Quantum Physics Problem Sheet 1

Note that these questions are ordered by topic, not by difficulty. Do not get disheartened if you find some of the questions near the beginning of the sheet hard.

Units and Magnitudes

1. Metallic aluminium has atomic weight 27 and density 2700 kg m$^{-3}$. Estimate the radius of an aluminium atom in Ångstroms.

2. Electrons (mass $m = 0.511$ MeV/c$^2$) in a television tube are accelerated through a potential difference of 25 kV. Using relativistic formulae and units (MeV/c$^2$ for energies, MeV/c for momenta, and so on), calculate the final speed of the electrons as a fraction of the speed of light. Are relativistic effects important?

3. The ionisation energy of a hydrogen atom is 13.6 eV. By equating the thermal energy scale, $k_B T$, to the ionisation energy, estimate the temperature of the universe when the first neutral H atoms formed.

Travelling Waves

4. Without referring to your lecture notes, show that the wave
   \[ \psi(x, t) = a \cos(-k x - \omega t + \phi) \]
   has wavelength $\lambda = 2\pi/k$ and frequency $\nu = \omega/2\pi$. Show that the wave crests travel with speed $\omega/k$ in the $-x$ direction.

5. The dispersion relation of large deep ocean waves is $\omega = \sqrt{gk}$. What is the phase velocity (the velocity of the wave crests) when the wavelength is 10 m?

Complex Representation of Waves, Interference and Diffraction

6. Use complex numbers to show that $\cos \theta + \sin \theta = \sqrt{2} \cos(\theta - \pi/4)$. [Hint: $\sin \theta = \text{Re}(\sqrt{2}e^{i\theta})$.]
7. Consider the function
\[ \psi(x, t) = a \cos(kx - \omega t) + a \cos(-kx - \omega t) \]
obtained by superposing right- and left-going travelling waves of equal amplitude. Using (i) real numbers only, and (ii) complex numbers, show that \( \psi(x, t) \) is a standing wave and calculate its intensity \( I(x) \). Show that the position average of \( I(x) \) is \( 2a^2 \). Why is this result expected?

8. This question takes you through the theory of diffraction — the way that waves spread out as they emerge from a narrow opening. Although quite difficult, it provides good practice with the complex representation of waves and will prove important when we discuss the uncertainty principle. It is worth the effort.

[Diagram of a slit and wavefronts]

Parallel monochromatic plane waves of wavelength \( \lambda = 2\pi/k \) and frequency \( \nu = \omega/2\pi \) pass through a narrow slit stretching from \( y = -d/2 \) to \( y = d/2 \). To work out the wave emerging at angle \( \theta \), it is helpful to imagine dividing the slit into huge numbers of tiny segments, each of height \( \Delta y \). The amplitude of the wave emerging from each segment is proportional to the height of that segment, and so the wave emerging from the segment at \( y = 0 \) may be written as
\[ Ae^{i(k\zeta - \omega t)} \Delta y, \]
where \( \zeta \) measures the distance from the centre of the slit in the \( \theta \) direction and the complex constant \( A \) encodes the overall amplitude and phase of the wave.

(i) Write down the wave emerging from the segment \( \Delta y \) at height \( y \).

(ii) Show that the total wave emerging in the \( \zeta \) direction is
\[ \psi(\zeta, t) = Ae^{i(k\zeta - \omega t)} \int_{-d/2}^{d/2} e^{iky \sin \theta} dy. \]
(iii) Evaluate the integral and hence show that the intensity emerging at angle \( \theta \) is

\[
I(\theta) = \frac{|A|^2 d^2 \sin^2 \left( \frac{k d \sin \theta}{2} \right)}{\left( \frac{k d \sin \theta}{2} \right)^2}.
\]

(iv) Sketch \( I \) as a function of \( \frac{1}{2} k d \sin \theta \). What is the path-length difference between the top and the bottom of the slit when \( \theta \) corresponds to the first zero of the diffraction pattern?

Physical Constants

\[
\begin{align*}
m_e & \approx 9.11 \times 10^{-31} \text{kg} \approx 511 \text{keV}/c^2, \\
\text{atomic mass unit} & \approx 1.66 \times 10^{-27} \text{kg}, \\
h & \approx 6.63 \times 10^{-34} \text{Js}, \\
h & \approx 1.05 \times 10^{-34} \text{Js}, \\
c & \approx 3.00 \times 10^8 \text{ms}^{-1}, \\
e & \approx 1.60 \times 10^{-19} \text{C}, \\
g & \approx 9.8 \text{ms}^{-1}, \\
N_A & \approx 6.02 \times 10^{23}, \\
R & \approx 8.314 \text{JK}^{-1}, \\
k_B & \approx 1.38 \times 10^{-23} \text{JK}^{-1}.
\end{align*}
\]

Numerical Answers

1. Any radius between 1.25 Å and 1.6 Å acceptable.
2. \( v/c \approx 0.30 \).
3. 160,000 K.
4. 3.96 ms\(^{-1} \).
Quantum Physics Problem Sheet 2

Note that these questions are ordered by topic, not by difficulty. Do not get disheartened if you find some of the questions near the beginning of the sheet hard.

Photons

1. The energy flux of starlight reaching us from a sixth-magnitude star (approximately the faintest that can be seen by the naked eye) is \(1.4 \times 10^{-10}\) Wm\(^{-2}\). If you are looking at such a star, how many photons enter your eye per second? On average, how many photons are inside your eye at any one time? Do these estimates provide any evidence that the human eye can detect single photons? (Assume that the diameter of your dark-adapted pupil is 0.7 cm, the length of your eye is 4 cm, and the wavelength of the light is 500 nm.)

2. Electrons in an X-ray tube are accelerated by a potential difference of 30 kV. What is the minimum wavelength of the resulting X-rays?

3. In a photoelectric effect experiment, a current was observed when the metallic cathode was illuminated with light of wavelength 310 nm, but none was observed with longer wavelength light.
   
   (i) Estimate the work function of the cathode in eV.
   
   (ii) What is the energy (in eV) of a photon of wavelength 200 nm?
   
   (iii) Find the stopping potential (in Volts) at 200 nm. Hence find the maximum kinetic energy (in eV) of the electrons emitted at this wavelength.

4. In a photoelectric effect experiment, the cathode was illuminated with EM radiation of three different frequencies and the stopping potential \(V_0\) was measured for each.

<table>
<thead>
<tr>
<th>Frequency (\nu) (10(^{15}) Hz)</th>
<th>Stopping Potential (V_0) (V)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.75</td>
<td>1.0</td>
</tr>
<tr>
<td>1.00</td>
<td>2.0</td>
</tr>
<tr>
<td>1.25</td>
<td>3.0</td>
</tr>
</tbody>
</table>

   (i) Make a sketch of stopping potential against frequency and hence estimate the work function of the metal in eV.

   (ii) From the slope of your line, estimate the value of Planck’s constant \(h\). How close is this to the accepted value?
5. X-rays of energy 20 keV are Compton scattered by a thin metal foil through an angle of 60°. Find (i) the wavelength of the scattered photons, and (ii) the energy (in eV) lost by each photon. Would the kinetic energy of the recoiling electrons be sufficient to allow some of them to escape from the metal?

What value of the scattering angle θ would give the largest change in wavelength? What is the maximum possible wavelength of the scattered photon?

6. (Q40-63 from Young and Freedman) Nuclear fusion reactions at the centre of the sun produce gamma-ray photons with energies of order 1 MeV. By contrast, what we see emanating from the sun’s surface are visible photons with wavelengths of order 500 nm. Models of the solar interior explain this wavelength difference by suggesting that every photon is Compton scattered about $10^{26}$ times during its journey from the centre of the sun to the surface.

(i) Estimate the average increase in wavelength of a solar photon per Compton-scattering event.

(ii) Find the angle in degrees through which the photon is scattered in the scattering event described in part (i). (Hint: a useful approximation is $\cos \phi \approx 1 - \phi^2/2$, which is valid when $\phi$ is $\ll 1$ radian.)

(iii) It is estimated that a photon takes about $10^6$ years to travel from the core to the surface of the sun. Find the average distance that light can travel within the interior of the sun without being scattered.

As you can see, the sun is very opaque, and the radiation has plenty of opportunity to reach equilibrium with the matter before emerging. This explains why the sun emits black-body radiation.

Physical Constants

- $h \approx 6.63 \times 10^{-34}$ Js
- $\hbar \approx 1.05 \times 10^{-34}$ Js
- $c \approx 3.00 \times 10^8$ ms$^{-1}$
- $e \approx 1.60 \times 10^{-19}$ C
- $m_e \approx 9.11 \times 10^{-31}$ kg

Numerical Answers

1. Number of photons entering eye per second $\approx 13,500$; expected number inside eye at any one time $\approx 1.8 \times 10^{-6}$.
2. $4.14 \times 10^{-11}$ m.
3. (i) 4.00 eV; (ii) 6.22 eV; (iii) Stopping potential 2.22 V; maximum KE 2.22 eV.
4. (i) 2.0 eV.
5. (i) $6.34 \times 10^{-11}$ m; (ii) 378 eV (with large and difficult to estimate numerical uncertainty). Angle for largest change in wavelength is $\theta = 180°$; maximum possible scattered wavelength is $6.71 \times 10^{-11}$ m.
6. (i) $5 \times 10^{-33}$ m; (ii) $3.68 \times 10^{-9}$ degrees; (iii) $9.46 \times 10^{-5}$ m.
Quantum Physics Problem Sheet 3

Note that these questions are ordered by topic, not by difficulty. Do not get disheartened if you find some of the questions near the beginning of the sheet hard.

De Broglie Waves

1. Neutron diffraction is often used to study the atomic positions and atomic-scale magnetic fields in solids. Suggest a reasonable value for the de Broglie wavelength of the neutrons used in such experiments. Find the kinetic energy (in eV) of neutrons of this wavelength. Use the equation \( E = \frac{3k_B T}{2} \) to translate the kinetic energy into an equivalent temperature.

2. Electrons of energy 100 eV pass through a narrow slit of width 1 \( \mu \)m. What is the distance between the zeros of intensity on either side of the central peak of the electron diffraction pattern 1 m away from the slit? (Hint: you worked out the form of the diffraction pattern in Q8 of Problem Sheet 1.)

3. An electron and a positron (which has the same mass as an electron but the opposite charge) annihilate at rest to produce two photons.
   (i) The two photons have the same wavelength and travel in opposite directions. Why?
   (ii) Calculate the photon momentum and wavelength.

4. Show that the de Broglie wavelength of a particle of mass \( m \) and kinetic energy \( 3k_B T/2 \) is
   \[
   \lambda = \frac{h}{\sqrt{3mk_B T}}.
   \]

   The molar volume of liquid \(^4\)He is 27.6 cm\(^3\). Assuming that each atom occupies a cube of side \( d \), calculate the temperature at which \( \lambda = d \). Discuss the temperature range over which the wave-like properties of the atoms in liquid \(^4\)He are likely to be important.
   (Incidentally, liquid \(^4\)He becomes a superfluid, able to flow without friction, below 2.17 K. The above calculation suggests that superfluidity is almost certainly a QM phenomenon.)

5. When liquid \(^4\)He freezes, every atom is confined to a “box” (its lattice site in the crystal). Since liquids and solids normally have similar densities, the box size is similar to the value
of $d$ calculated in Q4. Assuming that the de Broglie wave of wavelength $\lambda$ associated with the $^4\text{He}$ atom has to equal zero at the box walls, show (perhaps by drawing a diagram) that

$$d = n\lambda/2,$$

where $n$ is any integer $> 0$. Hence calculate the smallest possible momentum and kinetic energy of the confined atom. At what temperature $T$ would the thermal kinetic energy $3k_B T/2$ equal the quantum mechanical KE of confinement?

Note: The idea behind this question is important and very general. Since the de Broglie wavelength of a confined particle has to “fit in” to the confining box, the particle must have a non-zero momentum and KE. The particle must therefore be moving — rattling backwards and forwards inside the box — even at zero temperature. This confinement-induced motion is called zero-point motion, and the corresponding kinetic energy is called zero-point energy. The orbits of electrons in atoms can be viewed as a type of zero-point motion.

In most solids, the atoms are heavy enough and the chemical bonds strong enough that the zero-point motion of the atoms (as opposed to the electrons) is unimportant. In $^4\text{He}$, however, where the atoms are comparatively light and the bonding is very weak, the zero-point motion alone is sufficient to melt the solid — there is no need to heat it up! This is why liquid $^4\text{He}$ remains liquid right down to $T = 0$ K. To solidify $^4\text{He}$, it is necessary to apply pressure.

The Bohr Atom

6. Light of frequency $7.316 \times 10^{14}$ Hz is emitted in a downward transition to the $n = 2$ energy level of a hydrogen atom. What was the initial energy level?

7. Generalise the derivation of the Bohr model given in lectures to obtain a formula for the energy levels of ions such as He$^+$ or Al$^{12+}$, which have atomic number $Z > 1$ ($Z = 2$ for He; $Z = 13$ for Al) but only one orbiting electron.

8. (i) What is the shortest wavelength photon that could be emitted by a hydrogen atom that did not end up in the ground state $n = 1$ level.

(ii) A spectral line at 121.5 nm is observed in both the H atom emission spectrum and, at exactly the same wavelength, in the spectrum from the singly ionized He$^+$ ion. Estimate the initial and final ($n_i$ and $n_f$) levels of this transition in the H and He$^+$ spectra.

9. An electron and a positron (same mass, opposite charge) can form a short-lived bound state called a positronium atom, in which the two particles orbit their centre of mass. Use the Bohr model to estimate the binding energy and the distance between the electron and the positron in the positronium ground state. (Hint: consider a single particle of mass $m_{\text{reduced}} = (1/m + 1/m)^{-1}$ orbiting a fixed origin.)
10. A particle of mass \( m \) is bound in a spherical potential of the form \( V(r) = Cr \). By adapting the derivation of the Bohr model to this case, show that:

(i) the quantised orbital radii are

\[
 r_n = \left( \frac{\hbar^2 n^2}{mC} \right)^{1/3} \quad (n \text{ any integer } \geq 1);
\]

(ii) the quantised energy levels are

\[
 E_n = \frac{3}{2} \left( \frac{C^2 \hbar^2 n^2}{m} \right)^{1/3} \quad (n \text{ any integer } \geq 1).
\]

Physical Constants

\[
\begin{align*}
m_e &\approx 9.11 \times 10^{-31} \text{ kg} \approx 511 \text{ keV}/c^2 \\
m_n &\approx 1.67 \times 10^{-27} \text{ kg} \\
\text{atomic mass unit} &\approx 1.66 \times 10^{-27} \text{ kg} \\
\hbar &\approx 6.63 \times 10^{-34} \text{ Js} \\
h &\approx 1.05 \times 10^{-34} \text{ Js} \\
c &\approx 3.00 \times 10^8 \text{ ms}^{-1} \\
e &\approx 1.60 \times 10^{-19} \text{ C} \\
k_B &\approx 1.38 \times 10^{-23} \text{ JK}^{-1} \\
\text{N}_A &\approx 6.02 \times 10^{23} \\
R_H &\approx 1.097 \times 10^7 \text{ m}^{-1} \\
e_0 &\approx 8.85 \times 10^{-12} \text{ C}^2\text{J}^{-1}\text{m}^{-1}
\end{align*}
\]

Numerical Answers

1. Any wavelength within an order of magnitude of the interatomic spacing OK. Choosing \( \lambda = 10^{-10} \text{ m} \) gives KE of 0.083 eV and temperature of 640 K.
2. 2.46 \times 10^{-4} \text{ m}.
3. (ii) \( p = 2.73 \times 10^{-22} \text{ kg m s}^{-1}; \lambda = 2.43 \times 10^{-12} \text{ m} \).
4. \( T = 12.5 \text{ K} \).
5. \( p_{\text{min}} = 9.26 \times 10^{-25} \text{ kg m s}^{-1}; \text{KE}_{\text{min}} = 6.46 \times 10^{-23} \text{ J}; T = 3.12 \text{ K} \).
6. Initial energy level \( n_i = 6 \).
7. (i) 3.65 \times 10^{-7} \text{ m}; (ii) For H, \( n_i = 2 \) and \( n_f = 1 \); for He, \( n_i = 4 \) and \( n_f = 2 \).
8. \( E_{\text{binding}} \approx 6.8 \text{ eV}; r \approx 1.06 \text{ Å} \).
Quantum Physics Problem Sheet 4

Note that these questions are ordered by topic, not by difficulty. Do not get disheartened if you find some of the questions near the beginning of the sheet hard.

Working with Wavefunctions

1. A particle of mass $m$ is confined within a potential well of the form:

   $$ V(x) = \begin{cases} 
   0 & |x| < d/2, \\
   \infty & \text{otherwise}. 
   \end{cases} $$

   The (unnormalised) ground-state wavefunction is

   $$ \psi(x,t) = \begin{cases} 
   \cos \left( \pi x / d \right) e^{-i(h\pi^2/2md^2)t} & |x| < d/2, \\
   0 & \text{otherwise}. 
   \end{cases} $$

   (i) A measurement is made of the particle position. Show that the probability that the measured value lies between $x$ and $x + dx$ is independent of time. Where is the particle most likely to be found?

   (ii) Normalise $\psi(x,t)$.

   (iii) Find $\langle x \rangle$ and $\langle x^2 \rangle$. Hence obtain the rms width $\Delta x$ of the probability density function.

   \[ \int_{-\pi/2}^{\pi/2} \theta^2 \cos^2 \theta \, d\theta = \frac{\pi}{4} \left( \frac{\pi^2}{6} - 1 \right) \]

2. The normalised wavefunction $\psi(r,t)$ of a particle moving in three dimensions has the following probability interpretation:

   $$ |\psi(r,t)|^2 \, dV = \left\{ \begin{array}{l}
   \text{probability that a measurement of the position of the particle yields a value in the volume element } dV \text{ at } r. \\
   \end{array} \right. $$

   The (unnormalised) ground-state wavefunction of the electron in a hydrogen atom is $e^{-r/a_0} e^{-iE_0 t/\hbar}$, where $a_0 \approx 0.53 \, \text{Å}$ is the Bohr radius and $E_0 \approx -13.6 \, \text{eV}$ is the ground-state energy.
(i) Show that the electron probability density is independent of time. Where is the probability density largest?

(ii) Write down the normalisation condition that must be satisfied by any wavefunction satisfying the probability interpretation described above. Show that the normalised ground-state wavefunction is

\[ \psi(r, t) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0} e^{-iE_0 t/\hbar}. \]

(Hint: you will have to evaluate the integral \( \int_0^\infty e^{-2r/a_0} 4\pi r^2 dr \). You may assume that \( \int_0^\infty \xi^n e^{-\xi} d\xi = n! \))

(iii) A measurement is made of the electron position \( r \). Write down an expression for the probability that the electron is found in the spherical shell with inner and outer radii \( r \) and \( r + dr \). Hence show that the most probable distance of the electron from the nucleus is \( a_0 \). Is this result consistent with the position of maximum probability density found in part (i)?

(iv) Show that the mean distance of the electron from the nucleus is \( 3a_0/2 \) and that the rms distance is \( \sqrt{3} a_0 \).

(v) Show that the probability of finding the electron on top of the nucleus, which you may assume has radius \( 10^{-15} \) m, is approximately \( 9 \times 10^{-15} \). (Hint: no integration is required for this part.)

Momentum Measurements

3. The (unnormalised) wavefunction of a quantum mechanical particle that is being reflected from an infinitely high potential barrier at \( x = 0 \) takes the form:

\[ \psi(x, t) = \begin{cases} \sin(kx)e^{-i\omega t} & x < 0, \\ 0 & \text{otherwise}. \end{cases} \]

where \( \omega = \hbar k^2/2m \). In the region \( x < 0 \), show that \( \psi(x, t) \) may be written as a sum of right- and left-going complex travelling waves.

The momentum of the particle is measured. What values might be obtained and what are their relative probabilities?

The Uncertainty Principle

4. When undergoing radioactive decay, nuclei often emit electrons with energies between 1 and 10 MeV. Use the position-momentum uncertainty principle to show that an electron of energy 1 MeV could not have been contained in the nucleus before the decay.
5. (i) Consider a QM simple harmonic oscillator of mass \( m \), spring constant \( s \) and natural frequency \( \omega = \sqrt{s/m} \). Assuming that the average displacement \( \langle x \rangle \) and average momentum \( \langle p \rangle \) are both zero, show that the mean energy,

\[
\langle E \rangle = \langle KE \rangle + \langle PE \rangle = \frac{\langle p^2 \rangle}{2m} + \frac{1}{2} s \langle x^2 \rangle ,
\]

may be expressed in the form:

\[
\langle E \rangle = \frac{(\Delta p)^2}{2m} + \frac{1}{2} m \omega^2 (\Delta x)^2 ,
\]

where \( \Delta x \) is the rms displacement and \( \Delta p \) is the rms momentum.

(ii) Use the uncertainty principle to show that the ground-state energy of the oscillator must be greater than or equal to \( \frac{1}{2} \hbar \omega \). (In fact, the ground-state energy is exactly \( \frac{1}{2} \hbar \omega \).)

6. When a short-lived excited state of an atom or molecule decays to the ground state, the wavelength of the photon emitted is uncertain. What is the minimum energy uncertainty (often known as the linewidth), measured in eV, of photons emitted from a state of lifetime \( 2.6 \times 10^{-10} \) s.

7. In 1935, Yukawa proposed that the nuclear force arises through the emission of an unknown virtual particle, a pion, by one of the nucleons and its absorption by the other.

Assuming that \( \Delta E \Delta t \sim \hbar \), relate the lifetime of the virtual pion to its mass \( m \). Hence show that the distance travelled by the pion during its lifetime is unlikely to exceed \( \hbar /mc \). Given that the range of the nuclear force is approximately \( 1.4 \times 10^{-15} \) m, estimate \( m \) in MeV/c\(^2\). (When the pion was finally discovered, its mass was found to be \( \sim 140 \) MeV/c\(^2\). The high accuracy of the estimate obtained in this question is of course fortuitous.)

The Schrödinger Equation

8. A particle of mass \( m \) is confined within a potential well of the form:

\[
V(x) = \begin{cases} 
0 & 0 < x < d \\
\infty & \text{otherwise}
\end{cases}
\]

Write down the time-independent Schrödinger equation for this system. Verify that the wavefunctions

\[
\psi_n(x) = \begin{cases} 
\sqrt{\frac{2}{d}} \sin \left( \frac{n \pi x}{d} \right) & 0 < x < d \\
0 & \text{otherwise}
\end{cases}
\]

where \( n = 1, 2, \ldots \), are normalised energy eigenfunctions. Find the corresponding energy eigenvalues.
9. Assuming that a nucleus can be modelled as an infinite potential well of width \( d = 10^{-15} \) m, estimate the energy a nucleon (mass \( 1.67 \times 10^{-27} \) kg) emits as it falls from the \( n = 2 \) to the \( n = 1 \) level. Is this a sensible number?

10. The bond in a carbon monoxide (\(^{12}\text{C}^{16}\text{O}\)) molecule acts like a spring with spring constant \( s = 1857 \) Nm\(^{-1}\). Calculate the angular frequency of vibration according to classical physics. (Hint: use the reduced mass.) Hence work out:

   (i) The wavelength of the photons emitted when a CO molecule makes a transition between neighbouring vibrational states.

   (ii) The vibrational zero-point energy of a CO molecule.

11. The time-independent Schrödinger equation for a simple harmonic oscillator of mass \( m \) and spring constant \( s \) is

\[
-\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + \frac{1}{2}sx^2 \psi(x) = E \psi(x).
\]

Verify by subsitution that the function

\[
\psi(x) = e^{-\alpha x^2}
\]

satisfies this equation if \( \alpha \) is chosen correctly. Show that the corresponding eigenvalue is \( \frac{1}{2} \hbar \omega \), where \( \omega \) is the classical angular frequency of the oscillator.

12. An electron of energy \( E \) encounters a potential barrier of height \( V > E \) and width \( a \). Show that the probability that the electron tunnels across the barrier is approximately \( \exp(-2\gamma a) \), where \( \gamma = \sqrt{2m(V-E)/\hbar^2} \). Estimate the probability of tunnelling across a \( 10^{-6} \) m gap between two slabs of metal of work function \( 5 \) eV.

**Extra Questions for Enthusiasts (not examinable)**

13. The probability density of the positions at which the electrons in a two-slit experiment hit the screen is the square modulus of a complex wave that travels through both slits. Which slit does the electron itself go through? As I tried to explain in lectures, this is not a question that quantum mechanics can answer. The point-like “particle” that we associate with the electron is simply a convenient abstraction or mental picture that we use to help interpret the results of certain types of measurement — clicks of an electron counter or flashes of light originating from specific points in space. QM tells us almost nothing about what “really” produces these experimental results, and does not allow us to ascribe a position to the particle abstraction between measurements.

To help make these ideas more concrete, consider the thought experiment illustrated below.
When the light bulb is switched off, the electron wave passing through the two slits interferes constructively at angles $\theta$ satisfying:

$$d \sin \theta = n \lambda_{\text{electron}} \quad (n = 0, 1, 2, \ldots).$$

If $\theta$ is small, this translates to $\theta \approx n \lambda_{\text{electron}} / d$.

Now imagine switching on the light bulb (which is shaded to ensure that it only sends light along the plane of the barrier containing the slits). If the light is bright enough, most of the electrons emerging from the slits will scatter one or more photons. By imaging the flashes of light corresponding to the scattered photons, we can find out where they came from and hence measure the positions of the electrons when they are just behind the slits.

What are the results of this experiment? As you might expect, we see flashes of light originating from both slits, but never from both at the same time. Every time we detect an electron, it is either behind one slit or behind the other. However, when we look at the screen, we find that the two-slit diffraction pattern has disappeared! The measurement of the electron position just behind the slits has somehow destroyed the interference pattern. We can have it one way (interference pattern but no measurement of electron path) or the other (measurement of electron path but no interference pattern), but not both.

(i) In order for this experiment to work, the photon wavelength must be less than or approximately equal to $d$. Why?

(ii) Assuming that the $x$ component of the momentum of the electron is small (i.e., assuming that $\theta$ is small), show that the impact of a single photon of wavelength $\lambda_{\text{photon}}$ deflects the electron by an angle $\Delta \theta$ of order $\lambda_{\text{electron}} / \lambda_{\text{photon}}$. 
(iii) Explain why the measurement of the electron position wipes out the electron diffraction pattern.

14. In next year’s QM course, you will prove that different energy eigenfunctions of the same physical system are “orthogonal” to each other, in the sense that
\[ \int_{-\infty}^{\infty} \psi_n^*(x) \psi_m(x) \, dx = 0 \quad n \neq m. \]

Verify this result for the energy eigenfunctions given in Q8. (Since these functions are real there is no need to worry about the complex conjugation.)

The following result may be used without proof:
\[ \sin(n\theta) \sin(m\theta) = \frac{1}{2} \left[ \cos((n - m)\theta) - \cos((n + m)\theta) \right] \]

15. Show that the function
\[ \phi(x) = \sum_{n=1}^{\infty} c_n \psi_n(x) \]

is normalised if and only if the (possibly complex) coefficients \( c_n \) satisfy \( \sum_{n=1}^{\infty} |c_n|^2 = 1. \) (Hint: use the orthogonality relation from Q14 and remember that \( \psi_n(x) \) is normalised.) Hazard a guess at the probability interpretation of \( |c_n|^2. \)

Physical Constants

\[
\begin{align*}
    m_e & \approx 9.11 \times 10^{-31} \text{ J} \approx 511 \text{ keV} / c^2 \\
    \text{atomic mass unit} & \approx 1.66 \times 10^{-27} \text{ kg} \\
    \hbar & \approx 6.63 \times 10^{-34} \text{ Js} \\
    \bar{\hbar} & \approx 1.05 \times 10^{-34} \text{ Js} \\
    c & \approx 3.00 \times 10^8 \text{ m} / \text{s} \\
    e & \approx 1.60 \times 10^{-19} \text{ C}
\end{align*}
\]

Numerical Answers

6. \( 1.26 \times 10^{-6} \text{ eV.} \)
7. \( 141 \text{ MeV} / c^2. \)
9. \( 610 \text{ MeV.} \)
10. (i) \( 4.67 \times 10^{-6} \text{ m; 0.13 eV.} \)
12. \( \sim e^{-23,000} \sim 10^{-10,000}. \)
Quantum Physics Answer Sheet 1

Units and Magnitudes

1. Number of atoms per unit volume is:

\[ n = \frac{\text{Mass per unit volume}}{\text{Mass per atom}} = \frac{2700}{27 \times 1.66 \times 10^{-27}} = 6.02 \times 10^{28} \text{ m}^{-3} . \]

Volume per atom is \( 1/n = 1.66 \times 10^{-29} \text{ m}^3 \). If we assume that each atom is a sphere of radius \( r \), we have

\[ \frac{4}{3} \pi r^3 = 1.66 \times 10^{-29} \]

and hence \( r = 1.58 \times 10^{-10} \text{ m} = 1.58 \text{ Å} \). This is an overestimate because it is impossible to fill space with spheres (there are always gaps between them). A different estimate may be obtained by assuming that the spheres are stacked in a simple cubic lattice. Each cube of side \( 2r \) then contains one atom of radius \( r \) and the volume per atom is \( 8r^3 \). This gives

\[ 8r^3 = 1.66 \times 10^{-29} \]

and hence \( r = 1.28 \times 10^{-10} \text{ m} = 1.28 \text{ Å} \). It is possible to pack spheres much more efficiently than in a simple cubic lattice (the packing in Al is actually face-centred cubic — the most efficient regular packing of spheres), and so the true answer must lie somewhere between these two estimates.

2. The KE of the electrons is 25 keV = 0.025 MeV. Since the KE is the total energy, \( m \gamma c^2 \), minus the rest energy, \( mc^2 \), this gives:

\[ 0.025 = \gamma mc^2 - mc^2 = \left( \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right) \times 0.51 . \]

Aside: I remember finding these “relativistic” units very confusing the first time I met them. The statement that the mass \( m \) of an electron is 0.51 MeV/c^2 is exactly equivalent to the statement that \( mc^2 \) (which is, of course, an energy) is equal to 0.51 MeV. Similarly, if you are told that the momentum \( p \) is equal to 1 MeV/c, the implication is that \( pc \) (which also has the dimensions of an energy) is equal to 1 MeV.

A little algebra now gives

\[ 1 - \frac{v^2}{c^2} = \frac{1}{\left( 1 + \frac{0.025}{0.51} \right)^2} , \]

and hence

\[ \frac{v}{c} \approx 0.30 . \]
The relativistic effects (which depend on $v^2/c^2$) are small but not negligible. Television designers presumably need to take them into account.

3. For a very rough estimate, equate $k_B T$ (or $3k_B T/2$ if you insist) to 13.6 eV. This gives:

$$T = \frac{13.6 \times 1.6 \times 10^{-19}}{1.38 \times 10^{-23}} \approx 160,000 \text{ K}.$$

**Travelling Waves**

4. To work out the wavelength, note that

$$
\psi \left(x + \frac{2\pi}{k}, t\right) = a \cos \left(-k \left[x + \frac{2\pi}{k}\right] - \omega t + \phi\right)
= a \cos(-kx - \omega t + \phi - 2\pi) = \psi(x, t).
$$

Hence, $\psi(x, t)$ changes by one full period as $x$ increases by $2\pi/k$ at constant $t$. In other words, the wavelength $\lambda = 2\pi/k$.

Similarly, to work out the time period, note that

$$
\psi \left(x, t + \frac{2\pi}{\omega}\right) = a \cos \left(-kx - \omega \left[t + \frac{2\pi}{\omega}\right] + \phi\right)
= a \cos(-kx - \omega t + \phi - 2\pi) = \psi(x, t).
$$

Hence, $\psi(x, t)$ changes by one full period as $t$ increases by $2\pi/\omega$ at constant $x$. In other words, the time period $T = 2\pi/\omega$. The frequency $\nu = 1/T$ is therefore given by $\nu = \omega/2\pi$.

To work out the velocity of the crests, consider the crest at the point $x_{\text{crest}}$ where the argument of the cosine function is zero:

$$-kx_{\text{crest}} - \omega t + \phi = 0.$$

This equation rearranges to

$$x_{\text{crest}} = \frac{\phi}{k} - \frac{\omega}{k} t.$$

Comparing with the equation $x_{\text{crest}} = x_0 + vt$ that describes uniform motion at velocity $v$, we see that the wave crest is at position $x_0 = \phi/k$ at time $t = 0$ and is moving at velocity $-\omega/k$.

5. The phase velocity is

$$v = \frac{\omega}{k} = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}} \approx \sqrt{\frac{9.8 \times 10}{2\pi}} = 3.96 \text{ ms}^{-1}.$$
Complex Representation of Waves, Interference and Diffraction

6. Since $e^{i\theta} = \cos \theta + i \sin \theta$, it follows that $\cos \theta = \text{Re}(e^{i\theta})$ and $\sin \theta = \text{Re}(-ie^{i\theta})$. Hence

$$\cos \theta + \sin \theta = \text{Re}(e^{i\theta} - ie^{i\theta}) = \text{Re}((1 - i)e^{i\theta}).$$

The prefactor $1 - i$ may be rewritten in the form $re^{i\phi}$, where:

$$r = \sqrt{[\text{Re}(1 - i)]^2 + [\text{Im}(1 - i)]^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2},$$

$$\phi = \tan^{-1}\left(\frac{\text{Im}(1 - i)}{\text{Re}(1 - i)}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}.$$  

(An alternative way to find $r$ and $\phi$ is to draw the complex number $1 - i$ in the Argand diagram and note its length and argument.)

Hence

$$\cos \theta + \sin \theta = \text{Re}\left(\sqrt{2}e^{-i\pi/4}e^{i\theta}\right) = \sqrt{2}\text{Re}\left(e^{i(\theta-\pi/4)}\right) = \sqrt{2}\cos(\theta - \pi/4),$$

as required.

7. (i) Using real arithmetic only:

$$\psi(x,t) = a \cos(kx - \omega t) + a \cos(-kx - \omega t) = a(\cos kx \cos \omega t + \sin kx \sin \omega t) + a(\cos kx \cos \omega t - \sin kx \sin \omega t) = 2a \cos kx \cos \omega t,$$

where the first step used the trigonometric identity

$$\cos(\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi,$$

and the information that $\cos \theta$ is an even function [$\cos(-\theta) = \cos \theta$] while $\sin \theta$ is an odd function [$\sin(-\theta) = -\sin \theta$].

(ii) Using complex arithmetic:

$$\psi(x,t) = ae^{i(kx-\omega t)} + ae^{i(-kx-\omega t)}.$$

(Actually, of course, $\psi(x,t)$ is the real part of the expression on the RHS. From now on we shall take this as understood.) Hence:

$$\psi(x,t) = ae^{-i\omega t} (e^{ikx} + e^{-ikx}) = 2ae^{-i\omega t} \cos kx.$$

This step used the identities

$$e^{i\theta} = \cos \theta + i \sin \theta \quad \text{and} \quad e^{-i\theta} = \cos \theta - i \sin \theta.$$
from which it follows that
\[ e^{i\theta} + e^{-i\theta} = 2 \cos \theta \quad \text{and} \quad e^{i\theta} - e^{-i\theta} = 2i \sin \theta. \]

Taking the real part of \( \psi(x, t) \) now gives the real wavefunction
\[ \psi(x, t) = 2a \cos kx \cos \omega t, \]

exactly as in part (i).

The amplitude \( a_{\text{total}} \) of \( \psi(x, t) \) depends on position,
\[ a_{\text{total}}(x) = 2a \cos kx, \]

and hence so does the intensity,
\[ I_{\text{total}}(x) = a_{\text{total}}^2(x) = 4a^2 \cos^2 kx. \]

The position average of the intensity is \( 4a^2 \) times the position average of \( \cos^2 kx \). As long as you average over a whole number of half periods, the average value of \( \cos^2 \theta \) (or \( \sin^2 \theta \)) is equal to \( 1/2 \) (this is easy to prove by starting from the trigonometric identity \( \cos^2 \theta = (1 + \cos(2\theta))/2 \) and noting that \( \cos(2\theta) \) averages to zero). Hence, the average intensity is \( 2a^2 \).

This is expected because the average intensity is proportional to the average energy per unit volume. Assuming that energy is conserved, the average energy density of the standing wave must equal the sum of the average energy densities of the two travelling waves. Both travelling waves have intensity \( a^2 \) (independent of position), and hence the average intensity of the standing wave must be \( 2a^2 \).

8. (i) The wave emerging from the segment \( \Delta y \) at height \( y \) is
\[ Ae^{i(k[\zeta - (-y \sin \theta)] - \omega t)} \Delta y = Ae^{i(k\zeta - \omega t + ky \sin \theta)} \Delta y. \]

(ii) The total wave emerging in the \( \zeta \) direction is the sum of the waves emerging from all the little segments:
\[ \psi(\zeta, t) = \sum_{\text{segments}} Ae^{i(k\zeta - \omega t + ky \sin \theta)} \Delta y, \]

where, for each segment (or, equivalently, for each term in the sum), \( y \) is the position of the centre of that segment. In the limit as \( \Delta y \to 0 \), the sum turns into the integral:
\[ \psi(\zeta, t) = \int_{-d/2}^{d/2} Ae^{i(k\zeta - \omega t + ky \sin \theta)} dy. \]

Since \( e^{i(k\zeta - \omega t + ky \sin \theta)} = e^{i(k\zeta - \omega t)} e^{iky \sin \theta} \), this is equivalent to
\[ \psi(\zeta, t) = Ae^{i(k\zeta - \omega t)} \int_{-d/2}^{d/2} e^{iky \sin \theta} dy, \]

as required.
If you are confused about the relationship between the sum and the integral, consider the diagram below. The integral of \( f(y) \) from \( a \) to \( b \) is the sum of the areas of the segments of width \( \Delta y \).

Since the area of each segment is approximately \( f(y) \Delta y \), it follows that

\[
\int_a^b f(y) \, dy \approx \sum_{\text{segments}} f(y) \Delta y.
\]

In fact, the integral is normally defined as the limit of this sum as \( \Delta y \) tends to zero.

(iii) The integral

\[
\int_{-d/2}^{d/2} e^{ikysin\theta} \, dy
\]

may be written as

\[
\int_{-d/2}^{d/2} e^{\alpha y} \, dy,
\]

where \( \alpha = ik \sin \theta \). This is easy to integrate:

\[
\int_{-d/2}^{d/2} e^{\alpha y} \, dy = \left[ \frac{e^{\alpha y}}{\alpha} \right]_{-d/2}^{d/2} = \frac{1}{\alpha} \left( e^{\alpha d/2} - e^{-\alpha d/2} \right)
\]

\[
= \frac{e^{\frac{1}{2}ikd\sin\theta} - e^{-\frac{1}{2}ikd\sin\theta}}{ik \sin \theta} = \frac{d \sin \left( \frac{kd \sin \theta}{2} \right)}{\frac{kd \sin \theta}{2}},
\]

where the last step used the identity \( e^{i\phi} - e^{-i\phi} = 2i \sin \phi \) discussed in the answer to question 7.

The wavefunction \( \psi(\zeta, t) \) is therefore given by:

\[
\psi(\zeta, t) = Ad \frac{\sin \left( \frac{kd \sin \theta}{2} \right)}{kd \sin \theta} e^{i(k\zeta - \omega t)}.
\]
The intensity $I$ is the square modulus of the complex amplitude (which is everything in front of the $e^{i(k\zeta - \omega t)}$ factor). Hence

$$I = |A|^2 d^2 \sin^2 \left( \frac{kd \sin \theta}{2} \right),$$

as required.

(iv) The diffraction pattern looks like this (note that the horizontal axis shows values of $\frac{kd \sin \theta}{2\pi}$ instead of values of $\frac{kd \sin \theta}{2}$):

The first zero occurs where $\frac{kd \sin \theta}{2\pi} = 1$ and hence where

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{d \sin \theta}.$$

This implies that

$$\lambda = d \sin \theta.$$

The difference in the lengths of the paths emerging from the top and bottom of the slit is one wavelength.
Quantum Physics Answer Sheet 2

Photons

1. The total energy entering each eye per second is the energy striking a unit area per second times the area of the pupil:

\[
\text{energy entering eye per second} = 1.4 \times 10^{-10} \times \pi (0.0035)^2 \\
\approx 5.39 \times 10^{-15} \text{ J}.
\]

Average number of photons entering eye per second is

\[
\frac{\text{energy entering eye per second}}{\text{energy per photon}} = \frac{5.39 \times 10^{-15}}{hc/\lambda}
\]

\[
= \frac{5.39 \times 10^{-15} \times 500 \times 10^{-9}}{6.63 \times 10^{-34} \times 3.00 \times 10^8}
\]

\[
\approx 13,500.
\]

The average number of photons inside eye at any one time is

\[
\frac{\text{number entering per second} \times \text{length of eye}}{\text{distance a photon travels per second}} = \frac{13500 \times 0.04}{3.00 \times 10^8}
\]

\[
\approx 1.8 \times 10^{-6}.
\]

The actual number of photons in the eye is almost always zero.

Since light always arrives as individual photons, all light detectors must be capable of detecting individual photons. A more interesting question is whether the arrival of a single photon is sufficient to trigger one of the detectors (rods and cones) in the retina, or whether it is necessary to bombard that detector with many photons in close succession. Given that the eye takes much less than 1 s to process a new image, it is reasonable to assume that the “memory” of the detectors is less than, say, 0.2 s. Any effects caused by photons that arrived more than 0.2 s ago can therefore be ignored. Within 0.2 s, only 2,700 photons enter the eye, all of which are focused onto the small area of the retina where the image is formed. Are there more than 2,700 detectors in this area? I have no idea, but I doubt it. In other words, the ability of the eye to see the star provides no convincing evidence that the detectors in the retina are triggered by single photons — it may be necessary to hit the same detector with several photons in quick succession.

2. The electron energy is 30 keV and so the energy \( h\nu = h\nu/\lambda \) of the X-ray photons produced must be less than or equal to 30 keV. The minimum photon wavelength is therefore

\[
\lambda = \frac{hc}{30 \text{ keV}} = \frac{6.63 \times 10^{-34} \times 3.00 \times 10^8}{30 \times 10^3 \times 1.6 \times 10^{-19}} = 4.14 \times 10^{-11} \text{ m}.
\]
3. (i) Light of wavelength greater than $\lambda_{\text{max}} = 310 \text{ nm}$ is incapable of producing a current. Hence the work function $W$ is given by:

$$W = \frac{hc}{\lambda_{\text{max}}} = \frac{6.63 \times 10^{-34} \times 3.00 \times 10^8}{310 \times 10^{-9}} \approx 6.42 \times 10^{-19} \text{ J}.$$ 

or

$$W = \frac{6.42 \times 10^{-19}}{1.60 \times 10^{-19}} \approx 4.00 \text{ eV}.$$ 

(ii) The energy of a photon of wavelength 200 nm is

$$E = \frac{hc}{\lambda} = \frac{6.63 \times 10^{-34} \times 3.00 \times 10^8}{200 \times 10^{-9}} \approx 9.95 \times 10^{-19} \text{ J},$$

or

$$E = \frac{9.95 \times 10^{-19}}{1.60 \times 10^{-19}} \approx 6.22 \text{ eV}.$$ 

(iii) The stopping potential $V_0$ at 200 nm is given by Einstein’s equation, $W + eV_0 = E$. Hence

$$V_0 = 6.22 - 4.00 = 2.22 \text{ V}.$$ 

The maximum KE of the emitted electrons is 2.22 eV.

4. (i) The data look like this:

Since the $y$ intercept is $-W$, the work function $W$ is 2.0 eV.

(ii) The slope of the line is

$$\frac{(3.0 - 1.0) \text{ eV}}{(1.25 - 0.75) \times 10^{15} \text{ Hz}} = \frac{2 \times 1.60 \times 10^{-19} \text{ J}}{0.5 \times 10^{15} \text{ s}^{-1}} = 6.4 \times 10^{-34} \text{ Js}.$$ 

Assuming that the experimental errors were reflected in the precision with which the measured values were quoted, this is consistent with the accepted value of $6.63 \times 10^{-34} \text{ Js}$. (The fact that the three data points lie on a perfect straight line is suspicious, though.)
5. (i) The wavelength of the incident photons is
\[ \lambda = \frac{hc}{E} \approx \frac{6.63 \times 10^{-34} \times 3.00 \times 10^8}{20 \times 10^3 \times 1.60 \times 10^{-19}} \approx 6.22 \times 10^{-11} \text{ m} . \]

The change in wavelength is given by the Compton formula with \( \theta = 60^\circ \):
\[ \lambda' - \lambda = \frac{h}{mc} (1 - \cos \theta) \approx \frac{6.63 \times 10^{-34} \times 0.5}{9.11 \times 10^{-31} \times 3.00 \times 10^8} \approx 1.21 \times 10^{-12} \text{ m} . \]

Combining the values of \( \lambda \) and \( \lambda' - \lambda \) gives the wavelength of the scattered photons:
\[ \lambda' \approx 6.34 \times 10^{-11} \text{ m} . \]

(ii) The energy lost by a photon as it scatters is:
\[ E - E' = hc \left( \frac{1}{\lambda} - \frac{1}{\lambda'} \right) \]
\[ = 6.63 \times 10^{-34} \times 3.00 \times 10^8 \left( \frac{1}{6.22 \times 10^{-11}} - \frac{1}{6.34 \times 10^{-11}} \right) \]
\[ \approx 6.05 \times 10^{-17} \text{ J} \]
\[ \approx 378 \text{ eV} . \]

Warning: this answer is the difference of two much larger numbers (the incoming and outgoing photon energies) and is subject to considerable rounding error. For example, if you store intermediate values such as \( \lambda \) and \( \lambda' \) to full calculator precision, the final result changes by several eV. Short of using more accurate values for the fundamental constants, there is little that can be done about this.

All this energy is transferred to the electron as recoil energy. The work function of a typical solid is only 5 or 10 eV, so some of the recoiling electrons will certainly escape from the metal.

The largest change in wavelength would be obtained when \( \theta = 180^\circ \), in which case \( \lambda' - \lambda = 2h/mc \approx 4.85 \times 10^{-12} \text{ m} \). The maximum possible wavelength of the scattered photon (assuming only one scattering) is \( 6.22 \times 10^{-11} + 4.85 \times 10^{-12} \approx 6.71 \times 10^{-11} \text{ m} \).

6. (i) The initial photon wavelength is
\[ \lambda_{\text{init}} = \frac{hc}{E} \approx \frac{6.63 \times 10^{-34} \times 3.00 \times 10^8}{10^6 \times 1.60 \times 10^{-19}} \approx 1.24 \times 10^{-12} \text{ m} . \]

The final photon wavelength after \( 10^{26} \) Compton scattering events is 500 nm. If we assume that each scattering event increases the photon wavelength by the same amount \( \Delta \lambda \), we obtain
\[ 10^{26} \Delta \lambda \approx (500 \times 10^{-9} - 1.24 \times 10^{-12}) \text{ m} , \]

and hence
\[ \Delta \lambda \approx 5 \times 10^{-33} \text{ m} . \]
(ii) The Compton formula says that

\[ \Delta \lambda = \frac{h}{mc} (1 - \cos \theta) . \]

Since \( \Delta \lambda (\approx 5 \times 10^{-33} \text{ m}) \ll h/mc (\approx 2.43 \times 10^{-12} \text{ m}) \), the average scattering angle \( \theta \) must be very small (so that \( \cos \theta \) is very close to 1). We can therefore make the approximation \( \cos \theta \approx 1 - \theta^2/2 \) to obtain \( \Delta \lambda \approx h\theta^2/2mc \), and hence

\[
\theta \approx \sqrt{\frac{2mc\Delta \lambda}{h}} \\
\approx \sqrt{\frac{2 \times 9.11 \times 10^{-31} \times 3.00 \times 10^8 \times 5 \times 10^{-33}}{6.63 \times 10^{-34}}} \\
\approx 6.42 \times 10^{-11} \text{ radians} \\
\approx 3.68 \times 10^{-9} \text{ degrees} .
\]

(iii) In 10⁶ years, a photon travels a distance:

\[ d = ct = 3.00 \times 10^8 \times 60 \times 60 \times 24 \times 365 \times 10^6 \\
\approx 9.46 \times 10^{21} \text{ m} .
\]

During this time, it scatters \( 10^{26} \) times. Hence, the average distance travelled by a photon between scattering events is \( 9.46 \times 10^{21}/10^{26} \approx 9.46 \times 10^{-5} \text{ m} \) or roughly 0.1 mm.
Quantum Physics Answer Sheet 3

De Broglie Waves

1. In order to use neutron diffraction to study atomic positions and atomic-scale magnetic fields, the neutron de Broglie wavelength must be comparable to the size of an atom:

\[ \lambda \approx 10^{-10} \, \text{m} . \]

The kinetic energy of the neutron is thus:

\[
\frac{1}{2} m v^2 = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2} \approx \frac{(6.63 \times 10^{-34})^2}{2 \times 1.67 \times 10^{-27} \times 10^{-20}} \\
\approx 1.32 \times 10^{-20} \, \text{J} \approx 0.083 \, \text{eV} .
\]

This is the same as the average energy \(3k_B T/2\) of a neutron in thermal equilibrium at temperature

\[ T \approx \frac{2 \times 1.32 \times 10^{-20}}{3 \times 1.38 \times 10^{-23}} \approx 640 \, \text{K} . \]

2. The de Broglie wavelength of a 100 eV electron is given by:

\[
100 \, \text{eV} = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2} ,
\]

and hence

\[
\lambda = \frac{6.63 \times 10^{-34}}{\sqrt{2} \times 9.11 \times 10^{-31} \times 100 \times 1.60 \times 10^{-19}} \approx 1.23 \times 10^{-10} \, \text{m} .
\]
From Q8 of Problem Sheet 1, the first zero in the diffraction pattern from a slit of width $d$ occurs where $\sin \theta = \lambda/d$. Hence,

$$\theta = \sin^{-1}(\lambda/d) \approx \sin^{-1}(1.23 \times 10^{-10}/10^{-6}) \approx 1.23 \times 10^{-4} \text{ radians}.$$ 

The width $w$ of the central diffraction peak is $2l \tan \theta$, where $l = 1 \text{ m}$ is the distance from the slit to the screen. Hence,

$$w = 2 \times 1 \times \tan \theta \approx 2\theta \approx 2.46 \times 10^{-4} \text{ m}.$$ 

3. (i) When the electron and positron annihilate, both momentum and energy must be conserved. Since the electron and positron are initially at rest, the total momentum is equal to zero and must remain equal to zero. This explains why the two photons have equal and opposite momentum. Since the energy and (magnitude of) momentum of a photon are related via $E = pc$, photons with the same (magnitude of) momentum must also have the same energy.

(ii) The photon energy can be obtained by conservation of energy:

$$2mc^2 = 2 \times 511 \text{ keV} = 2E_{\text{photon}},$$

and hence $E_{\text{photon}} \approx 511 \text{ keV}$. The photon momentum is

$$p_{\text{photon}} = E_{\text{photon}}/c \approx 511 \text{ keV}/c \approx \frac{511 \times 10^3 \times 1.6 \times 10^{-19}}{3.00 \times 10^8} \approx 2.73 \times 10^{-22} \text{ kg ms}^{-1},$$

and the photon wavelength is

$$\lambda = \frac{h}{p} \approx \frac{6.63 \times 10^{-34}}{2.73 \times 10^{-22}} \approx 2.43 \times 10^{-12} \text{ m}.$$ 

4. A particle of mass $m$ and momentum $p$ has kinetic energy $p^2/2m$. If the kinetic energy is equal to $3k_B T/2$:

$$\frac{p^2}{2m} = \frac{3k_B T}{2},$$

then

$$p = \sqrt{3mk_B T}.$$ 

Combining this result with de Broglie’s equation, $p = h/\lambda$, gives:

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{3mk_B T}},$$

as required.

Avogadro’s number of He atoms occupy a volume of $27.6 \times 10^{-6} \text{ m}^3$. Hence, the volume per atom $d^3$ is

$$\frac{27.6 \times 10^{-6}}{6.02 \times 10^{23}} \approx 4.58 \times 10^{-29} \text{ m}^3.$$
Taking the cube root, we obtain \( d \approx 3.58 \times 10^{-10} \) m.

To find the temperature \( T \) at which \( \lambda = d \), we have to solve the equation

\[
\frac{\hbar}{\sqrt{3mk_B T}} = d.
\]

Hence

\[
T = \frac{1}{3mk_B} \left( \frac{\hbar}{d} \right)^2 \approx \frac{1}{3 \times 4 \times 1.66 \times 10^{-27} \times 1.38 \times 10^{-23}} \left( \frac{6.63 \times 10^{-34}}{3.58 \times 10^{-10}} \right)^2
\]

\[
\approx 12.5 \text{ K}.
\]

When the temperature is comparable to or smaller than this value, the de Broglie wavelength of the He atoms will be the same as or greater than the interparticle spacing, and the wave-like properties of the atoms will be important.

5. The figure below shows that in order for the de Broglie wave of wavelength \( \lambda \) to “fit in” to the box, the box side \( d \) must be an integer multiple of \( \lambda/2 \): \( d = n\lambda/2 \), where \( n = 1, 2, \ldots \)

The maximum possible de Broglie wavelength is therefore \( 2d \). The smallest possible momentum is

\[
p_{\text{min}} = \frac{h}{\lambda_{\text{max}}} = \frac{h}{2d} \approx \frac{6.63 \times 10^{-34}}{2 \times 3.58 \times 10^{-10}} \approx 9.26 \times 10^{-25} \text{ kg ms}^{-1}.
\]

The smallest possible KE is

\[
\text{KE}_{\text{min}} = \frac{p_{\text{min}}^2}{2m} \approx \frac{(9.26 \times 10^{-25})^2}{2 \times 4 \times 1.66 \times 10^{-27}} \approx 6.46 \times 10^{-23} \text{ J}.
\]

The thermal KE of \( 3k_B T/2 \) would equal \( \text{KE}_{\text{min}} \) when

\[
T = \frac{2 \text{KE}_{\text{min}}}{3k_B} \approx \frac{2 \times 6.46 \times 10^{-23}}{3 \times 1.38 \times 10^{-23}} \approx 3.12 \text{ K}.
\]

3
The Bohr Atom

6. The Rydberg formula is
\[ \frac{1}{\lambda} = R_H \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right), \]
where \( R_H = 1.097 \times 10^7 \text{ m}^{-1} \). In this case, \( n_f = 2 \) and \( \lambda = c/\nu = (3.00 \times 10^8)/(7.316 \times 10^{14}) = 4.10 \times 10^{-7} \text{ m} \).

Rearranging the Rydberg formula gives:
\[ n_i^2 = \left( \frac{1}{n_f^2} - \frac{1}{R_H \lambda} \right)^{-1} \approx \left( \frac{1}{4} - \frac{1}{1.097 \times 10^7 \times 4.10 \times 10^{-7}} \right)^{-1} \approx 36.1. \]
Hence, the initial energy level was the \( n = 6 \) level.

7. The Bohr orbit of the electron must still contain a whole number of de Broglie wavelengths. The angular momentum \( L = mvr \) must therefore be quantised just as in a hydrogen atom:
\[ L = n\hbar, \quad n = 1, 2, 3, \ldots. \]
However, since the Coulomb attraction between the orbiting electron and the nucleus is larger by a factor \( Z \) than in a hydrogen atom, the equation linking centripetal force and centripetal acceleration becomes:
\[ \frac{Ze^2}{4\pi\epsilon_0 r^2} = \frac{mv^2}{r} = \frac{(mvr)^2}{m r^3}. \]
Replacing \( mvr \) by \( n\hbar \) and rearranging gives the following formula,
\[ r_n = \frac{4\pi\epsilon_0 (n\hbar)^2}{Ze^2}, \]
for the radius of the \( n^{th} \) Bohr orbit.

The total energy of this orbit is
\[ E_n = KE_n + PE_n \]
\[ = \frac{1}{2} mv_n^2 - \frac{Ze^2}{4\pi\epsilon_0 r_n} \]
\[ = \frac{1}{2} \frac{Ze^2}{4\pi\epsilon_0 r_n} - \frac{Z^2 e^4}{2(4\pi\epsilon_0 \hbar)^2 n^2} \]
\[ \approx -\frac{Z^2 	imes 13.6 \text{ eV}}{n^2}. \]
8. (i) The shortest wavelength (highest energy) photon that a hydrogen atom can emit without ending up in the ground-state ($n_f = 1$) energy level is produced when the atom decays from an initial state with very large $n_i$ to the $n_f = 2$ final state. The wavelength of the photon emitted in this transition satisfies the equation
\[
\frac{1}{\lambda} = R_H \left( \frac{1}{4} - \frac{1}{n_i^2} \right) \approx \frac{R_H}{4},
\]
and hence
\[
\lambda \approx \frac{4}{R_H} \approx 3.65 \times 10^{-7} \text{ m}.
\]
(ii) For the H atom, the normal Rydberg formula applies:
\[
\frac{1}{\lambda} = R_H \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right),
\]
with $\lambda = 121.5 \text{ nm}$. Hence
\[
\frac{1}{n_f^2} - \frac{1}{n_i^2} = \frac{1}{\lambda R_H} \approx \frac{1}{121.5 \times 10^{-9} \times 1.097 \times 10^7} \approx 0.75.
\]
The only solution of this equation with $n_i$ and $n_f$ integers is $n_i = 2$ and $n_f = 1$. In other words, the transition is from the first excited state to the ground state.

For the He$^+$ ion, the energies of the states are $Z^2 = 2^2 = 4$ times what they were in the $H$ atom (see Q8). Hence, the Rydberg formula becomes
\[
\frac{1}{\lambda} = 4R_H \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right),
\]
and the equation for $n_f$ and $n_i$ is
\[
\frac{1}{n_f^2} - \frac{1}{n_i^2} \approx \frac{0.75}{4} = \frac{3}{16}.
\]
This equation can be solved by doubling the values of $n_i$ and $n_f$ obtained for the H atom. In other words, the He$^+$ transition is from the $n_i = 4$ level to the $n_f = 2$ level.

9. According to the Bohr model, the binding energy and radius of an H atom in its ground state are
\[
E_{\text{binding}} = \frac{me^4}{2(4\pi\varepsilon_0\hbar)^2} \approx 13.6 \text{ eV},
\]
\[
r = \frac{4\pi\varepsilon_0\hbar^2}{me^2} \approx 0.53 \text{ Å}.
\]

In the case of a positronium atom, the electron mass $m$ has to be replaced by the reduced mass $m_{\text{reduced}} = (1/m + 1/m)^{-1} = 0.5m$. Hence
\[
E_{\text{binding}} = \frac{0.5me^4}{2(4\pi\varepsilon_0\hbar)^2} \approx 6.8 \text{ eV},
\]
\[
r = \frac{4\pi\varepsilon_0\hbar^2}{0.5me^2} \approx 1.06 \text{ Å}.
\]
10. (i) The potential in this example is peculiar, but a Bohr orbit of radius \( r \) still has length \( 2\pi r \), and the de Broglie wavelength of the particle must still “fit in” to this length:

\[
2\pi r = n\lambda \quad (n \text{ any integer } > 0).
\]

Since \( p = mv = h/\lambda \), this condition translates to

\[
mvr = \frac{mv n\lambda}{2\pi} = \frac{p n\lambda}{2\pi} = \frac{n\hbar}{2\pi} = n\hbar.
\]

In other words, the angular momentum must still be a multiple of \( \hbar \).

The next step in deriving the Bohr model of the hydrogen atom is to write down Newton’s second law: force = mass \( \times \) centripetal acceleration. In this case, the force is

\[
F = -\frac{dV}{dr} = -C,
\]

where the \(-\)ve sign shows that the force acts towards the origin (in the \(-r\) direction). Hence, Newton’s second law reads:

\[
C = \frac{mv^2}{r} = \frac{(mvr)^2}{mr^3}.
\]

Using the angular momentum quantisation conditions, \( mvr = n\hbar \), then gives

\[
C = \frac{\hbar^2 n^2}{mr_n^3},
\]

or

\[
r_n = \left( \frac{\hbar^2 n^2}{mC} \right)^{1/3},
\]

as required.

(ii) The energy \( E_n \) of the \( n^{\text{th}} \) orbit is:

\[
E_n = KE + PE = \frac{1}{2}mv_n^2 + Cr_n = \frac{mv_n r_n}{2mr_n^2} + Cr_n = \frac{\hbar^2 n^2}{2mr_n^2} + Cr_n = \frac{\hbar^2 n^2}{2m \left( \frac{\hbar^2 n^2}{mC} \right)^{2/3}} + C \left( \frac{\hbar^2 n^2}{mC} \right)^{1/3} = \frac{C}{2} \left( \frac{\hbar^2 n^2}{mC} \right)^{1/3} + C \left( \frac{\hbar^2 n^2}{mC} \right)^{1/3} = \frac{3}{2} \left( \frac{C^2 \hbar^2 n^2}{m} \right)^{1/3},
\]

as required.
Quantum Physics Answer Sheet 4

Working with Wavefunctions

1. (i) The probability density $f(x,t)$ is proportional to $|\psi(x,t)|^2$. Hence, the probability $f(x,t)\,dx$ that the particle is found between $x$ and $x+dx$ at time $t$ is proportional to

$$|\psi(x,t)|^2 \,dx = \begin{cases} \cos^2(\pi x/d)e^{-i(h\pi^2/2md^2)t}e^{i(h\pi^2/2md^2)t} \,dx & |x| < d/2 \\ 0 & \text{otherwise} \end{cases}$$

which is independent of time. The particle is most likely to be found at $x = 0$, where $f(x)$ is largest.

(ii) In order to normalise $\psi(x,t)$ we have to evaluate

$$N = \int_{-\infty}^{\infty} |\psi(x,t)|^2 \,dx = \int_{-d/2}^{d/2} \cos^2(\pi x/d) \,dx = \frac{d}{2}.$$

(The integration is easy because $\cos^2 \theta$ repeats every $\pi$ radians and averages to $1/2$ over any whole number of repeats.)

Hence, the normalised ground-state wavefunction is:

$$\psi(x,t) = \begin{cases} \sqrt{\frac{2}{d}} \cos(\pi x/d)e^{-i(h\pi^2/2md^2)t} & |x| < d/2 \\ 0 & \text{otherwise} \end{cases}.$$

(iii) $\langle x \rangle = \int_{-d/2}^{d/2} x \frac{2}{d} \cos^2 \left(\frac{\pi x}{d}\right) \,dx = 0$ (integrand is odd).

$$\langle x^2 \rangle = \int_{-d/2}^{d/2} x^2 \frac{2}{d} \cos^2 \left(\frac{\pi x}{d}\right) \,dx$$

$$= \frac{d^3}{\pi^3} \int_{-\pi/2}^{\pi/2} \frac{2}{d} \theta^2 \cos^2 \theta \,d\theta$$

$$= \left(\frac{1}{12} - \frac{1}{2\pi^2}\right) d^2$$ (using integral given in question).

Hence

$$\Delta x = \sqrt{\langle (x-\langle x \rangle)^2 \rangle} = \sqrt{\langle x^2 \rangle} = d \sqrt{\frac{1}{12} - \frac{1}{2\pi^2}} \approx 0.181 \,d.$$
2. (i) The electron probability density \( f(r, t) \) is the square modulus of the normalised wavefunction and is proportional to the square modulus of the unnormalised wavefunction. Hence

\[
f(r, t) \propto \left( e^{-r/a_0} e^{-iE_0 t/\hbar} \right) \left( e^{-r/a_0} e^{-iE_0 t/\hbar} \right)^* = e^{-2r/a_0},
\]

which is independent of time. The probability density is largest at \( r = 0 \) (when the electron is right on top of the nucleus).

(ii) The normalisation condition in three dimensions is

\[
\int_{\text{all space}} |\psi(r, t)|^2 dV = 1.
\]

To verify that the wavefunction given in the question is normalised, we have to show that the normalisation integral,

\[
N = \int_{\text{all space}} \frac{1}{\pi a_0^3} e^{-2r/a_0} dV,
\]

is equal to 1. Since the integrand only depends on the length \( r \) of the vector \( r \), the integral can be evaluated by summing contributions from spherical shells. The contribution from the shell with inner and outer radii \( r \) and \( r + dr \) is the volume of that shell, \( 4\pi r^2 dr \), times the value of the integrand for that shell. Hence

\[
N = \int_{r=0}^{\infty} \frac{1}{\pi a_0^3} e^{-2r/a_0} 4\pi r^2 dr
\]

\[
= \frac{4\pi}{\pi a_0^3} \left( \frac{a_0}{2} \right)^3 \int_{\xi=0}^{\infty} e^{-\xi} \xi^2 d\xi
\]

(\text{where } \xi = 2r/a_0)

\[
= \frac{1}{2} \quad 2! = 1 \quad \text{(using integral given in question)}
\]

as required.

(iii) Let \( p(r) dr \) be the probability that the electron is found in the spherical shell with inner and outer radii \( r \) and \( r + dr \). Then

\[
p(r) dr = |\psi(r, t)|^2 4\pi r^2 dr = \frac{1}{\pi a_0^3} e^{-2r/a_0} 4\pi r^2 dr = \frac{4}{a_0^3} r^2 e^{-2r/a_0} dr.
\]

The most probable distance of the electron from the nucleus is the value of \( r \) for which \( p(r) \) is maximised. To find the maximum of \( p(r) \), we set \( dp/dr \) equal to zero:

\[
\frac{dp}{dr} = \frac{4}{a_0^3} \left( 2r - 2 \frac{r^2}{a_0} \right) e^{-2r/a_0} = 0.
\]

The only solution to this equation is \( r = r^2/a_0 \) and hence \( r = a_0 \). The most probable distance of the electron from the nucleus is therefore \( a_0 \). (Since \( p(r) \) is a non-negative function which is zero at \( r = 0 \), tends to zero as \( r \to \infty \), and has only a single
stationary point in between, there is no need to check the type of that stationary point. It can only be a maximum. Why?)

In part (i) we found that the probability density is maximised at the origin. Here we found that the most likely distance of the electron from the origin is $a_0$. These results are not inconsistent. The probability density $f(r)$ is defined as follows:

$$\text{prob. electron is found in volume element } dV \text{ at } r = f(r) \, dV.$$ 

In other words, $f(r)$ is the probability per unit volume. The probability of finding the electron in the spherical shell with inner and outer radii $r$ and $r + dr$ is the product of the probability density, $f(r)$, and the volume $4\pi r^2 \, dr$. Although the probability density peaks at the origin and decreases as $r$ increases, the volume of a shell of fixed thickness $dr$ increases with $r$. In fact, for $r < a_0$, the shell volume increases faster than the probability density decreases. The probability of finding the electron within a spherical shell of thickness $dr$ therefore increases as $r$ increases from 0 to $a_0$.

(iv) The mean distance from the nucleus is

$$\langle r \rangle = \int_{\text{all space}} r |\psi(r, t)|^2 \, dV = \int_{r=0}^{\infty} r \frac{e^{-2r/a_0}}{\pi a_0^3} 4\pi r^2 \, dr = \frac{4}{a_0^3} \left(\frac{a_0}{2}\right)^4 \int_0^\infty \xi^3 e^{-\xi} \, d\xi \quad \text{(where } \xi = 2r/a_0)$$

$$= \frac{a_0}{4} \frac{3!}{2} \quad \text{(using integral given in question)}$$

$$= \frac{3a_0}{2}.$$ 

The mean square distance from the nucleus is

$$\langle r^2 \rangle = \int_{\text{all space}} r^2 |\psi(r, t)|^2 \, dV = \int_{r=0}^{\infty} r^2 \frac{e^{-2r/a_0}}{\pi a_0^3} 4\pi r^2 \, dr = \frac{4}{a_0^3} \left(\frac{a_0}{2}\right)^5 \int_0^\infty \xi^4 e^{-\xi} \, d\xi \quad \text{(where } \xi = 2r/a_0)$$

$$= \frac{a_0^2}{8} \frac{4!}{2} \quad \text{(using integral given in question)}$$

$$= 3a_0^2.$$ 

Hence, the root mean square distance (the square root of the mean square distance) of the electron from the nucleus is $\sqrt{3}a_0$.

(v) The probability of finding the electron within a sphere of radius $10^{-15}$ m is given exactly by the following integral:

$$\int_{r=0}^{10^{-15} \text{ m}} |\psi(r, t)|^2 4\pi r^2 \, dr.$$
However, since $10^{-15}$ m is such a small radius, the probability density $|\psi(r, t)|^2$ is almost constant throughout the region of integration. A very good approximation to the probability may therefore be obtained by multiplying the probability density at the origin, $|\psi(0, t)|^2 = 1/\pi a_0^3$, by the volume of the tiny sphere, $\frac{4}{3}\pi(10^{-15})^3$ m$^3$. Using this method, the estimate of the probability of finding the electron on top of the nucleus is

$$\frac{\frac{4}{3}\pi(10^{-15})^3}{\pi a_0^3} = \frac{4}{3} \frac{10^{-45}}{(0.53)^3 \times 10^{-30}} \approx 9 \times 10^{-15}.$$ 

**Momentum Measurements**

3. Since 

$$e^{i\theta} = \cos \theta + i \sin \theta \quad \text{and} \quad e^{-i\theta} = \cos \theta - i \sin \theta,$$

it follows that

$$\sin(kx) = \frac{1}{2i} \left( e^{ikx} - e^{-ikx} \right).$$

Hence, in the region $x < 0$, we have:

$$\psi(x, t) = -\frac{i}{2} e^{ikx-\omega t} + \frac{i}{2} e^{-ikx-\omega t}.$$ 

The first term is a right-going travelling wave and the second term is a left-going travelling wave.

In lectures, we considered a general superposition of travelling waves,

$$\psi(x, t) = \sum_n A(k_n) e^{i(k_n x - \omega t)},$$

and argued that $|A(k_n)|^2$ was proportional to the probability of obtaining the result $\hbar k_n$ in a measurement of momentum. The wavefunction in this question is a simple two-term example of the general form considered in lectures. The wavevectors of the two terms are $k$ and $-k$ and their complex amplitudes are $A(k) = -i/2$ and $A(-k) = i/2$. Hence, the two possible results of a momentum measurement are $\hbar k$ and $-\hbar k$. Since the two terms have the same intensity, $|A(k)|^2 = |A(-k)|^2$, the probabilities of measuring $\hbar k$ and $-\hbar k$ are both 1/2.

Physically, the wavefunction represents a particle in a very (infinitely) spread out wavepacket of momentum $\hbar k$, which is being reflecting elastically from a potential barrier at $x = 0$. Since the wavepacket is so spread out, it seems reasonable that the probability of measuring the incident momentum $\hbar k$ is equal to the probability of measuring the reflected momentum $-\hbar k$. 

4
The Uncertainty Principle

4. The size of a nucleus is about $10^{-15}$ m and so the position uncertainty $\Delta x$ of an electron contained in a nucleus is also about $10^{-15}$ m. According to the uncertainty principle, the momentum uncertainty of such an electron satisfies

$$\Delta p \geq \frac{\hbar}{2\Delta x} \approx 5.25 \times 10^{-20} \text{ kg m s}^{-1}.$$ 

The confined electron is not going anywhere on average, so its average momentum $\langle p \rangle$ must be zero. Hence

$$(\Delta p)^2 = \langle (p - \langle p \rangle)^2 \rangle = \langle p^2 \rangle.$$ 

Since $c\Delta p = 1.575 \times 10^{-11} \text{ J} \approx 98 \text{ MeV} \gg mc^2 = 511 \text{ keV}$, the kinetic energy of confinement is sufficient to make the electron highly relativistic. The kinetic energy of the electron is therefore given by the relativistic formula:

$$KE = \sqrt{p^2 c^2 + m^2 c^4} - mc^2 \approx |p|c.$$ 

To obtain an order of magnitude estimate of this quantity, we approximate $|p|$ by $\Delta p = \sqrt{\langle p^2 \rangle}$ to obtain

$$KE \sim 98 \text{ MeV}.$$ 

This argument shows that the KE of an electron confined within a nucleus is of order $100 \text{ MeV}$. Since the kinetic energy of the electron emitted is between 1 and 10 MeV, it seems more likely that the electron was created during the radioactive decay process than that it was confined within the nucleus all along.

5. (i) Starting from the definition of the rms momentum,

$$(\Delta p)^2 = \langle (p - \langle p \rangle)^2 \rangle,$$

and using the result $\langle p \rangle = 0$ given in the question, we obtain $(\Delta p)^2 = \langle p^2 \rangle$. Similarly, we can show that $(\Delta x)^2 = \langle x^2 \rangle$. The expression for the total energy,

$$\langle E \rangle = \frac{\langle p^2 \rangle}{2m} + \frac{1}{2}s\langle x^2 \rangle,$$

may therefore be rewritten as

$$\langle E \rangle = \frac{(\Delta p)^2}{2m} + \frac{1}{2}s(\Delta x)^2 = \frac{(\Delta p)^2}{2m} + \frac{1}{2}m\omega^2(\Delta x)^2,$$

where the last step followed because $\omega = \sqrt{s/m}$.

(ii) The uncertainty principle states that $\Delta x \Delta p \geq \hbar/2$. If we use this to eliminate $\Delta p$ from the expression for $\langle E \rangle$ we obtain

$$\langle E \rangle \geq \frac{1}{2m} \frac{\hbar^2}{4(\Delta x)^2} + \frac{1}{2}m\omega^2(\Delta x)^2 = \frac{\hbar^2}{8m(\Delta x)^2} + \frac{1}{2}m\omega^2(\Delta x)^2.$$
The value of $\Delta x$ that makes the right-hand side as small as possible (and hence imposes the weakest possible condition on $\langle E \rangle$) may be found by differentiating with respect to $\Delta x$ and setting the result equal to zero:

$$\frac{d}{d(\Delta x)} \left[ \frac{\hbar^2}{8m(\Delta x)^2} + \frac{1}{2}m\omega^2(\Delta x)^2 \right] = \frac{-\hbar^2}{4m(\Delta x)^3} + m\omega^2 \Delta x = 0.$$ 

Solving this equation gives $\Delta x = \sqrt{\hbar/2m\omega}$. Substituting back into the inequality for $\langle E \rangle$ gives

$$\langle E \rangle \geq \frac{\hbar^2}{8m(h/2m\omega)} + \frac{1}{2}m\omega^2 \left( \frac{h}{2m\omega} \right) = \frac{1}{4}\hbar\omega + \frac{1}{4}h\omega,$$

and hence

$$\langle E \rangle \geq \frac{1}{2}h\omega,$$

as required.

6. The energy-time uncertainty principle,

$$\Delta E \Delta t \geq \frac{\hbar}{2},$$

relates the lifetime $\Delta t$ of a state to its energy uncertainty $\Delta E$. For $\Delta t = 2.6 \times 10^{-10}$ s we obtain

$$\Delta E \geq \frac{1.05 \times 10^{-34}}{2 \times 2.6 \times 10^{-10}} \approx 2.02 \times 10^{-25} \text{ J} \approx 1.26 \times 10^{-6} \text{ eV}.$$ 

Hence, the minimum energy uncertainty of the emitted photons is $1.26 \times 10^{-6}$ eV.

7. In order for a virtual particle of energy $mc^2$ to pop out of the vacuum, the energy uncertainty $\Delta E$ must be (at least) $mc^2$. According to the energy-time uncertainty principle in the form $\Delta E \Delta t \sim \hbar$, the lifetime of such a particle is $\Delta t \sim \hbar/\Delta E \sim h/mc^2$. The speed of the particle must be less than the speed of light, and hence the distance it travels during its lifetime must be less than $c\Delta t \sim \hbar/mc$.

Equating this distance to the range of the nuclear force gives

$$\frac{\hbar}{mc} \approx 1.4 \times 10^{-15} \text{ m},$$

and hence

$$mc^2 \approx \frac{\hbar c}{1.4 \times 10^{-15}} = \frac{1.05 \times 10^{-34} \times 3.00 \times 10^8}{1.4 \times 10^{-15}} = 2.25 \times 10^{-11} \text{ J}.$$ 

In eV, this becomes

$$mc^2 \approx \frac{2.25 \times 10^{-11}}{1.60 \times 10^{-19}} \approx 141 \times 10^6 \text{ eV} = 141 \text{ MeV},$$

or, equivalently, $m \approx 141 \text{ MeV}/c^2$. 

6
The Schrödinger Equation

8. The time-independent Schrödinger equation is

\[-\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)\,.

If the wavefunction

\[\psi_n(x) = \begin{cases} \sqrt{\frac{2}{d}} \sin \left(\frac{n\pi x}{d}\right) & 0 < x < d \\ 0 & \text{otherwise}\end{cases}\]

is a normalised energy eigenfunction, the following conditions hold:

(a) \(\psi_n(x)\) satisfies the boundary conditions: \(\psi_n(0) = \psi_n(d) = 0\).

(b) \(\psi_n(x)\) satisfies the time-independent Schrödinger equation for \(0 < x < d\).

(c) \(\psi_n(x)\) is normalised.

Consider these conditions one by one:

(a) By inspection, \(\psi_n(x)\) satisfies the boundary conditions for \(n = 1, 2, \ldots\).

(b) Substitute \(\psi_n(x)\) into the left-hand side of the Schrödinger equation for \(0 < x < d\):

\[-\frac{\hbar^2}{2m} \frac{d^2 \psi_n(x)}{dx^2} + V(x)\psi_n(x) = -\frac{\hbar^2}{2m} \frac{d^2 \psi_n(x)}{dx^2}\]

\[= -\frac{\hbar^2}{2m} \sqrt{\frac{2}{d}} \frac{d^2}{dx^2} \sin \left(\frac{n\pi x}{d}\right)\]

\[= \frac{\hbar^2}{2m} \left(\frac{n\pi}{d}\right)^2 \sqrt{\frac{2}{d}} \sin \left(\frac{n\pi x}{d}\right)\]

\[= E_n\psi_n(x)\, , \quad \text{where} \quad E_n = \frac{\hbar^2}{2md^2}\left(\frac{n^2\pi^2}{2md^2}\right)\.

Hence, \(\psi_n(x)\) satisfies the time-independent Schrödinger equation in the region \(0 < x < d\). The corresponding energy eigenvalue is \(E_n = \frac{\hbar^2}{2md^2}\left(\frac{n^2\pi^2}{2md^2}\right)\).

(c) If \(\psi_n(x)\) is normalised, the integral

\[N = \int_0^d |\psi_n(x)|^2 dx\]

must be equal to 1. Check this by evaluating the integral:

\[N = \frac{2}{d} \int_0^d \sin^2 \left(\frac{n\pi x}{d}\right) dx = \frac{2}{d} \times \frac{d}{2} = 1\,.

Hence, \(\psi_n(x)\) is normalised.
9. Using the result of Q8, the energy eigenvalues are:

\[ E_n = \frac{\hbar^2 n^2 \pi^2}{2md^2} \quad n = 1, 2, \ldots \]

The energy emitted as the nucleon falls from the \( n = 2 \) level to the \( n = 1 \) level is

\[ E_2 - E_1 = \frac{3\hbar^2 \pi^2}{2md^2} \approx \frac{3 \times (1.05 \times 10^{-34})^2 \times \pi^2}{2 \times 1.67 \times 10^{-27} \times (10^{-15})^2} \]

\[ \approx 9.8 \times 10^{-11} \text{ J} \approx 610 \text{ MeV} . \]

This is a sensible number. The energy released in fission is about 200 MeV per nucleus.

10. The classical angular frequency of vibration is \( \omega_{\text{vib}} = \sqrt{s/m_{\text{reduced}}} \), where

\[ m_{\text{reduced}} = \left( \frac{1}{m_C} + \frac{1}{m_O} \right)^{-1} = \left( \frac{1}{12} + \frac{1}{16} \right)^{-1} \approx 6.86 \text{ atomic mass units} . \]

Hence

\[ \omega_{\text{vib}} = \sqrt{\frac{1857}{6.86 \times 1.66 \times 10^{-27}}} \approx 4.04 \times 10^{14} \text{ Radians s}^{-1} . \]

(i) The energy of the photon emitted when a CO molecule makes a transition between neighbouring vibrational states is the difference between the energies of those states. The energy levels of a QM simple harmonic oscillator are \((n + \frac{1}{2})\hbar\omega_{\text{vib}}, \text{ where } n = 0, 1, 2, \ldots \) Hence, the energy \( E \) of the emitted photon is \( \hbar\omega_{\text{vib}} \). Since the energy \( E \) and angular frequency \( \omega \) of the photon are related via \( E = \hbar\omega \), it follows that \( \omega = \omega_{\text{vib}} \). The frequency of the emitted photon is therefore the same as the classical vibrational frequency of the molecule, which makes sense. The photon wavelength is

\[ \lambda = \frac{c}{\nu} = \frac{c}{\nu_{\text{vib}}} = \frac{2\pi c}{\omega_{\text{vib}}} \approx 4.67 \times 10^{-6} \text{ m} . \]

(ii) The vibrational zero-point energy of a CO molecule is

\[ \frac{1}{2} \hbar \omega_{\text{vib}} \approx 2.12 \times 10^{-20} \text{ J} \approx 0.13 \text{ eV} . \]

11. Substitute the trial solution \( \psi(x) = e^{-\alpha x^2} \) into the left-hand side of the Schrödinger equation:

\[ -\frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} + \frac{1}{2} sx^2\psi(x) = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} e^{-\alpha x^2} + \frac{1}{2} sx^2 e^{-\alpha x^2} \]

\[ = -\frac{\hbar^2}{2m} \frac{d}{dx} (-2\alpha x e^{-\alpha x^2}) + \frac{1}{2} sx^2 e^{-\alpha x^2} \]

\[ = \frac{\hbar^2 \alpha}{m} \frac{d}{dx} (xe^{-\alpha x^2}) + \frac{1}{2} sx^2 e^{-\alpha x^2} \]
\[
\psi(x) = \frac{\hbar^2}{m} \left( e^{-\alpha x^2} - 2\alpha x^2 e^{-\alpha x^2} \right) + \frac{1}{2} s x^2 e^{-\alpha x^2} \\
= \frac{\hbar^2}{m} e^{-\alpha x^2} + \left( \frac{1}{2} s - \frac{2\hbar^2 \alpha^2}{m} \right) x^2 e^{-\alpha x^2}.
\]

If \(\psi(x)\) is to be a solution of the Schrödinger equation, this must equal a constant, \(E\), times \(e^{-\alpha x^2}\). The value of \(\alpha\) must therefore be chosen to ensure that the coefficient of the \(x^2 e^{-\alpha x^2}\) term is zero. Hence

\[
\alpha^2 = \frac{ms}{4\hbar^2} \Rightarrow \alpha = \frac{\sqrt{sm}}{2\hbar}.
\]

If this condition is met, \(\psi(x)\) is an energy eigenfunction with energy eigenvalue

\[
E = \frac{\hbar^2 \alpha}{m} = \frac{\hbar^2 \sqrt{sm}}{2\hbar} = \frac{1}{2} \hbar \sqrt{\frac{s}{m}} = \frac{1}{2} \hbar \omega,
\]

where \(\omega = \sqrt{s/m}\) is the classical angular frequency of the oscillator.

Q5 showed that the energy of a quantum mechanical simple harmonic oscillator must be \(\geq \frac{1}{2} \hbar \omega\), and so the eigenfunction found here must be the (unnormalised) ground state.

12. Inside the barrier, the Schrödinger equation is

\[
-\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + V \psi(x) = E \psi(x),
\]

with \(V > E\). A simple rearrangement gives

\[
\frac{d^2 \psi(x)}{dx^2} = \frac{2m(V - E)}{\hbar^2} \psi(x) = \gamma^2 \psi(x),
\]

where

\[
\gamma = \sqrt{\frac{2m(V - E)}{\hbar^2}}
\]

is real because \(V - E\) is greater than zero. By inspection, the two independent solutions of this second-order differential equation are \(e^{\gamma x}\) and \(e^{-\gamma x}\), and so the general solution is

\[
\psi(x) = Ae^{-\gamma x} + Be^{\gamma x},
\]

where \(A\) and \(B\) are arbitrary constants to be determined from the boundary conditions.

Assume now that the barrier stretches from \(x = 0\) to \(x = a\) and that the particles are incident from the left. We discard the \(e^{\gamma x}\) solution on physical grounds: it would make little sense if the wavefunction, and hence the probability density, increased with distance into the barrier. (In fact, as you will find out next year, the exact wavefunction inside a barrier of finite width does include a small amount of the \(e^{\gamma x}\) solution. However, as long as the barrier is wide and the tunnelling probability low, the value of \(B\) is so small that this contribution can be ignored.) Inside the barrier, then, the wavefunction \(\psi(x)\) is equal to \(Ae^{-\gamma x}\), as shown in the figure.
It follows that
\[
\left| \psi(x = a) \right|^2 \approx \left( e^{-\gamma a} \right)^2 = e^{-2\gamma a} .
\]
Since \( \left| \psi(x) \right|^2 \) is proportional to the probability density at position \( x \), this equation shows that the probability density at the right-hand edge of the barrier is \( e^{-2\gamma a} \) (which is normally a very small number) times the probability density at the left-hand edge. The tunnelling probability is therefore equal to \( e^{-2\gamma a} \).

The work function \( W \) is the minimum energy required to remove an electron from a piece of metal. If additional energy \( W \) is supplied to a metallic electron with energy \( E \), that electron is only just able to escape from the surface. This implies that the potential energy \( V \) of the electron outside the metal must be \( E + W \). If no extra energy is supplied (so that the electron from the metal still has energy \( E \)), the Schrödinger equation in the gap region takes the form,
\[
- \frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + (E + W) \psi(x) = E \psi(x) ,
\]
and hence \( \gamma = \sqrt{2m(E + W - E)/\hbar^2} = \sqrt{2mW/\hbar^2} \).

In the case when \( a = 10^{-6} \) m and \( W = 5 \) eV, we obtain
\[
\gamma = \frac{\sqrt{2 \times 9.11 \times 10^{-31} \times 5 \times 1.60 \times 10^{-19}}}{1.05 \times 10^{-34}} \approx 1.15 \times 10^{10} \text{ m}^{-1} .
\]

The tunnelling probability is therefore
\[
e^{-2\gamma a} \approx e^{-23,000} = 10^{-23,000 \times \log_{10} e} \approx 10^{-10,000} .
\]

This is very small! Electron tunnelling is important in scanning tunnelling microscopes, where the gap is comparable to the size of an atom (\( 10^{-10} \) m), but not for the much larger gap considered here.

**Extra Questions for Enthusiasts**

13. (i) In order to tell which slit the electron went through, the optical instrument that detects the scattered photons must be able to resolve features of size \( d \). Since the resolution cannot be much better than the wavelength of the light used, \( \lambda_{\text{photon}} \) must be less than or approximately equal to \( d \).
(ii) The figure shows a photon moving in the $x$ direction scattering from an electron moving in the $z$ direction.

The $x$ momentum transferred to the electron may take any value from 0 (if the photon is not deflected at all) to $2h/\lambda_{\text{photon}}$ (if the photon scatters back towards its source), but the typical value is around $h/\lambda_{\text{photon}}$. The initial $z$ momentum of the electron is $p_z = h/\lambda_{\text{electron}}$. The value of $p_z$ changes slightly when the photon scatters, but if the change in electron direction is small the change in $p_z$ will also be small. Hence, even after the photon has been scattered, $p_z \approx h/\lambda_{\text{electron}}$.

After the scattering event, the electron’s momentum vector looks something like this:

If the scattering angle $\Delta \theta$ is small, so that $\tan(\Delta \theta) \approx \Delta \theta$, it follows that

$$\Delta \theta \approx \tan(\Delta \theta) = \frac{p_x}{p_z} \sim \frac{\lambda_{\text{electron}}}{\lambda_{\text{photon}}}.$$  

(iii) The angular spacing between adjacent diffraction maxima or minima is $\lambda_{\text{electron}}/d$. Since $\lambda_{\text{photon}} \leq d$ (see part (i)), the angular spacing must be $\leq \lambda_{\text{electron}}/\lambda_{\text{photon}}$. But this is exactly equal to $\Delta \theta$, the uncertainty in the direction of the electron caused by scattering the photon (part (ii)). Hence, the scattering of the photon is sufficient to smear out the diffraction pattern completely.

14. The integral we have to evaluate is

$$\int_{-\infty}^{\infty} \psi_n^*(x) \psi_m(x) \, dx = \frac{2}{d} \int_0^d \sin\left(\frac{n \pi x}{d}\right) \sin\left(\frac{m \pi x}{d}\right) \, dx$$

$$= \frac{2}{d} \int_0^\pi \sin(n \theta) \sin(m \theta) \, d\theta \quad \text{(where } \theta = \pi x/d)$$

$$= \frac{1}{d} \int_0^\pi \left[ \cos((n-m)\theta) - \cos((n+m)\theta) \right] \, d\theta ,$$

$$= 2d \frac{1}{ \pi } \left[ \frac{\sin((n-m)\theta)}{n-m} - \frac{\sin((n+m)\theta)}{n+m} \right]_0^\pi$$
where the last step used the expression for \( \sin(n\theta) \sin(m\theta) \) given in the question. For any non-zero integer \( j \), the integral

\[
\int_0^\pi \cos(j\theta) \, d\theta = \left[ \frac{1}{j} \sin(j\theta) \right]_0^\pi = 0.
\]

Hence, if \( n \) and \( m \) are unequal positive integers,

\[
\int_{-\infty}^{\infty} \psi_n^*(x) \psi_m(x) \, dx = 0.
\]

If \( n = m \), so that \( \cos((n-m)\theta) = 1 \), we obtain

\[
\int_{-\infty}^{\infty} \psi_n^*(x) \psi_n(x) \, dx = \frac{1}{\pi} \int_0^\pi [1 - \cos(2n\theta)] \, d\theta = \frac{1}{\pi} \int_0^\pi d\theta = 1,
\]

demonstrating that \( \psi_n(x) \) is normalised.

15. If \( \phi(x) = \sum_{n=1}^{\infty} c_n \psi_n(x) \) is normalised, then

\[
\int_{-\infty}^{\infty} \phi^*(x) \phi(x) \, dx = 1.
\]

Substituting the expansion of \( \phi(x) \) into the normalisation integral gives

\[
\int_{-\infty}^{\infty} \phi^*(x) \phi(x) \, dx = \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} c_n^* \psi_n^*(x) \sum_{m=1}^{\infty} c_m \psi_m(x) \, dx
\]

\[
= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_n^* c_m \int_{-\infty}^{\infty} \psi_n^*(x) \psi_m(x) \, dx.
\]

The integral in this expression is equal to 1 if \( n = m \) and to 0 otherwise. Hence, all terms with \( n \neq m \) vanish and only the \( n = m \) terms are left:

\[
\int_{-\infty}^{\infty} \phi^*(x) \phi(x) \, dx = \sum_{n=1}^{\infty} c_n^* c_n \int_{-\infty}^{\infty} \psi_n^*(x) \psi_n(x) \, dx = \sum_{n=1}^{\infty} c_n^* c_n.
\]

This shows that \( \phi(x) \) is normalised if and only if

\[
\sum_{n=1}^{\infty} c_n^* c_n = 1.
\]

Since \( c_n^* c_n \) is greater than or equal to zero, and since the sum of \( c_n^* c_n \) over all \( n \) gives 1, it is reasonable (and right) to guess that \( c_n^* c_n \) is the probability that a measurement of the energy gives the result \( E_n \) (where \( E_n \) is the energy eigenvalue corresponding to the energy eigenfunction \( \psi_n(x) \)).