Current Sheet.

The earth’s magnetic field is distorted by the solar wind – field lines look like:

![Diagram of the earth's magnetic field with solar wind and field lines](image)

Figure 1: Conductor

The field in the tail – on the right hand side of the picture above – has the form of a current sheet centred about the equator. It is often modelled by a form called the *Harris Sheet*:

\[
B = -B_0 \tanh \left( \frac{y}{L} \right)
\]

where \( B_0 \) and \( L \) are constants. The \( y \) direction is vertical (roughly northwards) in the picture and \( x \) is horizontal.

(i) Draw the field lines given by Eq. (1) in the x-y plane and plot the field strength against \( y \).

(ii) Calculate the current in the *Harris Sheet* assuming that \( \frac{\partial E}{\partial t} = 0 \).

You may need \( \frac{d}{ds} \tanh(s) = \frac{1}{\cosh^2(s)} \)

This sheet of current becomes unstable when the current becomes large – it attracts itself and breaks up into blobs called plasmoids. This is believed to trigger the *aurora borealis – the northern lights*. 
Inductive Energy Storage.

One way to store energy is in magnetic fields:

Consider a \( l \) metre long solenoid with \( n \) turns per metre, a radius of \( R \) metres and a current of \( I \) amps.

(i) derive an expression for the magnetic energy stored in the solenoid – ignore any field outside the solenoid. Calculate a numerical value for \( l = 20m, n = 10^6, R = 1m \) and \( I = 10 \) amps.

(ii) The solenoid is an **inductor** so that the voltage across the wires leading to the coil can be written as

\[
V = L \frac{dI}{dt}
\]

where \( L \) is a constant. The power going to the solenoid is \( VI \) from simple circuit theory. Show that the energy delivered to the solenoid ramping up the current from zero to \( I \) is:

\[
\int dt(VI) = \frac{1}{2} LI^2
\]

(iii) Where is this energy stored? Give an expression for \( L \).

(iv) The current in the solenoid is slowly ramped up so that the magnetic field in the solenoid is given at all times by the usual formula for a steady current. What is the direction of the Poynting vector? What is the magnitude of the Poynting vector at radius \( r \) where \( r < R \)? You may ignore energy in the electric field.

**Note.** \( \mu_0 = 4\pi \times 10^{-7} \) and Poynting’s theorem for the conservation of Electromagnetic energy is:

\[
\frac{\partial W}{\partial t} + \nabla \cdot \mathbf{N} = -\mathbf{J} \cdot \mathbf{E}
\]

where the electromagnetic energy density and the Poynting vector are:

\[
W = \frac{1}{2} \mu_0 E^2 + \frac{1}{2\mu_0} B^2, \quad \mathbf{N} = \frac{\mathbf{E} \times \mathbf{B}}{\mu_0}.
\]
Ultra High Fields.

Ultra high fields have been made by compressing fields inside a plasma inside a "flux conservor". In this question we will examine the basic physics.

\[
\text{Consider a } l \text{ metre long cylindrical plasma with radius } a \text{ and an axial current: }
\]

\[
J = \frac{2B_0}{a\mu_0} \left(1 - \frac{r^2}{a^2}\right)\hat{z}.
\]

(i) Derive an expression for the magnetic field for \( r < a \). You may want to use \( \frac{1}{r} \frac{d(rB_\theta)}{dr} = \mu_0 J_z \).

(ii) Calculate the magnetic flux per unit length crossing a surface bounded by the axis and the edge of the cylinder – e.g. the surface could be the \( y = 0 \) plane between \( 0 < z < a \) and \( 0 < z < 1 \).

(iii) Calculate the magnetic energy per unit length

The plasma and field is surrounded by a conducting shell and explosives around the shell. The explosives are detonated and the shell is compressed – the conductor and plasma ensure that (\( E = 0 \) in the moving conductor) and the magnetic flux is conserved. We can assume that the magnetic field preserves its form given in (i) and that \( a \) decreases.

(iv) Just before the detonation \( B_0 = B_{00} \) and \( a = a_0 \). Derive an expression for \( B_0 \) when the radius has become \( a \) and give an expression for \( B_\theta \) in terms of \( B_{00}, a_0 \) and \( a \). Does the magnetic energy change during compression?

(iv) If the maximum value of field is initially 20 Tesla and the cylinder is compressed by a factor of ten i.e. \( a = 0.1a_0 \), what is the maximum field after compression?
Classwork IV
The Wave Equation in 3D

So far, we have resolved the components of Maxwell’s equations to be able to determine wave solutions. Here, we practise using vector notation to obtain solutions that are general in 3D, and we explore again (?) the properties of these solutions. This classwork is long, but it is important - you should finish it at home, if necessary.

(i) Write down Maxwell’s equations in differential form (in vacuum) from memory! Give each one a name. Now check with your lecture notes to make sure you got them right.

(ii) Take the $\nabla \times$ of Faraday’s Law. Remembering that $t$ and $r$ are independent values, give the result in terms of $\nabla \times \mathbf{B}$.

(iii) Simplify $\nabla \times (\nabla \times \mathbf{E})$ using the vector triple product, $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$, you will need another of Maxwell’s equations to eliminate one of the terms.

(iv) Finally use another one of Maxwell’s equations (say which!) to substitute for $\mathbf{B}$ and show that you obtain,

$$\nabla^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Give a value for $c$.

(v) It is common to take a function $\mathbf{E} = E_0 e^{i(k \cdot r - \omega t)}$ as a solution, and then to construct more complicated solutions as a Fourier superposition of these solutions. Show that $\nabla^2 \mathbf{E} = -k^2 \mathbf{E}$.

Hint: Write the wavenumber in component form, $k = (k_x, k_y, k_z)$, so that $k \cdot r = k_x x + k_y y + k_z z$.

(vi) Show that the dispersion relation (which relates $\omega$ to $k$) is given for all Fourier components by,

$$\omega = ck$$

(vii) By considering the length $r = \lambda$ for which the real part of $\mathbf{E}$ performs one cycle (for $t = 0$) calculate the wavelength. Similarly for $r = 0$, consider the time taken for the $\mathbf{E}$ to again pass through one cycle and hence give an expression for the frequency $f$. 

Please Turn Over
We can investigate the form of this solution given in (v) in space by once more rewriting k in component form at a given time (say take \( t = 0 \)). Consider a wave with wave-vector, \( k = (1, 1, 0) \). Sketch in the \( xy \) plane, lines of constant phase \( \phi \), i.e. where \( \phi = k \cdot r = 0, \lambda, 2\lambda \) ... Can you say something about the direction in which the wave propagates (think about what happens when \( t \) increases)?

We can investigate further properties of the solution given in (v) by applying it to Maxwell’s equations. Apply this solution to Gauss’ Law. You will need to write \( E_0 = (E_{x0}, E_{y0}, E_{z0}) \). What does it say about the direction of the fields relative to the waves propagation?

You should have noticed that \( \frac{\partial}{\partial x} \left( E_n e^{i(k \cdot r - \omega t)} \right) = ik_x E_n e^{i(k \cdot r - \omega t)} \) etc. for all the components \( E_n \) of \( E \), and similarly for the derivatives with respect to \( y \) and \( z \). Use this to show that \( \nabla \times E = i k \times E \). (Note: it’s easiest if you use the determinant form of the curl, and do not explicitly multiply it out, but use properties of determinants).

Hence applying our above solution to Faraday’s Law, obtain a general expression relating the \( E \) and \( B \) fields in vacuum (leave it in vector form). Use this to relate the magnitudes of \( E \) and \( B \).

By considering a point of constant phase, where phase is defined as \( \phi = (k \cdot r - \omega t) \), show that the phase velocity (the rate at which this point of constant phase moves) is given by \( v_\phi = c \).

Consider two solutions with slightly different values of \((\omega, k)\):

\[
E_1 = E_0 e^{i(k_1 \cdot r - \omega_1 t)} \quad \text{and} \quad E_2 = E_0 e^{i(k_2 \cdot r - \omega_2 t)}
\]

where \( k_1 = k + \Delta k \) and \( k_2 = k - \Delta k \) and where \( \omega_1 = \omega + \Delta \omega \) and \( \omega_2 = \omega - \Delta \omega \).

Create a solution \( E = E_1 + E_2 \), and show that it can be written as,

\[
E(r, t) = 2E_0 e^{i(k \cdot r - \omega t)} \cos(\Delta k \cdot r - \Delta \omega t)
\]

By considering the limit \( (\Delta \omega, \Delta k) \to 0 \), show that the group velocity (the velocity of the “beat” for a continuous range of \((\omega, k)\)), is

\[
v_g = \lim_{(\Delta \omega, \Delta k) \to 0} \left( \frac{\Delta \omega}{\Delta k} \right) = \frac{\partial \omega}{\partial k} = c
\]
Classwork V
Why is the sky blue?

Information needed for this Classwork

- \( \mathbf{E} = \frac{q}{4\pi \epsilon_0 r c^2} \sin \theta \hat{\theta} \)  
  electric field radiated by accelerating charge

- \( \nabla \times \mathbf{E} = \begin{vmatrix} \frac{1}{r^2 \sin \theta} & \frac{1}{r \sin \theta} & \frac{1}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ E_r & r E_\theta & r \sin \theta E_\phi \end{vmatrix} \)  
  curl in spherical co-ordinates

- \( \oint_S \mathbf{dS} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} R^2 \sin \theta \mathbf{d}\theta \mathbf{d}\phi \)  
  integral over surface of sphere \( S \) radius \( R \)

There are some things every physicists should know; the big bang theory, what LHC stands for, what instrument Richard Feynman used to play, and of course why the sky is blue? Well at the very least we will try and address the last one today:

(i) Give an expression for the electric field at a distance \( r \) from an electron performing simple harmonic motion \( z = z_0 \cos \omega t \), where \( r \gg z_0 \).

(ii) From Faraday’s law, show that the corresponding magnetic field is given by:

\[
\mathbf{B} = \frac{\varepsilon_0 \omega^2}{4\pi \epsilon_0} \frac{\cos \omega(t - r/c)}{r c^3} \sin \theta \hat{\phi}
\]

(Hint: I suggest leaving the curl in determinant form until you’ve figured out which components of \( \mathbf{E} \) are present and what they are functions of).

(iii) Hence write down an expression for the Poynting vector. (This is the rate at which energy is radiated by the particle).

(iv) First take the time average and then integrate the energy flux over a complete sphere of radius \( r \) to show that the total mean radiated power is given by:

\[
P = \frac{\varepsilon^2 z_0^2 \omega^4}{12\pi \epsilon_0 c^3}
\]
An electron is bound to an atom at the origin with a linear restoring force \( F = -K_s z \), where \( K_s \) is the spring constant. It is perturbed by a light wave polarised in the \( z \) direction: \( E = E_z \hat{z} = E_0 \cos(kx - \omega t) \hat{z} \). Write down the equation of motion. (You may ignore the contribution of the \( B \)-field.) Hence show that the oscillation amplitude varies as:

\[
z_0 = \frac{eE_0}{m(\omega^2 - \omega_0^2)}
\]
giving an appropriate expression for \( \omega_0 \).

We can take the two limits of this expression. Give an expression for \( z_0 \) in the limit \( \omega \gg \omega_0 \). Hence show that the ratio of power radiated to the incident energy (Poynting) flux, is given by the (Thomson scattering) cross section, \( \sigma_T = \frac{8}{3} \pi r_e^2 \) where \( r_e \) is called the classical radius of the electron defined by \( r_e = \frac{e^2}{4\pi\epsilon_0 mc^2} \).

(Remember the Poynting vector of the incident beam will be \( \langle N \rangle = \frac{1}{2}\epsilon_0 E_0^2 c \).)

Note that this cross-section is independent of the frequency. Calculate \( \sigma_T \) (numerically). It is small, but can be used for example in fusion plasmas to calculate its temperature (from the Doppler shift of the scattered beam).

In the opposing limit, \( \omega \ll \omega_0 \), calculate once more the Rayleigh scattering cross-section. How does it vary with frequency?

This scattering is the case where the electron is still more influenced by its parent atom, and is what happens when light passes through the atmosphere. Can you explain now why the sky is blue?
Information needed for this Classwork

- $eE_\perp$ is discontinuous by $\rho_c$, $E_\parallel$ is continuous; boundary conditions for $E$
- $B_\perp$ is continuous; $B_\parallel/\mu$ is discontinuous by $j'_c$; boundary conditions for $B$
- $\eta \simeq \sqrt{\epsilon_r \mu_r}$
- $\eta_{glass} = 1.5$; refractive index of glass

We derived the transmission and reflection coefficients across a single dielectric boundary at normal incidence in lectures. Things get more interesting when we consider multiple boundaries, as then the phase becomes important, allowing control of reflection and transmission.

(i) Consider the arrangement shown in figure 1 where a dielectric of permittivity $\epsilon_1$ is separated from a dielectric of permittivity $\epsilon_3$, by a thin dielectric layer of permittivity $\epsilon_2$ of thickness $d$. An electromagnetic wave is incident normally on boundary 1 from the left. The waves shown in the figure can be represented by:

$$E_1 = A_1 e^{i(\omega_1 t - k_1 z)}; \quad E_1' = A_1' e^{i(\omega_1 t + k_1 z)}; \quad \text{(1)}$$
$$E_2 = A_2 e^{i(\omega_2 t - k_2 z)}; \quad E_2' = A_2' e^{i(\omega_2 t + k_2 z)}; \quad \text{(2)}$$
$$E_3 = A_3 e^{i(\omega_3 t - k_3 z)} \quad \text{(3)}$$

What is the condition for the angular frequency $\omega$ in each region, explaining why? Also give an expression for $k$ in each medium in terms of $\eta$. How are the magnetic components of the waves related to the electric components given in terms of the respective refractive indices (no need to calculate explicitly or even to write them out for each!).

![Figure 1: EM wave incident on a dielectric with a surface coating.](image-url)
(ii) Taking boundary 2 as $x = 0$, write down continuity equations for $E_\parallel$ and $B_\parallel$. Hence show that the amplitudes of the transmitted and reflected waves at this boundary are given by:

$$A_3 = \frac{2\eta_2}{\eta_2 + \eta_3} A_2; \quad A_2' = \frac{\eta_2 - \eta_3}{\eta_2 + \eta_3} A_2.$$

(iii) Now we consider boundary 1. Write out the boundary conditions once more at this boundary, and eliminate the incoming beam ($E_1$) to obtain:

$$2\eta_1 E_1' = (\eta_1 - \eta_2) E_2 + (\eta_1 + \eta_2) E_2'.$$

(iv) OK, now to plug in the phase dependent terms from eq.(2) above, and the expression for the amplitude of $E_2$ found in part (ii). Show that if the thickness of the layer is given by $d = \lambda/4$, where $\lambda = 2\pi/k_2$, then the reflected wave $E_1'$ can be extinguished, provided $\epsilon_2^2 = \epsilon_1 \epsilon_3$.

(iv) Without calculation, what would you expect the corresponding reflection coefficient to be if $\epsilon_2^2 = \epsilon_1 \epsilon_3$ but the layer is now $d = \lambda/2$? What would happen if you used multiple layers which satisfied these conditions?

(v) A single dielectric layer as in the diagram is commonly used on camera lenses to reduce the light reflected from its surface. Why would it be advantageous? Calculate the thickness of the layer and its refractive index for green light ($\lambda = 532$ nm). Why does this anti-blooming layer usually have a slight purple hue?
Classwork VII
Reflection from the Ionosphere

Information needed for this Classwork

- $\omega^2 = \omega_p^2 + c^2 k^2$ dispersion relation in a plasma
- $R = 6400$ km radius of the Earth

*Derived from a past exam question*\(^1\): This question demonstrates how the ionosphere, which is a tenuous plasma, can increase the range of some radio transmissions.

(i) The general equation of motion for an electron (mass $m$) moving with a velocity $v$ in a plasma is given by:

$$m_e \frac{dv}{dt} = -eE - \frac{mv}{\tau_c}$$

where $\tau_c$ is the characteristic time on which the electron undergoes collisions; and $E$ is the electric field. Explain the physical significance of each of the terms, as well as the reason we can ignore the effects of the magnetic field on the motion of the electron.

(ii) (a) Assuming that at any position, the time dependence of $v$ and $E$ is described by $e^{-i\omega t}$, show that the current density, $j$, satisfies:

$$j = \frac{ne^2E}{m\left(\frac{1}{\tau_c} - i\omega\right)}$$

where $n$ is the number density of the free electrons.

(b) Write down the expressions for $j$ for an ohmic conductor, and for a collisionless plasma.

(c) In the latter case, what implications does the result have for the mean power dissipated in the plasma?

(iii)* Starting from the dispersion relation for em waves, calculate the refractive index $\eta$ of a collisionless plasma of electron density $n$. What happens to $\eta$ when $\omega < \omega_p$.

(iv)* We showed (in lectures) that an em wave is reflected off a plasma boundary, if it satisfies the condition $\omega < \omega_p / \cos \theta$. Hence for given values of $\omega$ and $\omega_p$, show that the critical angle, $\cos \theta_c = \omega_p / \omega$, is the same as given by the condition for total internal reflection.

(v) Convince yourself, by simple geometrical considerations, that long distance communication can be extended via ionospheric reflection at an angle of reflection of $\sin \theta = R / (R + h)$, where $R$ is the radius of the Earth, and $h$ is the height of the ionosphere above the surface.

(It should help to draw the earth and ionosphere, with the atmosphere a little exaggerated!).

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\(^1\)starred questions are my additions!
(vi) The Earth’s ionospheric electron density peaks at a value of $10^{12} \text{ m}^{-3}$ at a height, $h = 300 \text{ km}$. Find the value of the highest frequency $\nu_{cr}$, that can be used at this extreme angle. Also find how much the range be extended by?

(You may assume that below the 300 km level the effects of the ionosphere can be ignored and the refractive index = 1).
Current Sheet.

The earth’s magnetic field is distorted by the solar wind – field lines look like:

![Diagram of the Earth's magnetic field distorted by the solar wind]

Figure 1: Conductor

The field in the tail – on the right hand side of the picture above – has the form of a current sheet centred about the equator. It is often modelled by a form called the *Harris Sheet*:

$$\mathbf{B} = -B_0 \tanh \left( \frac{y}{L} \right) \mathbf{\hat{x}}$$  \hspace{1cm} (1)

where $B_0$ and $L$ are constants. The $y$ direction is vertical (roughly northwards) in the picture and $x$ is horizontal.

(i) Draw the field lines given by Eq. (1) in the $x$-$y$ plane and plot the field strength against $y$.

(ii) Calculate the current in the *Harris Sheet* assuming that $\frac{\partial \mathbf{E}}{\partial t} = 0$.

You may need $\frac{d}{ds} \tanh(s) = \frac{1}{\cosh^2(s)}$.

dropping time derivatives we get ampere’s law:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$  \hspace{1cm} (2)

only the $z$ component of the curl survives so $J_x = J_y = 0$ and:

$$-\frac{\partial B_z}{\partial y} = \mu_0 J_z.$$  \hspace{1cm} (3)

$$J_z = \frac{B_0}{\mu_0 L \cosh^2 \left( \frac{y}{L} \right)}.$$  \hspace{1cm} (4)
This sheet of current becomes unstable when the current becomes large – it attracts itself and breaks up into blobs called plasmoids. This is believed to trigger the *aurora borealis* – *the northern lights.*
Inductive Energy Storage.

One way to store energy is in magnetic fields:

Consider a \( l \) metre long solenoid with \( n \) turns per metre, a radius of \( R \) metres and a current of \( I \) amps.

(i) derive an expression for the magnetic energy stored in the solenoid – ignore any field outside the solenoid. Calculate a numerical value for \( l = 20 \text{m} \), \( n = 10^6 \), \( R = 1 \text{m} \) and \( I = 10 \text{amps} \).

The field in a solenoid is \( B = \mu_0 n I \) and uniform so the energy density is \( \frac{1}{2} \mu_0 n \frac{B^2}{n} = \frac{1}{2} \mu_0 n^2 I^2 \pi R^2 l \). Plugging in the numbers we get the energy stored in the coil to be \( 4 \times 10^9 J = 4 \text{GJ} \) about one tonne of TNT.

(ii) The solenoid is an inductor so that the voltage across the wires leading to the coil can be written as

\[ V = L \frac{dI}{dt} \]

where \( L \) is a constant. The power going to the solenoid is \( VI \) from simple circuit theory. Show that the energy delivered to the solenoid ramping up the current from zero to \( I \) is:

\[ \int dt(VI) = \frac{1}{2} LI^2 \]

(iii) Where is this energy stored? Give an expression for \( L \).

The energy is the energy in the magnetic field. Thus \( \frac{1}{2} LI^2 = \frac{1}{2} \mu_0 n^2 I^2 \pi R^2 l \) and therefore \( L = \mu_0 n^2 \pi R^2 l \)

(iv) The current in the solenoid is slowly ramped up so that the magnetic field in the solenoid is given at all times by the usual formula for a steady current. What is the direction of the Poynting vector? What is the magnitude of the Poynting vector at radius \( r \) where \( r < R \)? You may ignore energy in the electric field. The magnetic field is increasing so the Poynting vector must show the flow of energy inwards from coil towards
the centre of the coil. Using Poynting’s theorem we can use that the flow of energy through a cylinder of radius $r$ must equal the rate of change of energy inside $r$.

$$|\mathbf{N}|2\pi rl = \frac{d}{dt}\left(\frac{1}{2}\mu_0 B^2\right) \times \pi r^2 l = \frac{d}{dt}\left(\frac{1}{2}\mu_0 n^2 I^2\right) \times \pi r^2 l,$$

(1)

$$|\mathbf{N}| = \frac{d}{dt}\left(\frac{1}{4\mu_0 n^2 I^2}\right) \times r$$

Note. $\mu_0 = 4\pi \times 10^{-7}$ and Poynting’s theorem for the conservation of Electromagnetic energy is:

$$\frac{\partial W}{\partial t} + \nabla \cdot \mathbf{N} = -\mathbf{J} \cdot \mathbf{E}$$

where the electromagnetic energy density and the Poynting vector are:

$$W = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0} B^2, \quad \mathbf{N} = \frac{\mathbf{E} \times \mathbf{B}}{\mu_0}.$$
Ultra High Fields.

Answers in red

Ultra high fields have been made by compressing fields inside a plasma inside a "flux conserver". In this question we will examine the basic physics.

Consider a 1 metre long cylindrical plasma with radius \( a \) and an axial current:

\[
J = \frac{2B_0}{\mu_0 a} \left(1 - \frac{r^2}{a^2}\right) \hat{z}.
\]  
(1)

(i) Derive an expression for the magnetic field for \( r < a \). You may want to use \( \frac{1}{r} \frac{d(rB_\theta)}{dr} = \mu_0 J_z \) usual Ampere argument or use \( \frac{1}{r} \frac{d(rB_\theta)}{dr} = \mu_0 J_z \)

\[
B_\theta = \frac{\mu_0}{r} \int_0^r r \, dr \, J_z = \frac{aB_0}{r} \int_0^{r^2} dy (1 - y) = B_0 \frac{r}{a} (1 - \frac{r^2}{2a^2})
\]

I will use the substitution \( y = \frac{y^2}{a^2} \) several times.

(ii) Calculate the magnetic flux per unit length crossing a surface bounded by the axis and the edge of the cylinder – e.g. the surface could be the \( y = 0 \) plane between \( 0 < x < a \) and \( 0 < z < 1 \).

\[
\text{Flux per unit length} = \int_0^a \, dr \, B_\theta = B_0 a \int_0^a \frac{r}{a^2} \left(1 - \frac{y^2}{2a^2}\right) dy = \frac{3}{8} B_0 a
\]

(iii) Calculate the magnetic energy per unit length

\[
\text{Energy per unit length} = \frac{1}{2 \mu_0} \int_0^a 2\pi r \, dr \, B_\theta^2 = \frac{B_0^2 a^2 \pi}{2 \mu_0} \int_0^1 y(1 - \frac{y^2}{2}) \, dy = \frac{11}{48} \frac{B_0^2 a^2 \pi}{2 \mu_0}
\]

The plasma and field is surrounded by a conducting shell and explosives around the shell. The explosives are detonated and the shell is compressed – the conductor and plasma ensure that \( \mathbf{E} = 0 \) in the moving conductor and the magnetic flux is conserved. We can assume that the magnetic field preserves its form given in (i) and that \( a \) decreases.

(iv) Just before the detonation \( B_0 = B_{00} \) and \( a = a_0 \). Derive an expression for \( B_0 \) when the radius has become \( a \) and give an expression for \( B_\theta \) in terms of \( B_{00}, a_0 \) and \( a \). Does the magnetic energy change during compression?
conservation of flux means that $B_0 a$ stays constant so $B_0 = B_{00} a_0 / a$ thus

$$B_0 = B_{00} \frac{a_0}{a} \frac{r}{a}(1 - \frac{r^2}{2a^2}).$$

Magnetic Energy $= \frac{11}{45} \frac{B_0^2 a^2}{2\rho_0}$ does not change as $B_0 a$ stays constant

(iv) If the maximum value of field is initially 20 Tesla and the cylinder is compressed by a factor of ten i.e. $a = 0.1a_0$, what is the maximum field after compression? increases by factor 10 - i.e. 200 Tesla
Classwork IV
The Wave Equation in 3D

So far, we have resolved the components of Maxwell’s equations to be able to determine wave solutions. Here, we practise using vector notation to obtain solutions that are general in 3D, and we explore again (?) the properties of these solutions. This classwork is long, but it is important - you should finish it at home, if necessary.

(i) Write down Maxwell’s equations in differential form (in vacuum) from memory! Give each one a name. Now check with your lecture notes to make sure you got them right.

(1) \( \nabla \cdot \mathbf{E} = 0 \) (Gauss)  
(2) \( \nabla \cdot \mathbf{B} = 0 \) (no monopoles)  
(3) \( \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \) (Faraday)  
(4) \( \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \) (Ampère-Maxwell)

(ii) Take the \( \nabla \times \) of Faraday’s Law. Remembering that \( t \) and \( r \) are independent values, give the result in terms of \( \nabla \times \mathbf{B} \).

\[
\nabla \times (\nabla \times \mathbf{E}) = -\nabla \times \frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial (\nabla \times \mathbf{B})}{\partial t}
\]

(iii) Simplify \( \nabla \times (\nabla \times \mathbf{E}) \) using the vector triple product, \( \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \), you will need another of Maxwell’s equations to eliminate one of the terms.

From \( \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \) \( \rightarrow \) \( \nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \mathbf{E}(\nabla \cdot \nabla) \)

Where we have cancelled the first term by Gauss’ Law in vacuum \( (\nabla \cdot \mathbf{E} = 0) \). Also \( \nabla \cdot \nabla = \nabla^2 \) (the Laplacian) which is an operator so must precede \( \mathbf{E} \), so finally,

\[
\nabla \times (\nabla \times \mathbf{E}) = -\nabla^2 \mathbf{E}
\]

(iv) Finally use another one of Maxwell’s equations (say which!) to substitute for \( \mathbf{B} \) and show that you can obtain the following solution,

\[
\nabla^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}
\]

Give a value for \( c \).

Substituting for \( \nabla \times \mathbf{B} \) from Ampère-Maxwell,

\[
-\nabla^2 \mathbf{E} = -\frac{\partial}{\partial t} \left( \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \rightarrow \nabla^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}
\]
where \( c = (\varepsilon_0 \mu_0)^{-1/2} \).

(v) It is common to take a function \( E = E_0 e^{i(k \cdot r - \omega t)} \) as a solution, and then to construct more complicated solutions as a Fourier superposition of these solutions. Show that \( \nabla^2 E = -k^2 E \).

Hint: Write the wavenumber in its components, \( k = (k_x, k_y, k_z) \), so that \( k \cdot r = k_x x + k_y y + k_z z \).

\[
\nabla^2 E = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) E_0 e^{i(k_x x + k_y y + k_z z - \omega t)} = \left( (ik_x)^2 + (ik_y)^2 + (ik_z)^2 \right) E_0 e^{i(k_x x + k_y y + k_z z - \omega t)}
\]

but \( i^2 = -1 \) and \( k_x^2 + k_y^2 + k_z^2 = k^2 \), So finally,

\[
\nabla^2 E = -k^2 E_0 e^{i(k_x x + k_y y + k_z z - \omega t)} = -k^2 E
\]

(vi) Show that the dispersion relation (which relates \( \omega \) to \( k \)) is given for all Fourier components by,

\[
\omega = ck
\]

\[-k^2 E = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \left( E_0 e^{i(k \cdot r - \omega t)} \right) = \frac{1}{c^2} (i\omega)^2 \left( E_0 e^{i(k \cdot r - \omega t)} \right) = -\frac{\omega^2}{c^2} E
\]

hence, \( k^2 = \omega^2 / c^2 \) \( \rightarrow \) \( \omega = ck \)

(vii) By considering the length \( r = \lambda \) for which the real part of \( E \) performs one cycle (for \( t = 0 \)) calculate the wavelength. Similarly for \( r = 0 \), consider the time taken for the \( E \) to again pass through one cycle and hence give an expression for the frequency \( f \).

\[
\Re (E) = E_0 \cos(k \cdot r - \omega t)
\]

For \( t = 0 \), \( \Re (E) = E_0 \cos(k \cdot r) \) maxima at \( k \cdot r = 2\pi n \)

\[
\rightarrow k(n\lambda) = 2\pi n \rightarrow \lambda = 2\pi / k
\]

For \( \omega = 0 \), \( \Re (E) = E_0 \cos(-\omega t) \) maxima at \( \omega t = 2\pi n \)

\[
\rightarrow \omega(n/f) = 2\pi n \rightarrow f = \omega / 2\pi
\]
(viii) We can investigate the form of this solution given in (v) in space by once more rewriting \( k \) in component form at a given time (say take \( t = 0 \)). Consider a wave with wave-vector, \( \mathbf{k} = (1, 1, 0) \). Sketch in the \( xy \) plane, lines of constant phase \( \phi \), i.e. where \( \phi = \mathbf{k} \cdot \mathbf{r} = 0, \lambda, 2\lambda \ldots \) Can you say something about the direction in which the wave propagates (think about what happens when \( t \) increases)?

![Figure 1: Lines of constant phase for \( \mathbf{k} = (1, 1, 0) \), Arrow shows phase fronts normal to \( \mathbf{k} \), hence \( \mathbf{k} \) is the direction of propagation, since as \( t \) increases, phase fronts would move in that direction.]

(ix) We can investigate further properties of the solution given in (v) by applying it to Maxwell’s equations. Apply this solution to Gauss’ Law. You will need to write \( \mathbf{E}_0 = (E_{x0}, E_{y0}, E_{z0}) \). What does it say about the direction of the fields relative to the waves propagation?

Gauss’ Law

\[
\nabla \cdot \mathbf{E} = \frac{\partial (E_{x0} e^{i(k_x x + k_y y + k_z z - \omega t)})}{\partial x} + \frac{\partial (E_{y0} e^{i(k_x x + k_y y + k_z z - \omega t)})}{\partial y} + \frac{\partial (E_{z0} e^{i(k_x x + k_y y + k_z z - \omega t)})}{\partial z}
\]

\[
= (ik_x E_{x0} + ik_y E_{y0} + ik_z E_{z0}) e^{i(k_x x + k_y y + k_z z - \omega t)} = i\mathbf{k} \cdot \mathbf{E} = 0
\]

which means \( \mathbf{k} \perp \mathbf{E} \), which since \( \mathbf{k} \) is the direction of propagation means that \( \mathbf{E} \) is purely transverse.

(x) You should have noticed that

\[
\frac{\partial}{\partial x} \left( E_n e^{i(k \cdot \mathbf{r} - \omega t)} \right) = i k_x E_n e^{i(k \cdot \mathbf{r} - \omega t)}
\]

for all the components \( E_n \) of \( \mathbf{E} \). Use this to show that \( \nabla \times \mathbf{E} = i\mathbf{k} \times \mathbf{E} \). (Note: it’s easiest if you use the determinant form of the curl, and do not explicitly multiply it out, but use properties of determinants).

Hence applying our above solution to Faradays’ Law, obtain a general expression relating the \( \mathbf{E} \) and \( \mathbf{B} \) fields in vacuum (leave it vector form). Use this to relate the magnitudes of \( \mathbf{E} \) and \( \mathbf{B} \).

\[
\nabla \times \mathbf{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ ik_x & ik_y & ik_z \\ k_x & k_y & k_z \end{vmatrix} = i \mathbf{k} \times \mathbf{E}
\]
So, \( \nabla \times \mathbf{E} = i(\mathbf{k} \times \mathbf{E}_0)e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} = -\mathbf{B} \)

Integrating gives, \( \mathbf{B} = \frac{1}{\omega} \int_{k_0}^{k} i(\mathbf{k} \times \mathbf{E}_0)e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}\, \text{d}k = \left( \frac{1}{c} \right) (\mathbf{k} \times \mathbf{E}) \)

Hence \( |\mathbf{E}| = c|\mathbf{B}| \)

(xi) By considering a point of constant phase, where phase is defined as \( \phi = (\mathbf{k} \cdot \mathbf{r} - \omega t) \), show that the phase velocity (the rate at which this point of constant phase moves) is given by \( v_\phi = c \).

\[ \phi = (\mathbf{k} \cdot \mathbf{r} - \omega t) = \text{const.} \rightarrow \frac{\partial \phi}{\partial t} = (\mathbf{k} \cdot \frac{\partial \mathbf{r}}{\partial t} - \omega) = 0. \rightarrow \quad v_\phi = \frac{\partial r_\phi}{\partial t} = \frac{\omega}{\mathbf{k}} = c \]

(xii) Consider two solutions with slightly different values of \( (\omega, k) \);

\[ \mathbf{E}_1 = \mathbf{E}_0 e^{i(k_1 \cdot \mathbf{r} - \omega_1 t)} \quad \text{and} \quad \mathbf{E}_2 = \mathbf{E}_0 e^{i(k_2 \cdot \mathbf{r} - \omega_2 t)} \]

where \( k_1 = k + \Delta k \) and \( k_2 = k - \Delta k \) and where \( \omega_1 = \omega + \Delta \omega \) and \( \omega_2 = \omega - \Delta \omega \).

Create a solution \( \mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 \), and show that it can be written as,

\[ \mathbf{E}(\mathbf{r}, t) = 2 \mathbf{E}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \cos(\Delta \mathbf{k} \cdot \mathbf{r} - \Delta \omega t) \]

By considering the limit \( (\Delta \omega, \Delta k) \to 0 \), show that the group velocity (the velocity of the “beat” for a continuous range of \( (\omega, k) \)), is

\[ v_g = \lim_{(\Delta \omega, \Delta k) \to 0} \left( \frac{\Delta \omega}{\Delta k} \right) = \frac{\partial \omega}{\partial k} = c \]

\[ \mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \mathbf{E}_0 e^{i(k_1 \cdot \mathbf{r} - \omega_1 t)} + \mathbf{E}_0 e^{i(k_1 \cdot \mathbf{r} - \omega_1 t)} = \mathbf{E}_0 \left( e^{i((k+\Delta k) \cdot \mathbf{r} - (\omega+\Delta \omega)t)} + e^{i((k-\Delta k) \cdot \mathbf{r} - (\omega-\Delta \omega)t)} \right) \]

\[ = \mathbf{E}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \left( e^{i(\Delta \mathbf{k} \cdot \mathbf{r} - \Delta \omega t)} + e^{-i(\Delta \mathbf{k} \cdot \mathbf{r} - \Delta \omega t)} \right) \]

\[ = 2\mathbf{E}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \cos(\Delta \mathbf{k} \cdot \mathbf{r} - \Delta \omega t) \]

One can see a beat that propagates with \( \Delta \mathbf{k} \cdot \frac{\partial \mathbf{r}_{\text{beat}}}{\partial t} - \Delta \omega = 0 \rightarrow v_{\text{beat}} = \frac{\Delta \omega}{\Delta k} \)

In the limit \( (\Delta \omega, \Delta k) \to 0 \), \( v_g = \frac{\partial \omega}{\partial k} \)
Classwork V
Why is the sky blue?

**Information needed for this Classwork**

(i) Give an expression for the electric field at a distance \( r \) from an electron performing simple harmonic motion

\[ z = z_0 \cos \omega t \], where \( r \gg z_0 \).

\( z = z_0 \cos \omega t \) but need to consider retarded time at distance \( r \),

i.e. \([z] = z_0 \cos (t - r/c)\); \([\dot{z}] = -z_0 \omega \sin (t - r/c)\); \([\ddot{z}] = -z_0 \omega^2 \cos (t - r/c)\)

So, \( E = \frac{q [\ddot{z}]}{4\pi \varepsilon_0 r^2} \sin \theta \hat{\theta} = \frac{e}{4\pi \varepsilon_0} \frac{z_0 \omega^2 \cos (t - r/c)}{r^2} \sin \theta \hat{\theta} \)

(ii) From Faraday’s law, show that the corresponding magnetic field is given by:

\[ B = \frac{ez_0 \omega^2 \cos (t - r/c)}{4\pi \varepsilon_0} \frac{1}{r^3} \sin \theta \hat{\phi} \]

(Hint: I suggest leaving the curl in determinant form until you’ve figured out which components of \( E \) are present and what they are functions of).

Faraday’s Law, \( \nabla \times \mathbf{E} = -\mathbf{B} \)

But \( \mathbf{E} \) has only \( \theta \)-component and is dependent only on \( r \) and \( \theta \). So,

\[
\nabla \times \mathbf{E} = \begin{vmatrix}
\frac{1}{r^2 \sin \theta} & \frac{1}{r \sin \theta} & \frac{1}{r} \\
\frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\
E_r & rE_\theta & r \sin \theta E_\phi \\
\end{vmatrix} = \begin{vmatrix}
\frac{1}{r^2 \sin \theta} & \frac{1}{r \sin \theta} & \frac{1}{r} \\
\frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & 0 \\
0 & 0 & 0 \\
\end{vmatrix}
\]

So only one term remains:

\[
\nabla \times \mathbf{E} = \frac{1}{r} \frac{\partial}{\partial \phi} \left( rE_\theta \right) = \frac{1}{r} \frac{\partial}{\partial \phi} \left( \frac{e f}{4\pi \varepsilon_0} \frac{z_0 \omega^2 \cos (t - r/c)}{r^2} \sin \theta \right)
\]

\[
= \frac{e}{r} \frac{\omega}{c} \left( \frac{z_0 \omega^2 \sin \omega (t - r/c)}{4\pi \varepsilon_0} \sin \theta \right) = -\mathbf{B}
\]

Integrating gives,

\[ \mathbf{B} = \frac{e}{4\pi \varepsilon_0} \frac{z_0 \omega^2 \cos (t - r/c)}{r^3} \sin \theta \hat{\phi} \]

(note field must decay to zero at \( \infty \) so no constant of integration).

(iii) Hence write down an expression for the Poynting vector. (This is the rate at which energy is radiated by the particle).
\[ \mathbf{N} = \frac{1}{\mu_0 E} \times \mathbf{B} = \frac{1}{\mu_0} \frac{e^2}{16\pi \epsilon_0^2} \frac{z_0^2 \omega^4 \cos^2 \omega(t - r/c)}{r^2 c^3} \sin^2 \theta (\hat{\theta} \times \hat{\phi}) \]

where we cancelled a \( \mu_0 \epsilon_0 c^2 \), so,

\[ \mathbf{N} = \frac{e^2}{16\pi \epsilon_0^2} \frac{z_0^2 \omega^4}{r^2 c^3} \sin^2 \theta \cos^2 (\omega(t - r/c)) \hat{\mathbf{r}} \]

(iv) First take the time average and then integrate the energy flux over a complete sphere of radius \( r \) to show that the total mean radiated power is given by:

\[ P = \frac{e^2 z_0^2 \omega^4}{12 \pi \epsilon_0 c^3} \]

\[ \langle \mathbf{N} \rangle = \frac{e^2}{16\pi \epsilon_0^2} \frac{z_0^2 \omega^4}{r^2 c^3} \sin^2 \theta \langle \cos^2 (\omega(t - r/c)) \rangle \hat{\mathbf{r}} = \frac{e^2}{16\pi \epsilon_0^2} \frac{z_0^2 \omega^4}{r^2 c^3} \sin^2 \theta \frac{1}{2} \hat{\mathbf{r}} \]

Now integrating over a sphere of radius \( r \),

\[ P = \oint \langle \mathbf{N} \rangle \hat{\mathbf{r}} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^\pi \frac{e^2}{2 16\pi \epsilon_0^2} \frac{z_0^2 \omega^4}{r^2 c^3} \sin^2 \theta \rho \sin \theta \rho \sin \theta \rho d\theta d\phi \]

\[ = 2\pi \frac{e^2}{2 16\pi \epsilon_0^2} \frac{z_0^2 \omega^4}{c^3} \int_0^\pi \sin^3 \theta d\theta \]

\[ \int_0^\pi \sin^3 \theta d\theta = \int_0^\pi \sin \theta (1 - \cos^2 \theta) d\theta = [\cos \theta + \frac{1}{3} \cos^3 \theta]_0^\pi = \frac{4}{3} \]

So finally,

\[ P = \frac{e^2}{164 \pi \epsilon_0^2} \frac{z_0^2 \omega^4}{c^3} \frac{4}{3} = \frac{e^2 z_0^2 \omega^4}{12 \pi \epsilon_0 c^3} \]

(v) An electron is bound to an atom at the origin with a linear restoring force \( \mathbf{F} = -K_s \mathbf{z} \), where \( K_s \) is the spring constant. It is perturbed by a light wave polarised in the \( \mathbf{z} \) direction: \( \mathbf{E} = E_0 \mathbf{e}_z \mathbf{e}_z = E_0 \cos(kx - \omega t) \mathbf{e}_z \). Write down the equation of motion. (You may ignore the contribution of the \( \mathbf{B} \)-field.) Hence show that the oscillation amplitude varies as:

\[ z_0 = \frac{eE_0}{m(\omega^2 - \omega_0^2)} \]

giving an appropriate expression for \( \omega_0 \).

Try soln. \( z = z_0 \cos \omega t, \dot{z} = -z_0 \omega^2 \cos \omega t \) and noting that at the origin \( x = 0 \)

\[ -m z_0 \omega^2 \cos \omega t = -K_s z_0 \cos \omega t - eE_0 \cos (-\omega t) \]

\[ z_0 (\omega^2 - \omega_0^2) = \frac{eE_0}{m} \rightarrow z_0 = \frac{eE_0}{m(\omega^2 - \omega_0^2)} \]

where \( \omega_0^2 = K_s/m \).
(vi) We can take the two limits of this expression. Give an expression for \( z_0 \) in the limit \( \omega \gg \omega_0 \). Hence show that the ratio of power radiated to the incident energy (Poynting) flux, is given by the (Thomson scattering) cross section, \( \sigma_T = \frac{8 \pi}{3} r_e^2 \) where \( r_e \) is called the classical radius of the electron defined by \( r_e = \frac{e^2}{4 \pi \epsilon_0 mc^2} \).

(Remember the Poyting vector of the incident beam will be \( \langle N \rangle = \frac{1}{2} \epsilon_0 E_0^2 c \). )

Note that this cross-section is independent of the frequency. Calculate \( \sigma_T \) (numerically). It is small, but can be used for example in fusion plasmas to calculate its temperature (from the Doppler shift of the scattered beam).

\[ \omega \gg \omega_0, \quad z_0 = \frac{eE_0}{m \omega^2} \]

So \( P = \frac{e^2 z_0^2 \omega^4}{12 \pi \epsilon_0 c^3} = \frac{e^2 \omega^2}{12 \pi \epsilon_0 c^3 m^2 \omega^2} = \frac{8 \pi}{3} \left( \frac{e^2}{4 \pi \epsilon_0 mc^2} \right)^2 \frac{1}{2} \epsilon_0 E_0^2 c = \frac{8 \pi}{3} r_e^2 \langle N_{in} \rangle \)

\[ \langle N_{in} \rangle = \frac{1}{2} \epsilon_0 E_0^2 c \] is the incoming Poynting flux, (the half-comes from the time average).

But since the power radiated is \( P = \langle N_{in} \rangle \sigma_T \)

(\( \sigma_T \) - the cross-section is the effective “area” of the electron.).

\[ \sigma_T = \frac{P}{(\frac{1}{2} \epsilon_0 E_0^2 c)} = \frac{8 \pi}{3} r_e^2, \]

\( r_e = 2.8 \times 10^{-15} \text{ m}^{-1} \), so \( \sigma_T = 6.67 \times 10^{-29} \text{ m}^{-2} \)

(vii) In the opposing limit, \( \omega \ll \omega_0 \), calculate once more the Rayleigh scattering cross-section. How does it vary with frequency?

This scattering is the case where the electron is still more influenced by its parent atom, and is what happens when light passes through the atmosphere. Can you explain now why the sky is blue?

Now \( \omega \ll \omega_0, \quad z_0 = -\frac{eE_0}{m \omega^2} = -\frac{eE_0}{m \omega^2} \left( \frac{\omega^2}{\omega_0^2} \right) \)

Calculation is the same except for the bit in brackets multiplied on, (\( \omega^4 \) does not cancel now), so

\[ \sigma_R = \frac{8}{3} \pi r_e^2 (\omega^4/\omega_0^4). \]

Hence scattering efficiency is greatly enhanced for shorter wavelengths, for example for blue compared to red light \( \sigma_{blue}/\sigma_{red} \propto (\lambda_{red}^4/\lambda_{blue}^4) \approx (700^4/400^4) \approx 10 \)

So blue from sunlight scatters from air molecules (and also from airborne aerosols and dust) much more strongly than the red, meaning that when not directly looking at the sun (so you are only looking at scattered light), the sky looks blue.

But then why is it not purple? 3 reasons (may be more!): 1) Blackbody spectrum of sun is rapidly falling at high \( \omega \), 2) Response of eye is rapidly falling above the blue, 3) Absorption in the atmosphere starts to be a problem at higher frequencies.
(i) Consider the arrangement shown in figure 1 where a dielectric of permittivity $\epsilon_1$ is separated from a dielectric of permittivity $\epsilon_3$, by a thin dielectric layer of permittivity $\epsilon_2$ of thickness $d$. An electromagnetic wave is incident normally on boundary 1 from the left. The waves shown in the figure can be represented by:

\[
E_1 = A_1 e^{i(\omega_1 t - k_1 x)}; \quad E'_1 = A'_1 e^{i(\omega_1 t + k_1 x)}; \quad (1)
\]
\[
E_2 = A_2 e^{i(\omega_2 t - k_2 x)}; \quad E'_2 = A'_2 e^{i(\omega_2 t + k_2 x)}; \quad (2)
\]
\[
E_3 = A_3 e^{i(\omega_3 t - k_3 x)} \quad (3)
\]

What is the condition for the angular frequency $\omega$ in each region, explaining why? Also give an expression for $k$ in each medium in terms of $\eta$. How are the magnetic components of the waves related to the electric components given in terms of the respective refractive indices (no need to write them out for each!).

![Figure 1: EM wave incident on a dielectric with a surface coating.](image)

\[\omega_1 = \omega_2 = \omega_3\] to ensure phase matching at all times.

\[k_1 = \eta_1 \omega / c, \quad k_2 = \eta_3 \omega / c, \quad k_3 = \eta_3 \omega / c\]

\[B_1 = \eta_1 E_1 / c, \quad B_2 = \eta_2 E_2 / c\] etc. but note directions changes for reflected waves.

(ii) Taking boundary 2 as $x = 0$, write down continuity equations for $E_\parallel$ and $B_\parallel$. Hence show that the amplitudes of the transmitted and reflected waves at this boundary are given by:

\[A_3 = \frac{2\eta_2}{\eta_2 + \eta_3} A_2; \quad A'_2 = \frac{\eta_2 - \eta_3}{\eta_2 + \eta_3} A_2.\]

Continuity of $E_\parallel$, \[E_2 + E'_2 = E_3\] \[\text{(A)}\]

Continuity of $B_\parallel$, \[B_2 - B'_2 = B_3\]

Rewriting in terms of $E$, \[\eta_2 E_2 - \eta_2 E'_2 = \eta_3 E_3\] \[\text{(B)}\]

Add $\eta_2 \times (\text{A})$ to (B): \[2\eta_2 E_2 = (\eta_2 + \eta_3) E_3\]

We can take $t = 0$ and since $z = 0$, \[A_3 = \frac{2\eta_2}{\eta_2 + \eta_3} A_2\]
Subtract (B) from \( \eta_2 \times (A) \);
\[
2 \eta_2 E_2' = (\eta_2 - \eta_3) E_3
\]
\[
\rightarrow E_2' = \frac{(\eta_2 - \eta_3)}{2 \eta_2} \frac{2 \eta_2}{\eta_2 + \eta_3} E_2
\]  
\[
\rightarrow A_2' = \frac{(\eta_2 - \eta_3)}{\eta_2 + \eta_3} A_2
\]

(iii) Now we consider boundary 1. Write out the boundary conditions once more at this boundary, and eliminate the incoming beam \( E_1 \) to obtain;
\[
2 \eta_1 E_1' = (\eta_1 - \eta_2) E_2 + (\eta_1 + \eta_2) E_2'.
\]
Continuity of \( E_\| \),
\[
E_1 + E_1' = E_2 + E_2'
\]  
Continuity of \( B_\| \),
\[
B_1 - B_1' = B_2 - B_2'
\]
Rewriting in terms of \( E \),
\[
\eta_1 E_1 - \eta_1 E_1' = \eta_2 E_2 - \eta_2 E_2'
\]  
Subtract (D) from \( \eta_1 \times (C) \);
\[
2 \eta_1 E_1' = (\eta_1 - \eta_2) E_2 + (\eta_1 + \eta_2) E_2'.
\]

(iv) OK, now to plug in the phase dependent terms from eq.(2) above, and the expression for the amplitude of \( E_2 \) found in part (ii). Show that if the thickness of the layer is given by \( d = \lambda/4 \), where \( \lambda = 2\pi/k_2 \), then the reflected wave \( E_1' \) can be extinguished, provided \( \epsilon_2^2 = \epsilon_1 \epsilon_3 \).
\[
k_2 d = (\lambda/4) \cdot (2\pi/\lambda) = \pi/2
\]
\[
2 \eta_1 E_1' = (\eta_1 - \eta_2) A_2 e^{-i\pi/2} + (\eta_1 + \eta_2) A_2' e^{i\pi/2}
\]
\[
= (\eta_1 - \eta_2) A_2 e^{-i\pi/2} + (\eta_1 + \eta_2) \frac{(\eta_2 - \eta_3)}{\eta_2 + \eta_3} A_2 e^{i\pi/2}
\]
\[
= \frac{A_2 e^{-i\pi/2}}{\eta_2 + \eta_3} \left[ (\eta_1 - \eta_2)(\eta_2 + \eta_3) - (\eta_1 + \eta_2)(\eta_2 - \eta_3) \right]
\]
Where we used \( e^{i\pi} = -1 \), [ ] bracket must be 0 for \( E_1' = 0 \). Expanding out:
\[
(\eta_1 \eta_2 + \eta_1 \eta_3 - \eta_2^2 - \eta_3^2) - (\eta_1 \eta_2 - \eta_1 \eta_3 + \eta_2^2 - \eta_3^2) = 0
\]
\[
\rightarrow \eta_1 \eta_3 = \eta_2^2
\]
or \( \sqrt{\epsilon_1 \epsilon_3} = \epsilon_2 \) or also \( \epsilon_1 \epsilon_3 = \epsilon_2^2 \)

(iv) Without calculation, what would you expect the corresponding reflection coefficient to be if \( \epsilon_2^2 = \epsilon_1 \epsilon_3 \) but the layer is now \( d = \lambda/2 \)? What would happen if you used multiple layers which satisfied these conditions?

To cancel perfectly the first reflection, the wave passing through region 2, must have the same amplitude but be exactly \( \pi \) out of phase, since it travels \( 2 \times (\lambda/4) \). If it is passes through \( d = \lambda/2 \) it returns with a constructive phase, and so the transmission is minimised. The reflection is twice the single layer value. Multiple layers like this, each with a graded \( \eta \) to reduce the transmission at each layer, can be used together to make a dielectric mirror.

(v) A single dielectric layer as in the diagram is commonly used on camera lenses to reduce the light reflected from its surface. Why would it be advantageous? Calculate the thickness of the layer and its refractive index for green light (\( \lambda = 532 \text{ nm} \)). Why does this anti-blooming layer usually have a slight purple hue?
\[
d = \lambda/4 = 530/4 = 133 \text{ nm} \quad \text{and} \quad \eta = \sqrt{\eta_\text{air}} = \sqrt{1.5} = 1.22
\]
The lenses are often multi-element, and the anti-reflection coatings prevent ghost images (from unwanted reflections). At an angle, the path of the beam that passes through layer 2 is not quite \( \lambda/4 \) and the anti-reflection works less well, especially for either the red or purple ends of the spectrum, the difference from \( \lambda/4 \) is greater.
Reflection from the ionosphere

1 Derived from a past exam question: This question demonstrates how the ionosphere, which is a tenuous plasma, can increase the range of some radio transmissions.

(i) The general equation of motion for an electron (mass \( m \)) moving with a velocity \( \mathbf{v} \) in a plasma is given by:

\[
\frac{m}{\tau_c} \frac{d\mathbf{v}}{dt} = -e\mathbf{E} - \frac{m\mathbf{v}}{\tau_c},
\]

where \( \tau_c \) is the characteristic time on which the electron undergoes collisions; and \( \mathbf{E} \) is the electric field. Explain the physical significance of each of the terms, as well as the reason we can ignore the effects of the magnetic field on the motion of the electron.

The LHS is mass times acceleration.

The first term on the RHS comes from the Lorentz force \( \mathbf{F} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \). But since the magnetic field contribution is of magnitude \( F_B \approx evB = e(v/c)E = (v/c)F_E \), it can be seen that it is \( v/c \) times the electric field contribution, which is small since \( v \ll c \) typically.

The second term on the right is a drag term due to collisions which occur with a characteristic time \( \tau_c \).

(ii) (a) Assuming that at any position, the time dependence of \( \mathbf{v} \) and \( \mathbf{E} \) is described by \( e^{-i\omega t} \), show that the current density, \( \mathbf{j} \), satisfies:

\[
\mathbf{j} = \frac{ne^2\mathbf{E}}{m} \frac{1}{\tau_c - i\omega}
\]

where \( n \) is the number density of the free electrons.

\[\mathbf{v} = \mathbf{v}_0 e^{-i\omega t} \rightarrow \frac{d\mathbf{v}}{dt} = m(-i\omega)\mathbf{v}_0 e^{-i\omega t} = -im\omega \mathbf{v}\]

Substitute in above, \( -im\omega \mathbf{v} = -e\mathbf{E} - \frac{m\mathbf{v}}{\tau_c} \rightarrow m\mathbf{v} \left(\frac{1}{\tau_c} - i\omega\right) = -e\mathbf{E}\]

but \( \mathbf{j} = -ne \mathbf{v} \) (for electrons) = \( \frac{ne^2\mathbf{E}}{m} \frac{1}{\tau_c - i\omega} \)

(b) Write down the expressions for \( \mathbf{j} \) for an ohmic conductor, and for a collisionless plasma.

In a conductor \( \frac{1}{\tau_c} \gg \omega \)

\[\mathbf{j} = \frac{ne^2\mathbf{E} \tau_c}{m}\]

In a collisionless plasma \( \frac{1}{\tau_c} \rightarrow 0 \)

\[\mathbf{j} = \frac{nie^2\mathbf{E}}{m\omega}\]

(c) In the latter case, what implications does the result have for the mean power dissipated in the plasma?

Power dissipated \( P = \mathbf{j} \cdot \mathbf{E} \), but since \( \mathbf{j} \) is \( \pi/2 \) out of phase with \( \mathbf{E} \). So if \( E = E_0 \cos(kz - \omega t) \) for example, then \( j = j_0 \sin(kz - \omega t) \)

then \( \langle \mathbf{j} \cdot \mathbf{E} \rangle \propto \langle \cos(kz - \omega t) \sin(kz - \omega t) \rangle = 0 \).

\(^1\)starred questions are my additions!
(iii)* Calculate the refractive index $\eta$ of a collisionless plasma of electron density $n$. What happens to $\eta$ when $\omega < \omega_p$.

$$\omega^2 = \omega_p^2 + c^2k^2$$

dividing by $\omega^2$ and rearranging: $c^2k^2/\omega^2 = 1 - \omega_p^2/\omega^2$

inverting LHS, (and multiplying by $c^2$): $\omega^2/k^2 = c^2(1 - \omega_p^2/\omega^2)^{-1}$

Taking square root gives: $\omega/k = v_{ph} = c/\eta = c(1 - \omega_p^2/\omega^2)^{-1/2}$

So $\eta = (1 - \omega_p^2/\omega^2)^{1/2}$

For $\omega < \omega_p$, $\eta$ comes imaginary as does $k = \eta\omega/c$, so propagation stops.

(iv)* We showed (in lectures) that an em wave is reflected off a plasma boundary if it satisfies the condition $\omega < \omega_p/\cos \theta$. Hence for given values of $\omega$ and $\omega_p$, show that the critical angle, $\cos \theta_c = \omega_p/\omega$, is the same as given by the condition for total internal reflection.

For total internal reflection,

$$\sin \theta_i = (\eta_1/\eta_2)\sin \theta_i = (1/\eta)\sin \theta_i > 1$$

where $\eta$ is the plasma refractive index.

Hence $\sin \theta_c = \eta$  $\rightarrow$  $\sin^2 \theta_c = \eta^2 = (1 - \omega_p^2/\omega^2)$

rearranging, $\omega_p^2/\omega^2 = 1 - \sin^2 \theta_c = \cos^2 \theta_c$.

So, $\cos \theta_c = \omega_p/\omega$, as before.

(v) Convince yourself, by simple geometrical considerations, that long distance communication can be extended via ionospheric reflection at an angle of reflection of $\sin \theta = R/(R + h)$, where $R$ is the radius of the earth, and $h$ is the height of the ionosphere above the surface.

(It should help to draw this configuration, with the atmosphere a little exaggerated!).

Maximum angle when radio waves graze along both outgoing and incoming surface.

From diagram, $\sin \theta = R/(R + h)$ as requested.

(vi) The Earth’s ionospheric electron density peaks at a value of $10^{12}$ m$^{-3}$ at a height, $h = 300$ km.

Given that the $R = $ radius of the Earth = 6400 km, find the value of the highest frequency $\nu_{cr}$, that can be used at this extreme angle. Also find how much the range be extended by?

(You may assume that below 300 km, the effects of the ionosphere can be ignored and $\eta = 1$).

From given figures, $\theta = \sin^{-1}(6400/6700) = 72.8^\circ$. $\omega_p = \sqrt{10^{12}\frac{e^2}{\epsilon_0 m_e}} = 5.6 \times 10^7$ rads$^{-1}$

So $\omega_{cr} = \omega_p/\cos(72.8) = 1.9 \times 10^8$ rads$^{-1}$

and $\nu_{cr} = \omega_{cr}/2\pi = 30.3$ MHz ($\lambda \approx 10$ m, so only useful for long wavelengths).

Range extended by the arc length in diagram = $2R\theta = 2 \cdot 6400 \cdot \pi((90 - 72.8)/180) = 3850$ km.

NB at night time, when the ionosphere is not “pushed” by the solar wind, it tends to go higher, which is why foreign transmissions can seem clearer at night - that is if you listen to long wave!