation of state of a gas/liquid is written
\[ F(p, V, T) = 0 \] (1)
\[ p = \text{pressure}, \quad V = \text{volume}, \quad T = \text{absolute temperature} \]

This implicitly defines \( p = p(V,T) \). Only for simple cases (e.g., the ideal gas \( p = RT/V \), van der Waals theory) can explicitly write \( p = p(V,T) \).

Nevertheless, from the general equation \( \Box \) one can show

\[
\left( \frac{\partial P}{\partial V} \right)_T \left( \frac{\partial V}{\partial T} \right)_P \left( \frac{\partial T}{\partial P} \right)_V = -1
\]

Example of an exact thermodynamic identity (see problem sheet).
4.7 Exact Differentials

We know that for a function of two variables, \( u = u(x, y) \), the total derivative is

\[
\frac{du}{dx} = \frac{\partial u}{\partial x} \, dx + \frac{\partial u}{\partial y} \, dy
\]  \( \text{(1)} \)

Both the \( \frac{\partial u}{\partial x} \) and \( \frac{\partial u}{\partial y} \) will, in general, be functions of \( x \) and \( y \).

Now consider opposite. Suppose we are treated with a differential

\[
P(x, y) \, dx + Q(x, y) \, dy
\]  \( \text{(2)} \)

Does this mean that this \( \text{(2)} \) is the total derivative of an unknown function of two variables \( u \) \( \text{a la (1)} \)? The answer is not necessarily.

If \( \text{(2)} \) is the total derivative of something, then we must have

\[
P = \frac{\partial u}{\partial x} \quad \text{and} \quad Q = \frac{\partial u}{\partial y}
\]
Now recall \( \frac{\partial u}{\partial x} \, dy = \frac{\partial u}{\partial y} \, dx \), thus consistency demands

\[
\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}
\]

(2)

If this condition fulfilled then eqn (2) is the total derivative of a function \( u = u(x, y) \) (which we can then try to find). In this case (2) is called an exact differential.

If (2) not obeyed, (2) is not exact and no function \( u(x, y) \) exists.

Exact and Inexact differentials are crucial in thermodynamics.

Example

1) Is \( y^2 \, dx + (x^2 + 2y) \, dy \) exact?

\( P = y^2 \) \( \Rightarrow \) \( \partial P/\partial y = 2y \)

\( Q = x^2 + 2y \) \( \Rightarrow \) \( \partial Q/\partial x = 2x \)

\[ \text{No} \]

not exact
\[ y \, dx + (x + 2y) \, dy \]
\[ P = y \implies \frac{\partial P}{\partial y} = 1 \quad \text{Exact} \]
\[ Q = x + 2y \implies \frac{\partial Q}{\partial x} = 1 \]
To find \( u \) we integrate and match
\[ P = \frac{\partial u}{\partial x} = y \implies u = xy + f(y) \]
\[ \uparrow \quad \text{unknown function of } y \]
\[ Q = \frac{\partial u}{\partial y} = x + 2y \implies u = xy + y^2 + g(x) \]
\[ \uparrow \quad \text{unknown function of } x \]
Comparing both requirements we see
\[ u = xy + y^2 + C \]
For a function of one variable \( u = u(x) \), stationary points are located at

\[
\frac{du}{dx} = 0
\]

Their nature is determined by the 2nd derivative:

- **Max**: \( \frac{d^2u}{dx^2} < 0 \)
- **Min**: \( \frac{d^2u}{dx^2} > 0 \)

Now consider a function of two variables \( u = u(x, y) \) e.g., height of some surface.
There are now **three** types of stationary point.

- **Maximum**

- **Minimum**

**Contour map**
- Rings of decreasing (constant) \( u \), away from \( SP \).
Saddle point

Eg
For functions of two variables

Stationary points located at simultaneous solutions of two equations

\[
\frac{\partial u}{\partial x} = 0 \quad \text{0}
\]
\[
\frac{\partial u}{\partial y} = 0 \quad \text{2}
\]

To determine nature calculate value of

\[
E = \left( \frac{\partial^2 u}{\partial x \partial y} \right)^2 - \frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial^2 u}{\partial y^2}
\]

at each S.P. Then

\[
E > 0 \implies \text{saddle}
\]

\[
E < 0 \implies \begin{cases} 
\text{Max}^m \quad b \left( \frac{\partial^2 u}{\partial x^2} \right) < 0 \\
\text{Min}^m \quad b \left( \frac{\partial^2 u}{\partial x^2} \right) > 0
\end{cases}
\]

This follows from Double Taylores series (see later)
Example: Find S.P.s of

\[ u = x^3 + xy^2 + x^2y - 6x + y^3 - 6y \]

de \[ \frac{\partial u}{\partial x} = 3x^2 + y^2 + 2xy - 6 \]

de \[ \frac{\partial u}{\partial y} = 2xy + x^2 + 3y^2 - 6 \]

- Locations

Simultaneous solution of

\[ 3x^2 + y^2 + 2xy - 6 = 0 \] \hspace{1cm} (1)

\[ 2xy + x^2 + 3y^2 - 6 = 0 \] \hspace{1cm} (2)

Subtract (1) - (2)

\[ y^2 = x^2 \]

\[ y = \pm x \] \hspace{1cm} \text{ (beware losing \( \pm \) signs)}

Substitute back into (1)

\[ 4x^2 \pm 2x^2 - 6 = 0 \] \hspace{1cm} \text{for } y = \pm x
\(-\sqrt{1} \Rightarrow x^2 = 1 \Rightarrow x = \pm 1 \quad (\text{recall} \ y = x)\)

\(\sqrt{3} \Rightarrow x^2 = 3 \Rightarrow x = \pm \sqrt{3} \quad (\text{recall} \ y = -x)\)

has stationary points at

\((1, 1)\), \((-1, -1)\), \((\sqrt{3}, -\sqrt{3})\), \((-\sqrt{3}, \sqrt{3})\)

Next determine

\[ E = u_{xy}^2 - u_{xx} u_{yy} \]

so have

\[ u_{xx} = 6x + 2y \quad \text{and} \quad u_{yy} = 2x + 6y \]

\[ u_{xy} = 2y + 2x \]

\[ \Rightarrow E = 4(y + x)^2 - 4(3x + y)(x + 3y) \]

\[ = -8(x^2 + y^2 + 4xy) \]
Hence

\((1, 1) : E = -48 \Rightarrow \text{max/min} \)
\[\text{but } u_{xx} = 8 \Rightarrow \text{Minimum}\]

\((-1, -1) : E = -48 \Rightarrow \text{max/min} \)
\[\text{but } u_{xx} = -8 \Rightarrow \text{maximum}\]

\((\sqrt{3}, -\sqrt{3}) : E = 48 \Rightarrow \text{saddle}\]

\((-\sqrt{3}, \sqrt{3}) : E = 48 \Rightarrow \text{saddle}\]

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Chapters 5 and 6

Maclaurin Series and Infinite Series etc

5.1 Taylor and Maclaurin series

5.2 Double Taylor series

6.1 Convergence and Divergence in General

6.2 Definitions and Theorems

6.3 Four tests for convergence and the Ratio Test
Chapter 5: Taylor and Maclaurin Series

5.1 Taylor Series

Consider a function \( f(x) \) whose value, and the value of all its derivatives, are known at a point \( x_0 \).

Let us try to evaluate function at a "nearby" point, \( x_0 + h \), using a power series

\[
f(x_0 + h) = \sum_{n=0}^{\infty} a_n h^n
\]

\[
= a_0 + a_1 h + a_2 h^2 + a_3 h^3 + \ldots
\]

with coefficients \( a_n \) to be determined.

Setting \( h = 0 \) we have
\[ f(x_0) = \alpha_0 \]  
\[ f'(x_0 + h) = \alpha_1 + 2\alpha_2 h + 3\alpha_3 h^2 + \ldots \]  

Next differentiate (1) w.r.t. \( h \)

\[ f'(x_0) = \alpha_1 \]  
\[ f''(x_0 + h) = 2\alpha_2 + 3 \cdot 2 \alpha_3 h + \ldots \]  

Continuing in this manner...

\[ f'''(x_0 + h) = 3 \cdot 2 \alpha_3 + 4 \cdot 3 \cdot 2 \alpha_4 h + \ldots \]  

\[ \Rightarrow \frac{f''(x_0)}{2} = \alpha_2 \]  

\[ \Rightarrow \alpha_3 = \frac{f'''(x_0)}{3 \cdot 2} \]  

General result

\[ \alpha_n = \frac{f^{(n)}(x_0)}{n!} \]
Hence we arrive at the Taylor series (an infinite power series)

\[ f(x_0 + h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} h^n \]

The Taylor series about the origin (set \( x_0 = 0 \) and write \( h = x \)) is

\[ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \]

\[ = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \ldots \]

and is called the \textbf{Maclaurin Series}.

The Maclaurin series is only valid for small enough \( |x| \) (see later).
Examples of \( f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \)

1) What is Maclaurin series for \( f(x) = e^x \)?

\[
 f^{(n)}(x) = e^x \quad \Rightarrow \quad f^{(n)}(0) = 1
\]

Hence

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots
\]

\[
= \sum_{n=0}^{\infty} \frac{x^n}{n!}
\]

2) \( f(x) = \sin x \quad \Rightarrow \quad f^{(n)}(0) = 0 \)

We have

\[
f^{(1)}(x) = \cos x, \quad f^{(2)}(x) = -\sin x, \quad f^{(3)}(x) = -\cos x
\]

\[
f^{(1)}(0) = 1, \quad f^{(2)}(0) = 0, \quad f^{(3)}(0) = -1
\]

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}
\]
Similarly

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Note. From above three Maclaurin series we find Euler's identity

$$e^{ix} = \cos x + i \sin x$$

$$i = \sqrt{-1}$$

(3) \( f(x) = \frac{1}{1-x} \)

The \( n \)-th derivative \( n! (1-x)^{-n-1} \)

$$\Rightarrow f^{(n)}(0) = n!$$

Hence Maclaurin for \( f(x) = (1-x)^{-1} \)

$$f(x) = 1 + x + x^2 + x^3 + \cdots$$

$$= \sum_{n=0}^{\infty} x^n$$
\( f(x) = \ln(1 + x) \)

Maclaurin

\[ x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \]

All above Maclaurin series worth remembering

Votes

1) We stated, but did not prove, that each Maclaurin series is only valid for certain ranges of \( |x| < 1 \) (because infinite power series don't always add up well).

Can see this in function \( f(x) = \frac{1}{1-x} \).

Putting \( x = -1 \) \( \Rightarrow \) \( f(-1) = \frac{1}{2} \) (true value)

But according to Maclaurin series:

\[ \sum_{n=0}^{8} x^n \]

\[ = 1 - 1 + (-1)^2 + (-1)^3 + \ldots \]

\[ = 1 - 1 + 1 - 1 + 1 - 1 + \ldots \]

which is meaningless! (see later)
(ii) L'Hopital's rule revisited

Consider the limit

\[
\lim_{x \to x_0} \frac{f(x)}{g(x)}
\]

\[
= \lim_{h \to 0} \frac{f(x_0 + h)}{g(x_0 + h)} \quad \text{provided} \quad g(x_0) \neq 0
\]

Assuming Taylor series exists for each function, we can write

\[
= \lim_{h \to 0} \frac{f(x_0) + hf(x_0) + \frac{h^2}{2} f''(x_0) + \cdots}{g(x_0) + hg'(x_0) + \frac{h^2}{2} g''(x_0) + \cdots}
\]

Now if

\[
f(x_0) = g(x_0) = 0 \quad \text{i.e.} \quad \frac{0}{0}
\]

\[
= \lim_{h \to 0} \frac{hf(x_0) + \frac{h^2}{2} f''(x_0) + \cdots}{hg'(x_0) + \frac{h^2}{2} g''(x_0) + \cdots}
\]

\[
= \lim_{h \to 0} \frac{f'(x_0) + \frac{h}{2} f''(x_0) + \cdots}{g'(x_0) + \frac{h}{2} g''(x_0) + \cdots}
\]

\[
= \frac{f'(x_0)}{g'(x_0)} \quad L'Hopital
\]
(i) **Double Taylor Series**

This is Taylor series for function of two variables \( u(x, y) \).

Suppose we know value of \( u \) and all partial derivatives at \((x_0, y_0)\).

What is value at \( u(x_0 + h, y_0 + k) \), with \( h \) and \( k \) "small" (in some sense)?

Answer is

\[
u(x_0 + h, y_0 + k) = u(x_0, y_0) + \left\{ \frac{h \partial u}{\partial x} + \frac{k \partial u}{\partial y} \right\} + \frac{1}{2!} \left\{ \frac{h^2 \partial^2 u}{\partial x^2} + 2hk \frac{\partial^2 u}{\partial x \partial y} + \frac{k^2 \partial^2 u}{\partial y^2} \right\} + \ldots
\]

with all derivatives evaluated at \((x_0, y_0)\).

More elegantly,

\[
u(x_0 + h, y_0 + k) = \sum_{n=0}^{\infty} \frac{1}{n!} (D^n u)
\]

with \( D = \frac{h \partial}{\partial x} + \frac{k \partial}{\partial y} \).

\[D^2 u = \left( \frac{h \partial}{\partial x} + \frac{k \partial}{\partial y} \right) \left( \frac{h \partial u}{\partial x} + \frac{k \partial u}{\partial y} \right) = \frac{h^2 \partial^2 u}{\partial x^2} + 2hk \frac{\partial^2 u}{\partial x \partial y} + \frac{k^2 \partial^2 u}{\partial y^2} \]
Proof

First, keep $y_0 + k$ constant and use normal Taylor on the "x" bit

$$u(x_0 + h, y_0 + k) = u(x_0, y_0 + k) + h \frac{\partial u}{\partial x} (x_0, y_0 + k) + \frac{h^2}{2!} \frac{\partial^2 u}{\partial x^2} (x_0, y_0 + k) + \ldots$$

Now expand each term on R.H.S. using normal Taylor for the "y" part

$$u(x_0, y_0 + k) = u(x_0, y_0) + k \frac{\partial u}{\partial y} (x_0, y_0) + \frac{k^2}{2!} \frac{\partial^2 u}{\partial y^2} (x_0, y_0) + \ldots$$

$$\frac{\partial u}{\partial x} (x_0, y_0 + k) = \frac{\partial u}{\partial x} (x_0, y_0) + \frac{\partial^2 u}{\partial x \partial y} (x_0, y_0) \bigg|_{x_0, y_0} + \frac{k^2}{2!} \frac{\partial^2 u}{\partial y^2} (x_0, y_0) + \ldots$$

$$\frac{\partial^2 u}{\partial x^2} (x_0, y_0 + k) = \frac{\partial^2 u}{\partial x^2} (x_0, y_0) + k \frac{\partial^2 u}{\partial x \partial y} (x_0, y_0) \bigg|_{x_0, y_0} + \frac{k^2}{2!} \frac{\partial^4 u}{\partial y^4} (x_0, y_0) + \ldots$$

Just add to get

$$u(x_0 + h, y_0 + k) = u(x_0, y_0) + \left\{ h \frac{\partial u}{\partial x} + k \frac{\partial^2 u}{\partial y^2} \right\} + \frac{h^2}{2!} \frac{\partial^2 u}{\partial x^2} + \frac{k^2}{2!} \frac{\partial^4 u}{\partial y^4} + \ldots$$
Chapter 6: Series and Convergence

Introduction

Our main goal here is to clarify the nature of Maclaurin expansion

\[ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \]

or what range of \( x \) values is this meaningful? (i.e., "small" does \( x \) have to be?).

Let's be more general and consider the infinite sum:

\[ S = \sum_{n=0}^{\infty} U_n \]  \hspace{1cm} (1)

What does this "infinite" sum mean?

To make sense we truncate and define the partial sum (series of sums actually)

\[ S_N = \sum_{n=0}^{N} U_n \]  \hspace{1cm} (2)
Then \( \lim_{N \to \infty} S_N \) exists (approach a number) we say \( \text{series converges} \).

Otherwise, \( \text{series diverges} \) and the infinite sum \( \sum_{n=0}^{\infty} u_n \) is meaningless.

Note. A series diverging means that \( S_N \) increases without bound as \( N \to \infty \) or does not approach a limit.

E.g. \( \sum_{n=0}^{\infty} (-1)^n \) meaningless because

\[
S_N = \sum_{n=0}^{N} (-1)^n \text{ does not approach a limit}
\]

\[
S_1 = (-1)^0 + (-1)^1 = 0
\]
\[
S_2 = (-1)^0 + (-1)^1 + (-1)^2 = 1
\]
\[
S_3 = (-1)^0 + (-1)^1 + (-1)^2 + (-1)^3 = 0
\]
\[
S_4 = \cdots \cdots \cdots \cdots = 1
\]
\[
S_5 = \cdots \cdots \cdots \cdots = 0
\]
Examples

1) The Geometric series

\[ S_N = 1 + x + x^2 + \cdots + x^N \]

can evaluate as follows:

\[ xS_N = x + x^2 + x^3 + \cdots + x^N + x^{N+1} \]

\[ S_N = 1 + x + x^2 + \cdots + x^N \]

\[ S_N (1-x) = 1 - x^{N+1} \]

\[ S_N = \frac{1 - x^{N+1}}{1-x} \]

Hence

Infinite sum \( S = \lim_{N \to \infty} S_N \) exists only for \(|x| < 1\). For \(|x| \geq 1\) (including cases \( x = 1 \), \( x = -1 \)) series diverges.

Hence Maclaurin expansion of \( f(x) = \frac{1}{1-x} \)

\[ = 1 + x + x^2 + \cdots \] only valid for \(|x| < 1\).
(2) A well known example of a diverging series is \( \sum_{n=1}^{\infty} \frac{1}{n} \).

Proof

\[
\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \ldots
\]

\[
= 1 + \frac{1}{2} + \left( \frac{1}{3} + \frac{1}{4} \right) + \left( \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \right) + \ldots
\]

\[
\sum_{n=1}^{\infty} \frac{1}{n} > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots
\]

thus \( \sum_{n=1}^{\infty} \frac{1}{n} \to \infty \) as \( N \to \infty \)

6.2 More Definitions + Theorems

An infinite series \( s = \sum_{n=0}^{\infty} u_n \) is convergent and is said to be absolutely convergent if

\[
\sum_{n=0}^{\infty} |u_n|
\]
is convergent. This definition is useful because of the following theorem:

**Absolute convergence implies convergence**

---

Note: If all $u_n$'s are positive then obviously $\sum u_n$ and $\sum |u_n|$ are same.

Theorem is useful if $u_n$'s change sign or are complex numbers.

**Example**

The alternating series

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots$$

is convergent because it is also absolutely convergent i.e.

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$$

$$= \frac{1}{2} \left( 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \right)$$

$$= \frac{1}{2} \left( 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots \right)$$
\[ \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{2}} = \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{2}} \]

Of course this does not mean that
\[ \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \ldots \] adds up to 1
but it does mean it converges (to something). In fact easy to see that
\[ \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \ldots = \frac{1}{3}, \quad \text{(Using result for geometric series)} \]

Alternatively if \( \sum_{n=0}^{\infty} u_n \) converges but
\[ \sum_{n=0}^{\infty} |u_n| \] does not, then the series \( \sum_{n=0}^{\infty} u_n \) is said to be conditionally convergent.

E.g. \( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \) is conditionally convergent because it converges (adds up to \( \ln 2 \)) but
$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges, as we saw earlier.

5.3 Four Tests for Convergence

$$\sum_{n=0}^{\infty} u_n$$

A) A necessary, but not sufficient, condition is that

$$\lim_{n \to \infty} u_n = 0$$

(i.e. numbers we are adding up must tend to zero).

Necessary means, if this condition is not obeyed then sum is definitely divergent.

Not sufficient means, even if $u_n \to 0$, sum may still diverge (recall $\sum \frac{1}{n}$ is divergent even though $\frac{1}{n} \to 0$)
Thus we can be sure that
\[ \sum_{n=1}^{\infty} \cos\left(\frac{1}{n^2}\right) \] is divergent because
\[ \cos \frac{1}{n^2} \to 1 \quad \text{as} \quad n \to \infty. \]

\[ \Box \quad \text{Comparison Test} \]

If \( \sum_{n}^{\infty} u_n \) is given and we can find a converging series \( \sum_{n}^{\infty} b_n \), with non-negative \( b_n \), such that
\[ |u_n| \leq b_n \quad \text{for all} \quad n \]
then \( \sum_{n}^{\infty} u_n \) is absolutely convergent.

Similarly, if we can find a diverging series \( \sum_{n}^{\infty} b_n \) with non-negative \( b_n \) such that
\[ u_n \geq b_n \]
then \( \sum_{n}^{\infty} u_n \) is divergent.
Thus \[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \] is divergent because

\[ \frac{1}{\sqrt{n}} \geq \frac{1}{n} \] for all \( n \) and we know \( \sum \frac{1}{n} \) is divergent.

ii) **Leibniz test** for alternating series.

If we have a series of form

\[ \sum_{n=0}^{\infty} (-1)^n a_n \]

with

1. \( a_n \) positive
2. \( a_n \) form a decreasing sequence: \( a_{n+1} \leq a_n \)
3. \( a_n \to 0 \) as \( n \to \infty \)

Then series \( \text{converges} \).

Thus \[ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \] \( \text{converges} \).
D) The Ratio Test

\[ \sum_{n=1}^{\infty} u_n \]

and suppose \( u_n \neq 0 \) for every \( n \).

Define

\[ L = \lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| \]

Then, if

\[ \begin{cases} L < 1 & \Rightarrow \text{absolute convergence} \\ L > 1 & \Rightarrow \text{divergence} \\ L = 1 & \text{don't know.} \end{cases} \]

If \( L = 1 \), don't know.

The ratio test allows us to determine the range of convergence of Maclaurin series.
Example 1

1. \( f(x) = e^x \)

Maclaurin

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} u_n
\]

with \( u_n = \frac{x^n}{n!} \). Then \( u_{n+1} = \frac{x^{n+1}}{(n+1)!} \)

and so

\[
\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \frac{1}{n+1}
\]

(recall \((n+1)! = (n+1)n(n-1)\ldots 1 = (n+1)n!\))

Thus

\[
L = \lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right|
\]

\[
= \lim_{n \to \infty} \frac{1}{n+1} = 0 \quad \forall x
\]

Since \( L < 1 \Rightarrow \) Maclaurin series for \( e^x \) converges \( \forall x \).
2) \( f(x) = \sin x \)

Maclaurin

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}
\]

\[U_n = \frac{(-1)^n x^{2n+1}}{(2n+1)!}\]

\[
\Rightarrow U_{n+1} = \frac{(-1)^n x^{2n+3}}{(2n+3)!(2n+1)!}
\]

\[
\left| \frac{U_{n+1}}{U_n} \right| = \left| \frac{-x^{2n+3}}{(2n+3)!(2n+1)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right|
\]

\[
= \frac{x^2}{(2n+3)(2n+2)}
\]

Thus \( L = \lim_{n \to \infty} \frac{x^2}{(2n+3)(2n+2)} = 0 \) \( \forall x \)

Expansion always convergent.
Same is true of Maclaurin expansion for \( \cos x \).
\[ f(x) = \frac{1}{1-x} \quad (\text{we know answer already}) \]

**Maclaurin**

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n
\]

\[
= 1 + x + x^2 + x^3 + \cdots
\]

\[ u_n = x^n \implies u_{n+1} = x^{n+1} \]

\[ L = \lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to \infty} 1 \times 1 = 1 \times 1 \]

Thus, expansion valid for \( |x| < 1 \) \( \checkmark \)

\[ f(x) = \ln(1+x) \]

**Maclaurin**

\[ \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \]

\[
\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}
\]

\[ u_n = \frac{(-1)^n x^{n+1}}{n+1} \implies u_{n+1} = \frac{(-1)^{n+1} x^{n+2}}{n+2} \]
\[
\left| \frac{u_{n+1}}{u_n} \right| = \left| -x \cdot \frac{n+1}{n+2} \right|
\]

Thus

\[
L = \lim_{n \to \infty} 1 \times 1 \frac{n+1}{n+2}
\]

\[
= \lim_{n \to \infty} \frac{n(1+\frac{1}{n})}{n(1+\frac{2}{n})}
\]

\[
= 1 \times 1 \lim_{n \to \infty} \frac{1+\frac{1}{n}}{1+\frac{2}{n}}
\]

\[
= 1 \times 1
\]

Thus, Maclaurin O.K. if \( |x| < 1 \).

(5) Find full range of convergence of

\[
\sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots
\]

First apply Ratio Test

\[
u_n = \frac{x^n}{n} \implies u_{n+1} = \frac{x^{n+1}}{n+1}
\]
Then
\[ \left| \frac{u_{n+1}}{u_n} \right| = \frac{1 \times 1 \cdot n}{n+1} \]

\[ L = \lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to \infty} \frac{n}{n+1} = 1 \times 1 \cdot \lim_{n \to \infty} \frac{1}{1 + \frac{1}{n}} = 1 \times 1 \]

This series converges if \( 1 \times 1 \times 1 \)
series diverges if \( 1 \times 1 \times 1 \)

Rahko Test does not work for \( L = 1 \)
\[ \Rightarrow 1 \times 1 \times 1 \text{ or } x = +1, -1 \]

For \( x = +1 \), \[ \sum \frac{x^n}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \]
\[ \text{Diverges} \]

For \( x = -1 \), \[ \sum \frac{(-1)^n}{n} = -1 \left( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \right) \]
\[ \text{Converges, by Leibniz} \]

Range of convergence \( -1 < x < 1 \)