PHYSICS I: MATHEMATICAL ANALYSIS I.
PROBLEMS 1

1. If \( f(x) = x^2 - 3x + 2, \) find \( f(0), f(x^2), f(x + 1) \). For what values of \( x \) does \( f(x) = 0 \)?
For what values of \( x \) does \( f(2x) = 0 \)?

2. Find the inverse of each of the functions:
   \begin{align*}
   (a) \quad & f(x) = 3x + 4, \quad \text{all real } x; \\
   (b) \quad & f(x) = 2x + x^2, \quad 0 < x < 1.
   \end{align*}

3. Are the following functions even, odd or neither?
   \begin{align*}
   (a) \quad & x^3 + 2 \sin x; \quad (b) \quad (1 + x^4)^{1/2} \cos 3x; \\
   (c) \quad & x + |x|; \quad (d) \quad \sin^2 x.
   \end{align*}

4. Evaluate the following limits:
   \begin{align*}
   (a) \lim_{x \to \infty} \frac{x^2 + 1}{x^2 - 1}; \quad & (b) \lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}.
   \end{align*}

5. Evaluate the limits:
   \begin{align*}
   (a) \lim_{x \to \infty} x \sin \left( \frac{1}{x} \right); \quad & (b) \lim_{x \to 0} \frac{x^3 + 2}{x^3 + x - 2}.
   \end{align*}
Hint for (b): Either use L'Hôpital's Rule or put \( x = 1 + h \) and use the binomial expansion.

Starred Question

6a. Given the definitions (from the lectures) of the hyperbolic functions
   \[
   \cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}, \quad \tanh x = \frac{\sinh x}{\cosh x}
   \]
show that
   \begin{enumerate}
   \item \( \cosh^2 x - \sinh^2 x = 1 \),
   \item \( \cosh^2 x + \sinh^2 x = \cosh 2x \),
   \item \( \sinh(x_1 + x_2) = \sinh x_1 \cosh x_2 + \sinh x_2 \cosh x_1 \),
   \item \( \frac{d}{dx} \tanh x = \text{sech}^2 x \) \quad (\text{sech} x = \frac{1}{\cosh x}).
   \end{enumerate}
Note the differences in the signs in 1) and 2) from the trigonometric cases.
PHYSICS 1: MATHEMATICAL ANALYSIS I.
PROBLEMS 2

1. Differentiate \( z^2 \cos(5x + 1); \) \( \ln(\sec x + \tan x); \) \( z/(z + 1). \)

2. Find \( dy/dx \) when (a) \( y^3 = x^3 - xy; \) (b) \( x^2y = \cos(xy). \)

3. Sketch the graphs of the following:
   (a) \( y = x + 1/x, \) \( x \neq 0; \)
   (b) \( y = \ln(1 - x^2), \) \(-1 < x < 1; \)
   (c) \( r = \alpha(1 - \cos \theta) \) where \( r \) and \( \theta \) are polar coordinates and \( \alpha \) is a positive constant.

Notes Plane polar co-ordinates \((r, \theta)\) are related to Cartesian co-ordinates \((x, y)\) by \( x = r \cos \theta \) and \( y = r \sin \theta; \) hence \( r^2 = x^2 + y^2 \) and \( \theta = \tan^{-1} \left( \frac{y}{x} \right). \)

4. Find the stationary points of the function \( f(x) = x^2(1 - x)^3 \) and determine their nature. Sketch the graph \( y = f(x). \)

5. If \( r(1 + \cos \theta) = 2, \) where \( r \) and \( \theta \) are plane polar coordinates, express the equation in terms of cartesian coordinates \((x, y); \) show that the graph is a parabola and sketch it.

STARRED PROBLEMS

6* Differentiate \( y = \sin^{-1}(x/(1 + x)) \) and \( y = \sec^{-1}(x). \)

7* Find where the function
\[
f(x) = \frac{2x^2 - 5x - 25}{x^2 + x - 2}
\]
is discontinuous. Find also the points where it is zero, its limiting values as \( x \to \pm \infty \) and its maximum and minima. Hence sketch its graph.
PHYSICS 1: MATHEMATICAL ANALYSIS I.
PROBLEMS 3

1. Integrate by parts:
   (a) $x^3 \sin x$;  
   (b) $\tan^{-1} x$.

2. Evaluate the integrals:
   (a) $\int_0^1 (1+x^2)^{-3/2} \, dx$  
   (b) $\int_0^\infty \left(1 + e^{2x}\right)^{-1} \, dx$  
   (c) $\int_1^3 (2-x)^{-1} (x-1)^{-1/2} \, dx$.

   Hint: In (a) substitute $x = \tan \theta$; in (b) substitute $u = e^{2x}$; in (c) use the substitution $(x-1) = u^2$.

3. Which of the following integrals are convergent?
   (a) $\int_0^1 \ln x \, dx$;  
   (b) $\int_0^2 (x-1)^{-2} \, dx$;

4. Show that
   $$\int x^k \ln x \, dx = \frac{x^{k+1}}{(k+1)^2} [(k+1) \ln x - 1] + c$$
   where $c$ is a constant and $k \neq -1$.

STARRED PROBLEMS

5* Which of the following integrals are convergent?
   (a) $\int_1^\infty \ln x \, dx$;  
   (b) $\int_0^\infty e^{-ax} \sin bx \, dx$, $(a > 0)$.

6* Integrate
   (a) $\frac{x^4}{x^2 + 1}$;  
   (b) $\frac{1}{x \ln x}$
PHYSICS I: MATHEMATICAL ANALYSIS I

PROBLEMS 4

1. Put into partial fractions and hence find the indefinite integral of

\[ f(x) = \frac{2x^2 - x + 2}{x(x - 1)(x + 1)} \]

2. By using the trigonometric formula \( \sin(A + B) + \sin(A - B) = 2 \sin A \cos B \) calculate the indefinite integral

\[ I = \int \sin 3x \cos 5x \, dx \]

3. Recall from the lectures that the mean value \( \bar{f} \) of a function \( f(x) \) over an interval \( 0 \leq x \leq a \) is given by

\[ \bar{f} = \frac{1}{a} \int_0^a f(x) \, dx \]

Find the mean value of \( f(x) = \sin x \) in the interval \( 0 \leq x \leq \pi \), and of \( f(x) = \sin^2 x \) in the interval \( 0 \leq x \leq 2\pi \).

4. If

\[ I_n = \int_0^{\pi/3} \sin^n x \, dx \]

where \( n \geq 0 \) is an integer, show that \( I_n = \frac{n-1}{n} I_{n-2} \), for \( n \geq 2 \). Hence show that

\[ I_8 = \int_0^{\pi/3} \sin^8 x \, dx = \frac{35}{233} \pi \]

STARRED PROBLEMS

8* Calculate the length of the curve

\[ y = \frac{x^3}{a^3} + \frac{a^2}{12x} \]

from \( x = a/2 \) to \( x = a \), where \( a \) is a positive constant.

6* If

\[ I = \int_0^{\pi/2} \frac{\sin^{1/3} x}{\sin^{1/3} x + \cos^{1/3} x} \, dx \]

use the substitution \( x = \pi/2 - y \) to show that

\[ I = \int_0^{\pi/2} \frac{\cos^{1/3} x}{\sin^{1/3} x + \cos^{1/3} x} \, dx \]

Hence show that \( I = \pi/4 \).
Recall from your notes that:

(i) Plane polar co-ordinates \((r, \theta)\) are related to Cartesian co-ordinates \((x, y)\) by \(x = r \cos \theta\) and \(y = r \sin \theta\) hence \(r^2 = x^2 + y^2\) and \(\theta = \tan^{-1} \left(\frac{y}{x}\right)\).

(ii) In Cartesian co-ordinates a small element of arc length \(ds\) is related to the small elements \(dx\) and \(dy\) by \((ds)^2 = (dx)^2 + (dy)^2\) (with an additional \((ds)^2\) in \(3D\)). In plane polar co-ordinates this converts to \((ds)^2 = (dr)^2 + r^2(d\theta)^2\).

(iii) Volume of revolution is \(\pi \int_a^b y^2\, dx\) whose surface area is \(2\pi \int_a^b y\, ds\).

1. Find the lengths of the following curves:
   (a) The catenary \(y = \cosh x\) from \(x = 0\) to \(x = 1\). [Answer: \(\text{sinh 1}\).]
   (b) The circular helix expressed in parametric form \(x = \cos t, y = \sin t\) and \(z = t\) from \(t = 0\) to \(t = 2\pi\). [Answer: \(2\sqrt{2}\pi\).]
   (c) The curve \(y = x^{3/2}\) from \((0, 0)\) to \((4, 8)\). [Answer: \(\frac{3}{2} \left(10^{3/2} - 1\right)\).]
   (d) One branch of the 4-cusped hypocycloid expressed in parametric form \(x = \cos^3 t, y = \sin^3 t\) from \(t = 0\) to \(t = \pi/2\). [Answer: \(3\pi/2\).]

2. Show that the area of one loop \((-\pi/4 \leq \theta \leq \pi/4)\) of the lemniscate \(r^2 = a^2 \cos 2\theta\) is \(a^2/2\).

3. Find the position of the centre of mass of a uniform thin wire in the form of a circular arc of radius \(a\), subtending an angle of \(2\gamma\) at the centre. [Answer: \(a \sin \gamma/\gamma\) from the origin.]  

4. Find the area enclosed by the ellipse \((x/a)^2 + (y/b)^2 = 1\). Assuming this elliptical area to be of uniform density, find also the position of the centre of gravity of the part that lies in the first quadrant. [Answers: \(a\sqrt{ab}\) and \((4a/3\pi, 4b/3\pi)\).]

**STARRED PROBLEMS**

5* Show that \(8a\) is the total length of the closed curve called the cardioid (heart shape)

\[ r = a(1 - \cos \theta). \]

6* Show that the length of one arch (\(0 \leq t \leq 2\pi\)) of the cycloid defined by

\[ x = a(t - \sin t) \quad y = a(1 - \cos t) \]

is \(8a\). Show that the area of the surface obtained by a complete revolution of this arch about the \(x\)-axis is \(64\pi a^2/3\).
PHYSICS I: MATHEMATICAL ANALYSIS I.

PROBLEMS 6

1. Calculate \( \partial u / \partial x \) and \( \partial u / \partial y \) if \( u = e^x y - y^2 + 3x - 1 \).

2. Find the relation between the constants \( \alpha \) and \( \beta \) if the function \( u = \cos x \cos y \) satisfies Laplace's equation:
   \[
   \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.
   \]
   
   Show also that the following function \( u(x, y) \) satisfies Laplace's equation:
   \[
   u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1.
   \]

3. If \( g = \tan^{-1}(y/x) \), calculate \( \partial g / \partial x \) and \( \partial g / \partial y \), and show that
   \[
   x \frac{\partial g}{\partial x} + y \frac{\partial g}{\partial y} = 0.
   \]

4. If \( u = x^2 + 3y^2 \) and \( x = s + t, y = 2s - t \), calculate \( \frac{\partial u}{\partial s} \) and \( \frac{\partial u}{\partial t} \) (i) by using the chain rule, and (ii) by first expressing \( u \) as a function of \( s \) and \( t \).

5. A closed box has variables sides of length \( x, y \) and \( z \) but a fixed volume \( V \). Show that the shape of the box is a cube when the surface area \( A \) is minimum. Note at a stationary point of a function of two variables \( u = u(x, y) \) the two partial derivatives \( \partial u / \partial x \) and \( \partial u / \partial y \) need to be zero simultaneously.

STARRED PROBLEMS

\( \star \) If \( u = x \ln(x^2 + y^2) - 2y \tan^{-1}(y/x) \), verify that
   \[
   x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = u + 2x.
   \]

\( \star \) The equation of state of a gas, relating pressure \( p \), volume \( V \) and temperature \( T \), is \( f(p, V, T) = 0 \) and hence
   \[
   \frac{\partial f}{\partial p} dp + \frac{\partial f}{\partial V} dV + \frac{\partial f}{\partial T} dT = 0.
   \]

Show that
   \[
   \frac{\partial f}{\partial V} \left( \frac{\partial V}{\partial T} \right)_p = - \frac{\partial f}{\partial p} \frac{\partial V}{\partial T} + \frac{\partial f}{\partial T} \frac{\partial V}{\partial p},
   \]
   and obtain similar expressions for \( \partial f / \partial V \) and \( \partial f / \partial p \). Deduce that
   \[
   \frac{\partial f}{\partial T} \left( \frac{\partial V}{\partial T} \right)_T = \left. \frac{\partial^2 f}{\partial V \partial p} \right|_V = -1.
   \]

---

\( ^1 \) In this problem we need not specify the function \( f \); it is left as an arbitrary function of the three independent variables \( p, V \) and \( T \) but for an ideal gas it would take the form \( f = pV = RT \). In fact, you can verify some of the above formulas using this relation.
1. If \( f \) and \( g \) are any twice-differentiable functions, use the chain rule, along with the new variables \( s = x + y \) and \( t = x + \frac{1}{2}y \), to show that
\[
V(x, y) = f(x + y) + g(x + \frac{1}{2}y)
\]
satisfies the partial differential equation
\[
V_{xx} - 3V_{xy} + 2V_{yy} = 0,
\]
where the suffixes denote partial derivatives.

2. If \( u = u(x, y) \) and \( x \) and \( y \) transform into two new variables \( s \) and \( t \) such that \( s = 2x^2 + y^2 \) and \( t = 3x^4 + y^2 \), show that
\[
u_s^2 + u_t^2 = \left( u_s^2 + u_t^2 \right) \left( x^2 + y^2 \right)^2.
\]

3. Are the following exact differentials? If so, of what functions?

   (i) \( e^y dx + x(e^x + 1) dy \)
   (ii) \( (e^x + ye^y) dx + (e^x + xe^x + 1) dy \)

**STARRED QUESTION**

4* If \( u = u(x, y) \) and \( x \) and \( y \) are related to two new independent variables \( s \) and \( t \) by
\[
x = xf, \\
y = yf,
\]
use the chain rule to find \( \frac{\partial u}{\partial x} \) in terms of \( \frac{\partial u}{\partial s} \) and \( \frac{\partial u}{\partial y} \) in terms of \( \frac{\partial u}{\partial s} \) and \( \frac{\partial u}{\partial t} \). Solve this to show that
\[
\frac{\partial u}{\partial x} = s \frac{\partial u}{\partial s} + t \frac{\partial u}{\partial t},
\]
and
\[
\frac{\partial u}{\partial y} = \left( s^2 - t^2 \right) \left( \frac{1}{s} \frac{\partial u}{\partial s} - \frac{1}{t} \frac{\partial u}{\partial t} \right).
\]
1. Use the ratio test to determine whether the following two series are convergent:

\[ \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{(3-4i)^n}{n!}. \]

2. Show that \( y = \tan x \) satisfies the equation

\[ \frac{dy}{dx} = 1 + y^2. \]

By repeated differentiation of this result, find the higher derivatives that are required to determine the first three non-zero terms of the Maclaurin series for \( \tan x \).

3. Show that there are two stationary values of the function

\[ u(x, y) = \frac{x^2 + y^2 + 2x + 1}{x + y}. \]

By considering the second partial derivatives \( u_{xx}, u_{yy} \) and \( u_{xy} \), show that one is a maximum and the other is a minimum.

4. Sketch contours (curves of constant \( u \)) for the function \( u = xy(x + y - 1) \) and indicate regions where \( u \) is zero, positive, and negative respectively. Locate the stationary points of the function and deduce their nature from the contour diagram. Now use the standard method of calculating the sign of \((u_{xy} - u_{xx}u_{yy})\) etc at each stationary point to confirm your findings.

STARRED PROBLEM

5* Show that the function

\[ u(x, y) = x^4 + 4x^2y^3 - 2x^2 + 2y^2 - 1 \]

has three stationary points, two of which are minima, the other being a saddle.
1) \( f(0) = 2; \quad f(x) = x^4 - 3x^2 + 2 \)
   \( f(x+1) = (x+1)^4 - 3(x+1)^2 + 2 = x^4 - x \)
   \( f(x) = 0 \) or roots of \( x^2 - 3x + 2 = 0 \), namely \( x = 1 \) and 2
   so \( f(x) = 0 \) at \( x = \frac{1}{2} \) and 1.

2) a) \( y = 5x + 4 \implies x = \frac{1}{5}(y-4) \)
   Hence \( f^{-1}(x) = \frac{1}{5}(y-4) \) for all \( x \).

b) \( y = 2x + x^2 \quad (0 < x < 1) \) so the range is \( 0 < y < 3 \).
   Solve the quadratic in \( x \), i.e. \( x^2 + 2x - y = 0 \),
   which gives \( x = -1 \pm \sqrt{1+y} \). Note that the
   2nd root \( x = -1 - \sqrt{1+y} \) is negative and out
   of the domain \( 0 < x < 1 \). We reject this root,
   leaving \( f^{-1}(x) = -1 + \sqrt{1+y} \) for \( 0 < x < 3 \).

3) (a) Neither (b) Even (c) Neither (d) Odd

4) a) \( \lim_{x \to 0} \frac{x^4}{x^2} = \lim_{x \to 0} \frac{1 + \frac{1}{x^2}}{1 - \frac{1}{x^2}} = 1 \)
   \( \lim_{x \to 0} \frac{(1+x)^{1/2} - (1-x)^{1/2}}{x} = \lim_{x \to 0} \frac{1}{x} \left[ (1 + \frac{1}{x^2} + \cdots) - (1 - \frac{1}{x^2} - \cdots) \right] \)
   \( = 1 \) Binomial Theorem.

b) \( \lim_{x \to 0} x \sin \left( \frac{1}{x} \right) = \lim_{y \to 0} (\sin y) = 1 \)
   \( y \to 0 \)

\( \lim_{x \to 1} \frac{x^4 + x - 2}{x^4 + x^2 - 2} = \lim_{x \to 1} \frac{9x^4 + 1}{4x^4 + 1} = \frac{10}{5} = 2 \)

Alternatively, \( \left[ x^4 + x - 2 \right] \) Rule gives,

\( \lim_{x \to 1} \frac{x^4 + x - 2}{x^4 + x^2 - 2} = \lim_{x \to 1} \frac{9x^4 + 1}{4x^4 + 1} = \frac{10}{5} = 2 \).
Solutions to Sheet 2

1. (i) \( \frac{dy}{dx} = 3x^2 \cos(5x+1) - 5x^3 \sin(5x+1) \) \quad \text{(Product Rule)}

(ii) \( \frac{dy}{dx} = \sec(x) \tan(x) + x \sec(x) \) \quad \text{(Note: \( \frac{d}{dx} \sec(x) = \sec(x) \tan(x) \))}

(iii) \( \frac{dy}{dx} = \frac{x+1-x}{(x+1)^2} = \frac{1}{x+1} \) \quad \text{(Quotient Rule)}

2. (a) \( y^2 = x^3 + yx \)  
    Hence \( 2y \frac{dy}{dx} = 3x^2 - y + x \frac{dy}{dx} \)

(b) \( x e^y = \cos(xy) \)  
    LHS: \( \frac{d}{dx} (xe^y) = e^y + x \frac{dy}{dx} e^y \)
    RHS: \( \frac{d}{dx} (\cos(xy)) = -\sin(xy)(y + x \frac{dy}{dx}) \)

\( \therefore \frac{d}{dx} (xe^y + x \sin(xy)) = -(y \sin(xy) + e^y) \)

so \( \frac{dy}{dx} = -\frac{y \sin(xy) + e^y}{x(e^y + \sin(xy))} \)

3. (a)

4. Stationary pt when \( f'' = 0 \):
   \( f'(x) = 2x(1-x)^3 - 3x^2(1-x)^2 \)
   \( x = 0 \) (min), \( x = 1 \) (inflexion), \( x = \frac{7}{5} \) (max)

5. \( r(1+e^x) = 2 \Rightarrow r+x = 2 \)
   because \( x = \cos \theta \), hence
   \( r = (2-x)^2 \)
   \( x^2 y'' = (2-x)^2 \)
   \( y'' = 4(1-x) \) \quad \text{(Parabola)}
\[ 1. \quad J \int x^3 \sin x \, dx = - \frac{1}{3} x^3 \cos x + 3 \int x^2 \cos x \, dx \\
= - \frac{1}{3} x^3 \cos x + 3 \int x^2 \cos x - 6 \int x \sin x \, dx \\
= - \frac{1}{3} x^3 \cos x + 3 x^2 \sin x - 6 \int x \sin x \, dx \\
= - \frac{1}{3} x^3 \cos x + 3 x^2 \sin x + 6 \int x \cos x \, dx \\
= - \frac{1}{3} x^3 \cos x + 3 x^2 \sin x + 6 \sin x + c \\
\]

\[ 4. \quad J \frac{\tan^{-1} x}{x} \, dx = \tan^{-1} x - \int \frac{x}{1 + x^2} \, dx \\
= \tan^{-1} x - \int \frac{dx}{1 + x^2} = \tan^{-1} x - \ln(1 + x^2) + c. \]

\[ 2. \quad \text{Use } u = \tan^{-1} x \]
\[ J \frac{du}{(1 + u^2) u} = \int \frac{1}{(1 + u^2) u} \, du = \int u \, dv = \int \frac{1}{1 + u^2} \, du \\
= \frac{1}{2} \left[ \ln 1 - \ln \frac{1}{u} \right] = \frac{1}{2} \ln 2. \]

\[ 3. \quad \int \frac{dx}{x(\ln x - 1)^2} = \int \frac{2x \, dx}{(1 - u^2)x} = \int \frac{dx}{1 - u^2} = \int \frac{1}{1 - u} \, du \\
= - \left[ \ln \left| \frac{1 + u}{1 - u} \right| \right]_0^1 = \ln \frac{2}{\sqrt{2}} = \ln 2. \]

\[ 5. \quad \text{Let } x = e^{\theta} \quad \text{and } \text{then } dx = e^{\theta} \, d\theta \]
\[ J \int e^{\theta} d\theta = [1 \times (e^\theta - 1)]_0^\infty = 1 - e \cdot (e^\infty - 1) = e - 1 - e \cdot \infty \\
\text{The integral does not converge} \quad \text{as } e \to \infty. \]

\[ 6. \quad J \frac{x \, dx}{(x + 1)^2} = \frac{1}{2} \int \frac{dx}{x + 1} = \frac{1}{2} \ln |x + 1| + c \\
= \frac{1}{2} \left[ \ln (x + 1) - \frac{1}{x + 1} \right] + c = (x + 1)^{-1} \left[ (x + 1) \ln x - 1 \right] x^{x_0} + c. \]
1. \( f(x) = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} \Rightarrow A(x-1)(x+1) + Bx(x+1) + Cx(x-1) = 2x^2 - x + 2. \)

\((A, B, C) = (-2, \frac{9}{6}, \frac{5}{6}) \). Hence

\[ \int f(x) \, dx = \int \left( \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} \right) \, dx = -2 \ln |x| + \frac{9}{2} \ln |x-1| + \frac{5}{6} \ln |x+1| + c. \]

2. \( \sin^3 x \cos x = \frac{1}{4} \left[ \sin 3x - 3 \sin x \right] \) from trig. formula

\( I = \frac{1}{4} \int (\sin 3x - 3 \sin x) \, dx = -\frac{1}{12} \cos 3x + \frac{3}{4} \cos 2x + c \)

3. \( \sin x = \frac{1}{2} \int_0^x \sin u \, du = -\frac{1}{2} \left[ \cos x \right]_0^x = -\frac{1}{2} [1 - 1] = \frac{1}{2} \pi \)

\( \sin^2 x = \frac{1}{2} \int_0^{2x} \sin^2 u \, du = \frac{1}{4} \pi \int_0^2 (1 - \cos 2u) \, du = \frac{1}{2} \)

4. \( J_n = \int_0^{\pi/2} \sin^n x \, dx = -\int_0^{\pi/2} \sin^{n-2} x \cos x \, dx \)

\[ = -\left[ \sin^{n-2} x \cos x \right]_0^{\pi/2} + (n-2) \int_0^{\pi/2} \sin^{n-2} x \cos^2 x \, dx \quad n \geq 1 \]

\[ = 0 + (n-2) \int_0^{\pi/2} \sin^{n-2} x \cos^n x \, dx \]

\[ = (n-2) \int_{n-2}^{n-1} \]

Solve for \( I_n : \quad n I_n = (n-1) I_{n-2} \quad n \geq 2 \)

\( I_1 = \frac{1}{1} I_0 = \frac{3}{2} \). \( I_2 = \frac{3}{4} \). \( I_3 = \frac{3}{8} \). \( I_4 = \frac{3}{6} \). \( I_5 = \frac{3}{8} \). \( I_6 = \frac{3}{10} \). \( I_7 = \frac{3}{12} \). \( I_8 = \frac{3}{14} \). \( I_9 = \frac{3}{16} \).

\( I_k = \frac{3}{2k+1} \).

\( I_0 = \int_0^{\pi/2} \sin x \, dx = \pi/2 \).

\( I_k = \frac{3}{2k+1} \).
7) a) \( y' = \sin x \)
\[ S = \int_0^1 (1 + \sin^2 x)'^2 \, dx = \int_0^1 \cos x \, dx = \sin 1 \]

b) \( x = \cos t, y = \sin t, z = t \)
\[(ds)^2 = [(-\sin t \, dt)^2 + (\cos t \, dt)^2 + (dt)^2] = \int_0^2 \sqrt{2} \, dt = 2\sqrt{2} \pi \]

c) \( y = x^{4/3} \)
\[ S = \int_0^1 \left(1 + \frac{4}{3} \, y \right)^{1/3} \, dy = \frac{3}{5} (10^{4/3} - 1) \]

d) \( u = \cos^2 t, y = \sin t \)
\[ S = \frac{1}{2} \int_0^{\pi/2} \cos^2 t \, dt = \frac{1}{2} \int_0^{\pi/2} \cos^2 t \, dt = \frac{\pi}{4} \]

8) Area \( = \frac{1}{2} \int_{-\pi/4}^{\pi/4} \cos 2 \theta \, d\theta = \frac{\pi}{2} \)

9) \[ \bar{x} = \frac{\int x \, p \, ds}{\int p \, ds} = \frac{\int x \, p \, ds}{\int p \, ds} \]

10) Area of ellipse \( = \pi \int_0^a y \, dx \)
\[ = 4\pi \int_0^a (1 - \frac{x^2}{a^2}) \, dx \]

Put \( x = a \cos \theta \)
\[ A = -4a \int_0^\pi \sin^2 \theta \, d\theta = \pi a^2 \]

11) \( \rho \bar{A} = \int x y \, dx \)
\[ = \int_0^\pi \sin^2 \theta \, d\theta \]

Mass \( \bar{m} = \rho \bar{A} \)
\[ \bar{A} = -a \int_0^\pi \cos \sin^2 \theta \, d\theta = \int_0^\pi \frac{a^{2/3}}{2} \]

Note: \( \bar{A} \) is the area of the 1st quadrant. Hence,
\[ \bar{A} = \frac{\pi a^2}{3} \]

By symmetry, \( y = \frac{\pi a^2}{2} \).
1) \( \frac{\partial u}{\partial y} = 8xy + 3 \), \( \frac{\partial u}{\partial y} = 4x^2 - 2y \\
\)

2) \( u = e^{xy} \cos y \), \( \frac{\partial u}{\partial x} = \alpha e^{xy} \cos y \), \( \frac{\partial u}{\partial y} = \beta e^{xy} \sin y \), \( \frac{\partial^2 u}{\partial y^2} = -\beta^2 e^{xy} \cos y \)

\( \alpha^2 - \beta^2 = 0 \) to satisfy Laplace's eqn. \( u = x \pm \beta y \)

3) \( u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1 \)
\( \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x \), \( \frac{\partial u}{\partial y} = 6x + 6 \) \( \frac{\partial^2 u}{\partial y^2} = -6x - 6 \)

Sum is zero.

3) \( f = \tan^{-1} (\frac{y}{x}) \):
\( \frac{\partial f}{\partial x} = \frac{\frac{1}{1+(\frac{y}{x})^2} \cdot \frac{y}{x}}{1+(\frac{y}{x})^2} = \frac{y}{x^2+y^2} \)
\( \frac{\partial f}{\partial y} = \frac{\frac{1}{1+(\frac{y}{x})^2} \cdot 1}{1+(\frac{y}{x})^2} = \frac{x}{x^2+y^2} \)

Hence \( \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 0 \)

4) i) Chain Rule:
\( \frac{\partial u}{\partial x} = \frac{3u}{x} + \frac{2y}{y} = \frac{3u}{x} + \frac{2y}{y} \)
\( \frac{\partial u}{\partial y} = 2x + 2 \frac{\partial u}{\partial y} \)
\( v = 2x + 2 \frac{\partial u}{\partial y} \)
\( v = 2x + 2 \frac{\partial u}{\partial y} \)
\( v = 2x + 2 \frac{\partial u}{\partial y} \)
\( v = 2x + 2 \frac{\partial u}{\partial y} \)

Now \( u = x^2 + y^2 = (x+y)^2 + 3(2x+y)^2 = x^2 + 2st + t^2 \)
\( + 3(8s^2 - 12st + 6st^2 - 6t^2) \)

So \( \frac{\partial u}{\partial x} = 2s + 2t + 3 (24x^2 - 24xt + 6t^2) \) Same as (4)

Do the same for \( \frac{\partial u}{\partial y} = 2t + 2s + 3 (24y^2 - 24yt + 6y^2) \)

5) \( V = xy \) \( (V \ fixed) \) \( A = 2(xy + y + x) \)

Eliminate \( x \) using \( x = \frac{V}{y} \).

\( \frac{\partial A}{\partial x} = 2 \left( \frac{y}{x} + \frac{x}{x} \right) \)
\( \frac{\partial A}{\partial y} = 2 \left( \frac{2y}{x} - \frac{y}{y} \right) \)

For \( A_x = 0 \) and \( A_y = 0 \) together we have
\( V = x \frac{y}{x} + y \frac{x}{y} \) with \( V = xy \). Only solutions
\( x = y = z = \frac{V}{x} \). Minimum by inspection.
1) \[ V(x, y) = f(x, y) + g(x + ty) \]
\[ u_x = f_x + g_x \]
\[ u_y = f_y + g_y \]
\[ v = f(u) + g(v) \]
\[ \frac{\partial V}{\partial u} = \frac{\partial f}{\partial u} + \frac{\partial g}{\partial u} = f_x + g_x \]
\[ \frac{\partial V}{\partial v} = \frac{\partial f}{\partial v} + \frac{\partial g}{\partial v} = f_y + g_y \]

Note that 1) implies that the derivative operator \( \frac{\partial}{\partial u} \)

can be written as
\[ \frac{\partial}{\partial u} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \]

and similarly
\[ \frac{\partial}{\partial v} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \]

2) \[ s = \frac{u}{x^2 + y^2} \]
\[ t = \frac{y}{x^2 + y^2} \]
\[ \frac{\partial s}{\partial u} = -\frac{2xy}{(x^2 + y^2)^2} \]
\[ \frac{\partial s}{\partial y} = \frac{2x}{(x^2 + y^2)^2} \]

Chain rule:
\[ u_x = u_s \frac{\partial s}{\partial u} + u_t \frac{\partial t}{\partial u} = [u_s (y^2 - x^2) - 2xy u_t] (x^2 + y^2)^{-2} \]
\[ u_y = u_s \frac{\partial s}{\partial u} + u_t \frac{\partial t}{\partial y} = [-2xy u_s + (x^2 - y^2) u_t] (x^2 + y^2)^{-2} \]

\[ (u_s^2 + u_t^2)(x^2 + y^2)^4 = u_s^2 (x^2 + y^2)^2 + u_t^2 (x^2 + y^2)^2 \]

3) Write the differential as \( P \, dx + Q \, dy \).

To be able to write this as \( df \) and find \( f(x, y) \)

we need (notes) \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \).

i) \( P = e^x \), \( Q = x(e^y + 1) \)

Clearly \( \frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} \) No!

ii) \( P = e^x + ye^x \), \( Q = e^x + e^y + 1 \)

\[ \frac{\partial P}{\partial y} = e^x + e^x, \quad \frac{\partial Q}{\partial x} = e^x + e^y. \]

Yes! \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \)

\[ f = xe^x + ye^x + 1 \]
1) \[ u_n = \frac{1}{(n+1)!} \implies \lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \lim_{n \to \infty} \frac{1}{(n+1)n} = 0. \text{ Converges.} \]

2) \[ u_n = \frac{2(-4)^n}{n!} \implies \lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \lim_{n \to \infty} \frac{2(-4)^n}{(n+1)n} \left( \frac{3-4}{1-(3/2)} \right) = \lim_{n \to \infty} \frac{2}{n+1} = 0. \text{ Converges.} \]

\[ \text{As } \frac{dn}{dx} = 4x^3 + 3x \text{, we need } \Delta x \to 0 \text{ as } n \to \infty. \]

Differentiate the MRF equation to find: \[ y'' = 2yy'; \quad y''' = 2(2yy'' + y'y). \]

At \( x = 0 \): \( y = 0, \ y' = 1, \ y'' = 0, \ y''' = 2, \ y^{(4)} = 0 \), \( y^{(6)} = 1 \).

\[ y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!} x^2 + \frac{y'''(0)}{3!} x^3 + \frac{y^{(4)}(0)}{4!} x^4 + \ldots = 0 + x + 0 + \frac{2}{6} x^3 + 0 + \ldots \]

\[ y = x + \frac{x^3}{3} + \frac{x^5}{5!} + \ldots \]

3) \[ u_n = \frac{x^3 + 2x^2 + 2x + 1}{(x+y)^2} \quad \text{and } \quad u_y = \frac{2yx + y^2 - x - 1}{(x+y)^2} \]

\[ u_x = u_y = 0 \text{ at } x = y = 0. \]

Together we have \( x = 0, \ y = 0, \) and \( x = 0, \ y = 1. \) Two points \((0,0), (0,1)\).

After a bit of work: \[ u_{xx} = \frac{6y^2 - 6xy + x}{(x+y)^3}, \quad u_{yy} = \frac{4x^2 + 4xy + 1}{(x+y)^3} \]

\[ u_{xy} = \frac{2(x^2 - 2xy + y)}{(x+y)^3} \]

\[ (-1,0): \ u_{xx} = -2, \ u_{yy} = -2, \ u_{xy} = 0 \text{ MAX.} \]

\[ (0,1): \ u_{xx} = 1, \ u_{yy} = 2, \ u_{xy} = 0 \text{ MIN.} \]

(Note: At the max \( u = 0 \) while at the min \( u = 2. \) How can this be? Consider that \( u \) becomes infinite along the line \( y = x. \))

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Signs in circles refer to the sign of \( u. \)

\[ u_x = y{(2xy - 1)}, \quad u_y = x{(x + y - 1)} \]

Shear in set at \((0,0), (0,1), (1,0) \) at \((1,1)\).

Consider changes of sign in \( u \) around each point. Clearly \((0,0), (0,1), (1,0) \) are SADDLES.

Check \( u_{xx}^2 - u_{xy}^2 > 0, \) This is 1 for three points, and \( -1 \) for \((1,0) \). This is MINIMUM as \( u_{xx} > 0 \) here.