Lecture #1. Particles & Fields - Classical Theory.

(i) **HEALTH WARNING:** Notes are an aid to your memory not a textbook; casual mistakes are inevitable.

(ii) **Elementary E&M:** Coulomb's Law. Stationary charge charge $q_1$ at $r_1$ produces a force $F_1$ on another stationary charge $q_2$ at $r_2$.

\[
F_{12} = \frac{q_1 q_2}{4\pi \varepsilon_0 r_{12}^2}
\]

\[
|r_{12}| = |r_2 - r_1|
\]

\[
F_{12} = \frac{q_1 q_2}{4\pi \varepsilon_0 r_{12}^2} \quad \text{"Inverse square law"}
\]

(iii) At first sight it is "action at a distance" - like Newtonian gravity.

(iv) Well not actually - we introduced the concept of a field.

**Electric field of stationary charge $q_1$ at $r$**

\[
E_{1e}(r) = \frac{q_1 (r - r_1)}{4\pi \varepsilon_0 |r - r_1|^3}
\]

\[
E_{12} = q_2 E_1(r_2)
\]

- Charge carries an invisible "force field" around it - the **ELECTRIC FIELD**.
- Electric field pushes $q_2$ locally.

(v) Now what happens if I move $q_1$ - jiggle it? Does the electric field move with $q_1$, rigidly?
(vi) Actually no! It takes a time \( r_{12}/c \) for the force on \( q_2 \) to change after I jiggle \( q_1 \). THERE IS NO INSTANTANEOUS ACTION AT A DISTANCE.

(vii) Although it is possible to find \( E_{12} \) in terms of the position and motion of \( q_1 \) at a previous time - THE RETARDED TIME \( t = r_{12}/c \), we think of the interaction as:

- MOTION OF CHARGE \( q_1 \), Emitted
- Energy and momentum in ELECTROMAGNETIC FIELDS, Propagation time \( r_{12}/c \)
- Absorbed

... photon jiggles electron on return.


CLÁSICO ELETRÔMAGNÉTISMO

All the equations - you don't know this yet we will build this up.

\( \text{sources} \)

- Charge density \( \rho (x,t) \) = charge per unit volume = \( q_1 \delta(x-x_i) \)
- Point charge.

\( \text{current density} \) = \( J(x,t) \) = "current per unit area" = \( q_1 \delta(x-x_i) \)

\( \text{fields} \)

- Maxwell's Equations
  \[ \begin{align*}
  & 1 \quad \nabla \cdot E = \frac{\rho}{\varepsilon_0} \quad , \quad 2 \quad \nabla \cdot B = 0 \\
  & 3 \quad \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} = \nabla \times B - \varepsilon_0 \frac{\partial J}{\partial t} \quad , \quad 4 \quad \frac{\partial B}{\partial t} = -\nabla \times E
  \end{align*} \]

\( \text{forces} \)

- Newton's laws + Lorentz force + Relativity
  \[ \frac{dp}{dt} = q(E + \gamma v \times B) \quad , \quad \frac{dr}{dt} = v \]

\( \text{momentum} \) \( p = \frac{mv}{\sqrt{1 - v^2/c^2}} \)

\( \text{location} \) \( r = \text{particle position} \)
Lecture #2. "Deriving" Maxwell's Equations: Part I

(last time) - Action at a distance?
(i) In the next 3 lectures we "derive" Maxwell's Equations - well almost.

\[
\text{INTEGRAL FORMS OF MAXWELL'S EQUATIONS} \quad \leftrightarrow \quad \text{DIFFERENTIAL FORMS OF MAXWELL'S EQUATIONS}
\]

Today

1. \( \nabla \cdot E = \rho / \varepsilon_0 \)
2. \( \nabla \cdot B = 0 \)

For any vector field: \( A(\mathbf{r}) \)

\[
\int (\nabla \cdot A) \, d^3x = \int_{\partial V} A \cdot d\mathbf{s} = \text{Flux of } A \text{ out of } V \text{ through } \partial V
\]

Volume \( V \) is enclosed inside closed volume \( S \)

I always think loosely of \( \nabla \cdot A \) as the "flux of \( A \) out of a point, per unit-volume" then summing the flux out of every point gives the L.H.S. and the flux out of the whole surface - the R.H.S.

(iii) Gauss's Law: you learned this from Prof. Schwartz.

\[
\int_S E \cdot d\mathbf{s} = \frac{\text{[CHARGE ENCLOSED BY } S\text{]}}{\varepsilon_0} = \frac{Q}{\varepsilon_0}
\]

\[
= \frac{1}{\varepsilon_0} \int d^3 \rho(x)
\]

\( \rho(x) \) = charge density.

But using Gauss's theorem we have \( \int_S E \cdot d\mathbf{s} = \int_V \nabla \cdot E \)

\[
\Rightarrow \int_V \left[ \nabla \cdot E - \frac{\rho}{\varepsilon_0} \right] d^3r = 0
\]

\( \nabla \cdot E = \frac{\rho}{\varepsilon_0} \)

MUST BE TRUE FOR ALL VOLUMES.

THE INTEGRAND MUST VANISH EVERYWHERE.

1st Maxwell Equation.

Proved integral => differential, reverse?
(iv) \( \mathbf{B} \) field:
\[
\oint_S \mathbf{B} \cdot d\mathbf{s} = 0
\]
no magnetic charges.

Using Gauss's Theorem
\[
\Rightarrow \int_V \left( \nabla \cdot \mathbf{B} \right) d^3x = 0
\]
for any volume.
\[
\Rightarrow \nabla \cdot \mathbf{B} = 0
\]
2nd Maxwell Equation.

(v) Field Lines
We visualize fields using field lines.

a) Freeze \( \mathbf{E}(r,t) \) in time.
b) Start at \( \mathbf{r}_0 \) and take a step \( \delta \mathbf{r} \) in the direction of \( \mathbf{E} \) (how do you know this?)
c) Now take step in the new direction of \( \mathbf{E} \).
   repeat etc. 

d) In the limit of very tiny/infinitesimal steps we trace out a field line.

(vi) Field lines do not indicate the strength of the field—just the direction. We cannot simply trace where a field line moves to the next moment in time.

(vii) a) \( \mathbf{E} \) field lines begin on the (positive) charges and end on the (negative) charges.

b) \( \mathbf{B} \) field lines never end or begin—one field line in universe?

\( \mathbf{E} \) field

\( \mathbf{B} \) field

what about points where \( \mathbf{B} = 0 \)?
2nd Year EMT I Lecture #3: Maxwell's Equations: Part II

Last time: \( \mathbf{\nabla} \cdot \mathbf{E} = \frac{q}{\varepsilon_0} \leftrightarrow \mathbf{\nabla} \cdot \mathbf{B} = 0 \) and \( \oint \mathbf{E} \cdot d\mathbf{s} = 0 \Rightarrow \mathbf{\nabla} \cdot \mathbf{B} = 0 \)

Today: 2 more equations

**Amperes Law:** \( \mathbf{\nabla} \times \mathbf{B} = \mathbf{\mu}_0 \mathbf{J} \)

**Faraday's Law:** \( \frac{d\mathbf{B}}{dt} = -\mathbf{\nabla} \times \mathbf{E} \)

**Not quite all of Maxwell @ lecture #1**

**Maxwell @**

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**Key Mathematical Theorem: Stokes's Theorem**

For any vector field \( \mathbf{A} \)

\[
\oint_{l} \mathbf{A} \cdot d\mathbf{l} = \int_{\mathbf{S}} \mathbf{\nabla} \times \mathbf{A} \cdot d\mathbf{S}
\]

For any surface \( \mathbf{S} \) bounded by a curve \( l \) (loop)

\( \mathbf{\nabla} \times \mathbf{A} = \text{curl} \mathbf{A} = \mathbf{\nabla} \times \mathbf{A} \cdot d\mathbf{s} \) amount \( A \) "circulates around \( d\mathbf{s} \)".

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**Current Density**

For a change in charge \( \Delta q \) in \( \Delta t \):

\[
\mathbf{J} = \text{flux of charge per unit area}.
\]

---

**Amperes Law**

\[
\oint_{l} \mathbf{B} \cdot d\mathbf{l} = \mathbf{\mu}_0 I \]

But \( I = \int_{S} \mathbf{J} \cdot d\mathbf{s} \)

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**Not quite all of Maxwell @**

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**Maxwell @**

not quite next time!
(v) Faraday's Law:
\[ \frac{d\Phi}{dt} = V = - \oint_E \cdot dl = - \mathcal{E} \]
where \( \Phi = \int_B \cdot ds \) = Magnetic Flux Through Surface

\[ \frac{d\Phi}{dt} = \frac{d}{dt} \left( \int_B \cdot ds \right) = \int \frac{dB}{dt} \cdot ds = - \int E \cdot dl = - \int (\nabla \times E) \cdot ds \]

\[ \implies \left\{ \frac{dB}{dt} + \nabla \times E \right\} ds = 0 \]

So, for all surfaces.

\[ \frac{dB}{dt} = - \nabla \times E \]

Faraday's Law

Maxwell (4)

(vi) Evolving \( B(r,t) \) — we can formally integrate Faraday's Law.

\[ B(r,t) - B(r,0) = - \int_0^t [\nabla \times E(s,t)] dt \]

so why do we need another equation for \( \mathcal{B} \)?

\( B \cdot \mathcal{B} = 0 \)

(vii) Let's use Faraday's law to calculate the evolution of \( \nabla \mathcal{B} \)

\[ \nabla \cdot (\frac{dB}{dt}) = \frac{d}{dt} (\nabla \mathcal{B}) = - \nabla \cdot (\nabla \times E) = 0 \]

Since for all vector fields \( \mathcal{A} \)

\[ \nabla \cdot \nabla \mathcal{A} = 0 \]

So equations for \( \mathcal{B} \) are consistent — if \( \nabla \mathcal{B} = 0 \) at \( t = 0 \) then evolving via Faraday's Law will ensure \( \nabla \mathcal{B} = 0 \) at all subsequent times.

(viii) How do we evolve \( E \) next lecture.
Lecture #4: Maxwell's Equations; Part III - Finish.

Last time:

\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \iff \oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I \quad - \text{AMPÈRE'S LAW} \]

\[ \frac{d\mathbf{B}}{dt} = -\nabla \times \mathbf{E} \iff \frac{d\mathbf{E}}{dt} = -\mathbf{E} = -\oint \mathbf{E} \cdot d\mathbf{s} \quad - \text{FARADAY'S LAW} \]

(i) Today: Ampère's law must be corrected to conserve charge.

We assume charge is always conserved - A law of nature.

(ii) Consider charge in a volume \( V \) enclosed by surface \( S \).

\[ Q = \text{charge in } V = \int d^3r \rho (x,t) \]

\[ \text{charge leaving through } ds \text{ in time } dt = \mathbf{J} \cdot ds \ dt \]

\[ \int d^3r \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} \right] = 0 \quad \text{for all volumes} \]

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \]

CHARGE CONSERVATION

(iii) But is this consistent with our equations, specifically \( \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \)

\[ \nabla \cdot \mathbf{J} = \frac{1}{\mu_0} \nabla \cdot (\nabla \times \mathbf{B}) \]

But \( \nabla \cdot \mathbf{B} = 0 \) for all \( B \)

\[ = 0 \]

So we cannot be right when \( \frac{\partial \rho}{\partial t} \neq 0 \)

(iv) We let \( \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mathbf{N} \) where \( \mathbf{N} \) is a term to be chosen to keep charge conserved.
(v) Then taking the divergence: \( \mu_0 \nabla \cdot J = -\nabla \cdot N = -\mu_0 \frac{\partial p}{\partial t} \)
and using charge conservation

(vi) Use \( \rho = \varepsilon_0 \nabla \cdot E \) Maxwell 1.

\[ \nabla \cdot (N - \varepsilon_0 \mu_0 \frac{\partial E}{\partial t}) = 0 \]

So we can choose \( N = \varepsilon_0 \mu_0 \frac{\partial E}{\partial t} \)

**Question:** Is this a unique solution?

(vii) From logic like this Maxwell conjectured that

\[ \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} = \nabla \times B - \mu_0 J \]

This is called the "displacement current"

(viii) **MAXWELL'S EQUATIONS**

\[ \nabla \cdot E = \frac{\rho}{\varepsilon_0} \quad 1 \]
\[ \nabla \cdot B = 0 \quad 2 \]

\[ \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} = \nabla \times B - \mu_0 J \quad 3 \]
\[ \frac{\partial B}{\partial t} = -\nabla \times E \quad 4 \]

Evolves \( E \).
Evolves \( B \).

Now consistent with:

\[ \frac{\partial p}{\partial t} + \nabla \cdot J = 0 \]

**CHARGE CONSERVATION.**
Lecture #5: "1D" Electromagnetism - Vacuum solutions.

Last time: Charge conservation \( \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \) and, \( \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{H} - \mathbf{M} \) (displacement current).

(8) Today simple 1D solutions - only \( z \) dependent.

(9) All vector components but only \( z \) variation \( \mathbf{E}(z,t), \mathbf{B}(z,t) \) fields.

\[
\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{\partial A_z}{\partial z}
\]

\[
\nabla \times \mathbf{A} = \left( \frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y} \right) \hat{x} + \left( \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) \hat{y} + \left( \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \right) \hat{z}
\]

\[
= -\frac{\partial A_y}{\partial z} \hat{x} + \frac{\partial A_x}{\partial z} \hat{y}
\]

(10) Now we write every (non-zero) component of Maxwell's Equations.

\[
\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} = 0 \Rightarrow \frac{\partial E_z}{\partial z} = 0 \quad 1 \quad \nabla \cdot \mathbf{B} = 0 \quad 2
\]

\[
\frac{\partial B_x}{\partial t} = \frac{\partial E_y}{\partial z} \quad 3a
\]

\[
\frac{\partial B_y}{\partial t} = -\frac{\partial E_x}{\partial z} \quad 3b
\]

\[
\mu_0 \varepsilon_0 \frac{\partial E_z}{\partial t} = \nabla \times \mathbf{B} \quad 4a
\]

\[
\mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t} = \nabla \times \mathbf{B} \quad 4b
\]

(11) Naturally splits into separate systems: for \( E_z \), 1; for \( B_z \), 2

for \( B_x \) and \( E_y \), 3a \& 4b: for \( E_x \) and \( B_y \)

(v) Boundary Conditions: \( E \parallel B \rightarrow 0 \) as \( z \rightarrow \pm \infty \).

Choose these as physically important.

(vi) From 1 \[
\frac{\partial E_z}{\partial z} = 0 \Rightarrow E_z = E_z(t)
\]

but \( E \rightarrow 0 \) as \( z \rightarrow \pm \infty \)

so \( E_z = 0 \) everywhere

clearly \( B_z = 0 \) too from 2

No longitudinal components without sources.

(vii) Only disturbances must be \underline{transverse} i.e. perpendicular to the direction of variation (z).

(viii) Only disturbances must be \underline{transverse} i.e. perpendicular to the direction of variation (z).
(vii) We treat pair \( E_x, B_y \) — the other pair is for your enjoyment!

This \textbf{Polarization} — called \textbf{Linearly Polarized} in \( X \) direction.

\[
\mu_0 \varepsilon_0 \frac{\partial E_x}{\partial t} = -\frac{\partial B_y}{\partial z} \quad \text{and} \quad \frac{\partial B_y}{\partial t} = -\frac{\partial E_x}{\partial z}
\]

\[
\mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2} = -\frac{\partial^2 B_y}{\partial z \partial t} = \frac{\partial^2 E_x}{\partial z^2} \Rightarrow \frac{\partial^2 E_x}{\partial t^2} = \frac{1}{\mu_0 \varepsilon_0} \frac{\partial^2 E_x}{\partial z^2} = c^2 \frac{\partial^2 E_x}{\partial z^2}
\]

\textbf{Wave Equation}

\[
c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}
\]

\textbf{Constants:}
- \( E_0 = \text{amplitude} \)
- \( k = \text{wave vector} \)
- \( \omega = \text{angular frequency} \)
- \( \phi = kz - \omega t = \text{phase} \)

\textbf{Revisiting:}

(ix) "Guess" \textbf{Traveling wave solution}.

\[
E_x = E_0 \cos \left( k z - \omega t \right)
\]

After time \( t_1 \), wave crest which was at \( z = 0 \) moves to \( z = z_1 \), this is the point with \( \phi = 0 = k z - \omega t \).

(xvii) \[ z_1 = \frac{\omega}{k} t_1 \Rightarrow \text{velocity of the crest is } \frac{\omega}{k} \]

(xviii) \[ \lambda = \text{WAVELENGTH} \quad k\lambda = 2\pi \Rightarrow \frac{\lambda}{k} = \frac{2\pi}{k} \]

(xix) \[ \tau = \text{PERIOD} \quad \omega \tau = 2\pi \Rightarrow \frac{\tau}{\omega} = \frac{2\pi}{\omega} \]
Lecture #6: More on Waves

Last time: 10 solutions, \( \frac{\partial^2 E_x}{\partial t^2} - \frac{\partial^2 B_y}{\partial z^2}, \frac{\partial B_y}{\partial t} - \frac{\partial E_x}{\partial z} \Rightarrow \frac{\partial^2 E_x}{\partial t^2} = c^2 \frac{\partial^2 E_x}{\partial z^2}, c^2 = \frac{1}{\mu_0 \epsilon_0} \)

Traveling wave: \( E_x = E_0 \cos \{kz - ct\} \)  
Wave speed: \( \frac{\omega}{k} \) \{Phase velocity\}

(i) Now we substitute our "guessed" solution into the wave equation:

\[ \phi = k z - \omega t \]  
\[ E_x = E_0 \cos \phi \]  
\[ \frac{\partial \phi}{\partial t} = -\omega, \frac{\partial \phi}{\partial z} = k. \]

"Phase"

\[ \Rightarrow \frac{\partial E_x}{\partial t} = -E_0 \sin \phi \frac{\partial \phi}{\partial t} = \omega E_0 \sin \{kz - ct\} \]

-chain rule

\[ \frac{\partial^2 E_x}{\partial t^2} = -\omega^2 E_0 \cos \{kz - ct\} = c^2 \frac{\partial^2 E_x}{\partial z^2} = -k c^2 E_0 \cos \{kz - ct\} \]

so our "guess" does indeed satisfy the equation if:

\[ \omega^2 = k c^2 \]

\[ \Rightarrow \frac{\omega}{k} = \pm c = \text{wave velocity}. \]

(ii) There are therefore two traveling wave solutions [for each k]

\[ E_x = E_+ \cos \{k(z - ct)\} \]  
wave going to right.

\[ E_x = E_- \cos \{k(z + ct)\} \]  
wave going to left.

(iii) Let's work out the magnetic field for these waves - \( B_y \).

\[ \frac{\partial B_y}{\partial t} = -\frac{\partial E_x}{\partial z} = k E_+ \sin \{k(z - ct)\} \]

"try" solution \( B_y = B_+ \cos \{k(z - ct)\} \)

\[ k c B_+ \sin \{k(z - ct)\} = k E_+ \sin \{k(z - ct)\} \Rightarrow \]

\[ B_+ = \frac{E_+}{c} \]

(iv) \[ E_x = E_+ \cos \{k(z - ct)\} + E_- \cos \{k(z + ct)\} \] \{Left and Right going waves\}

\[ B_y = \frac{E_+}{c} \cos \{k(z - ct)\} - \frac{E_-}{c} \cos \{k(z + ct)\} \]
(vi) How big is the magnetic field component? Look at the x force on an electron $F_x = -e\{E_x - V_y B_y\} = -eE_x \cos(kz - \omega t) (1 - \frac{V_y}{c})$
magnetic force is smaller by $\frac{V_y}{c}$ only important for relativistic particles.

(vii) GENERAL WAVES USING COMPLEX EXPONENTIALS.

We can write a $\frac{\text{REAL}}{\text{ID}} \times$ polarised wave as:

$$E = \left\{ E_o e^{ikz - i\omega t} + E_o^* e^{-ikz + i\omega t} \right\} \frac{1}{2}$$

if $E_o = |E_o| e^{i\psi}$ then

$$E = \frac{1}{2} |E_o| \cos\left(kz - \omega t + \psi\right)$$

(viii) But because Maxwell's Equations are real

if a complex $E_o B$ satisfy Maxwell's Equations so do $E_o^* B^*$ so

you only have to solve for $E_o B$.

(ix) This approach is useful for $\pi$ waves in media -

the second half of the course.

(x) In your homework you will show that the general solution to

the 1D equation $\frac{d^2 E_x}{d t^2} = c^2 \frac{d^2 E_x}{d z^2}$

$$E_x = E_+ (z - ct) + E_- (z + ct)$$

GENERAL FUNCTION GOING TO RIGHT

GENERAL FUNCTION GOING TO LEFT
Lecture #7: A 1D Radiation Problem.

Last time: 1D waves: \( E_x = E_x \cos(k(z-c)\tau) + E_y \cos(k(z+c)\tau) \)
\( B_y = E_x \cos(k(z-c)\tau) - E_y \cos(k(z+c)\tau) \)
\( \text{goes to right} \quad \text{goes to left} \)
using complex waves \( E_x = E_0 e^{ikz-i\omega t} + \text{c.c. etc.} \)

(i) Today: \( \mathbf{J} = \mathbf{J}_x(z, t) \) source current for waves take it as specified.

\[ \frac{\partial B_y}{\partial t} = -\frac{\partial E_x}{\partial z} \quad \text{and} \quad \mu_0\varepsilon_0 \frac{\partial E_x}{\partial t} = -\frac{\partial B_y}{\partial z} - \mu_0 \mathbf{J}_x(z, t) \]

(ii) We make the current \( \mathbf{J}_x(z, t) = I_0 \cos(\omega t) \delta(z) \)

\[ \int \delta(z) \, dz = 1 \]

\[ \text{SHEET OF CURRENT LOCALISED AROUND } z = 0. \]

\[ I(t) = \text{CURRENT PER UNIT LENGTH (IN } y) \]
\[ = \int_{-\Delta}^{\Delta} J_x(z, t) \, dz = I_0 \cos(\omega t) \]

- let \( \Delta \) shrink to zero

(iii) For \( z > \Delta \) \( J_x = 0 \) and we can use the vacuum solutions.

\[ E_x = E_+ \cos\left\{ k(z-c)\tau + \psi_+ \right\} + E_- \cos\left\{ k(z+c)\tau + \psi_- \right\} \]

- Clearly since \( J_x \) oscillates with frequency \( \omega \) we expect the solution to have the same frequency

\[ \Rightarrow \quad kc = \omega \]

- The \( E_- \) waves comes in towards the current from \( z = +\infty \)
  it can't be caused by the current so we drop it:

(iv) \[ \text{(CAUSALITY)} \]

\[ E_x = E_+ \cos\left\{ k(z-c)\tau + \psi_+ \right\} \quad B_y = E_+ \cos\left\{ k(z-c)\tau + \psi_+ \right\} \]

\[ \text{For } z > 0 \]

\[ \text{NOTE: WE IMPOSE CAUSALITY IT IS NOT A PROPERTY OF THE EQUATION.} \]
(v) Similarly for $z < 0$ we cannot have waves coming from $z = -\infty$

\[ E_x = E_- \cos \{kz + \omega t + \psi_2\} ; \quad B_y = -E_- \cos \{kz + \omega t + \psi_2\} \]

(vi) Now we must determine $E_+, \psi_+$ and $E_-, \psi_-$ from the conditions at $z = 0$. To do this we derive jump conditions. This involves a technique we will use often - watch carefully.

(vii) From Eq. (1) integrate $-\Delta < z < \Delta$.

\[ \int_{-\Delta}^{\Delta} \frac{dz}{d\tau} \frac{dE_x}{dt} = \left[ E_x (\Delta, t) - E_x (-\Delta, t) \right] \]

< 2\Delta \left( \frac{dB_y}{dt} \right)_{\text{MAXIMUM}} \quad \text{SO ASSUME} \quad \frac{dB_y}{dt} \text{DOES NOT BECOME INFINITE}

\[ \Rightarrow E_x (\Delta, t) = E_x (-\Delta, t) \quad \text{as} \quad \Delta \to 0 \]

1st JUMP CONDITION.

(viii) From Eq. (2) integrate $-\Delta < z < \Delta$.

\[ \int_{-\Delta}^{\Delta} \frac{dz}{d\tau} \frac{dB_y}{dt} = - \left[ B_y (\Delta, t) - B_y (-\Delta, t) \right] - M_0 I_0 \cos (\omega t) \]

\[ \Rightarrow B_y (\Delta, t) = B_y (-\Delta, t) - M_0 I_0 \cos (\omega t) \quad \text{as} \quad \Delta \to 0 \]

2nd JUMP CONDITION

1st J.C. \[ E_+ \cos \{ -\omega t + \psi_+ \} = E_- \cos \{ \omega t + \psi_2 \} \]

2nd J.C. \[ E_+ \cos \{ -\omega t + \psi_2 \} = -E_- \cos \{ \omega t + \psi_2 \} - M_0 I_0 \cos (\omega t) \]

\[ \Rightarrow \psi_+ = -\psi_- \]

\[ E_+ = E_- \quad \text{and} \quad \psi_+ = -\psi_- \]

\[ E_+ = \frac{1}{2} \sqrt{\frac{M_0 I_0}{\varepsilon_0}} \]

\[ E_x = \frac{1}{2} \sqrt{\frac{M_0 I_0}{\varepsilon_0}} \cos \{ \omega t + \psi_2 \} \]

\[ z < 0 \]

\[ z = 0 \]

\[ z > 0 \]
Lecture #8: Energy Conservation.

Last lecture: 1D radiation from an oscillating current sheet.

\[ E_x = \frac{1}{\mu_0} \cos[k(z - ct)] \]
\[ E_y = \frac{1}{\mu_0} \cos[k(x - ct)] \]

(i) There must be energy in the EM field - how else would it wiggle the electron. Today we derive the energy equation for EM.

(ii) Energy exchange of field and particle: (non-relativistic)

\[ \frac{dV}{dt} = q \left( E + v \times B \right) \]

Sometimes called the Lorentz Force.

Calculate the work done:

\[ m \frac{dV}{dt} = \frac{d}{dt} \left( \frac{1}{2} MV^2 \right) = q(V \cdot E) + qV \cdot (v \times B) \]

(iii) What is the current density of a point charge?

\[ J = \int \rho(x - y(t)) \Delta t \]

\[ V = \frac{d}{dt} \left( x(t) - x(t) \right) \]

Shrink blob to a point:

\[ \rho(x - y(t)) \Rightarrow q \delta(x - y(t)) \]

Lots, N, point charges:

\[ J = \sum_{i=1}^{N} q_i \delta(x - x_i(t)) \]

\[ \int \mathbf{E} \cdot d\mathbf{s} = \sum_{i=1}^{N} q_i \int \mathbf{E} \cdot \delta(x - x_i(t)) \, d\mathbf{s} = \sum_{i=1}^{N} q_i v_i \cdot E(x_i, t) \]

\[ = \sum_{i=1}^{N} \frac{d}{dt} \left( \frac{1}{2} M V_i^2 \right) \]

Rate of change of K.E. of all the particles.

(v) Where did this energy go?
(vi) Now let's calculate $\mathbf{J} \cdot \mathbf{E}$ from Maxwell's Equations.

Use \( \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B} - \mu_0 \mathbf{J} \)

\[ \implies \mathbf{J} \cdot \mathbf{E} = \frac{1}{\mu_0} \left( \mathbf{E} \cdot \nabla \mathbf{B} \right) - \varepsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\varepsilon_0} \left( \frac{\varepsilon_0 \mathbf{E}^2}{2} \right) \quad (1) \]

(vii) From Faraday's Law we do a "symmetrical" step: \( \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \)

\[ \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} = -\mathbf{B} \cdot \nabla \times \mathbf{E} \quad (2) \]

\( 0 + \frac{1}{\mu_0} \implies \)

\[ \mathbf{J} \cdot \mathbf{E} = \frac{1}{\mu_0} \left( \mathbf{E} \cdot \nabla \mathbf{B} - \mathbf{B} \cdot \nabla \mathbf{E} \right) = \frac{1}{\mu_0} \left[ \frac{\mathbf{B}^2}{2} + \frac{\varepsilon_0 \mathbf{E}^2}{2} \right] \]

Use vector identity: \( \nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{B} \)

(viii)

\[ \implies \frac{\partial W}{\partial t} + \nabla \cdot \mathbf{N} = -\mathbf{J} \cdot \mathbf{E} \]

"Poynting's theorem"

\[ W = \frac{1}{2} \frac{\mathbf{B}^2}{\mu_0} + \frac{\varepsilon_0 \mathbf{E}^2}{2} = \text{ELECTROMAGNETIC FIELD ENERGY DENSITY} \]

\[ \mathbf{N} = \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} = \text{FLUX OF ELECTROMAGNETIC ENERGY} \]

"POYNTING VECTOR"

\[ -\mathbf{J} \cdot \mathbf{E} = \text{SINK OF ELECTROMAGNETIC ENERGY GOES TO PARTICLE K.E.} \]

(ix) Now integrate Poynting's theorem over a volume \( V \) and we have \( \int_N \mathbf{N} \cdot d\mathbf{s} = \int_{V} dW \)

\[ \int_{N} \mathbf{N} \cdot d\mathbf{s} = \text{Energy Flux in EM Field out} \]

\[ V = \text{Volume containing} \]

\[ N \text{ charged particles} \]

\[ \frac{d}{dt} \int_{S} \mathbf{W} d^3r = -\int_{S} \mathbf{N} \cdot d\mathbf{s} - \int_{V} \mathbf{J} \cdot d^3s = -\int_{N} \mathbf{N} \cdot ds - \sum_{i=1}^{N} \int_{S_i} \mathbf{d} \left( \mathbf{W} \cdot \mathbf{E} \mathbf{V} \cdot \mathbf{N} \right) \]

RATE OF CHANGE OF EM ENERGY IN \( V \).

EM ENERGY FLUX ACROSS THE SURFACE "RADIATION".

LOSS OF EM ENERGY IN \( V \).

RADIATION K.E. GAIN OF PARTICLES IN \( V \).

UNITS? - Work them out!
(c) Today Solutions of the Stationary Equation $\frac{\partial}{\partial t} \equiv 0$.

**MAXWELL'S EQUATIONS BECOME:**

\[
\nabla \cdot \mathbf{E} = \frac{\rho(x)}{\varepsilon_0} \quad \nabla \cdot \mathbf{B} = 0 \\
\nabla \times \mathbf{E} = 0 \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}
\]

**ELECTROSTATICS**  **MAGNETOSTATICS**

(c) Today **ELECTROSTATICS** - First an important mathematical theorem:

IF $\nabla \times \mathbf{E} = 0$ then we can **ALWAYS** write $\mathbf{E} = \nabla \psi$

where $\psi$ is some scalar function of $\mathbf{r}$.

Converse: If $\mathbf{E} = \nabla \psi \Rightarrow \nabla \times \mathbf{E} = 0$ **ALWAYS**.

(c) Therefore $\nabla \times \mathbf{E} = 0 \Rightarrow \mathbf{E} = -\nabla V$ minus sign is convention.

$V$ is, as you know, called the "electrostatic potential".

(iv) We can find $V(x)$ from a known $\mathbf{E}(x)$ by integrating

\[
\int_a^b \mathbf{E} \cdot d\mathbf{r} = -\int_a^b \nabla V \cdot d\mathbf{r} = -\{V(a) - V(b)\}
\]

It doesn't matter which path you take between $a$ and $b$ (Proof?)

(v) Now we substitute $\mathbf{E} = -\nabla V$ into $\nabla \cdot \mathbf{E} = \rho/\varepsilon_0$.

$\Rightarrow \nabla \cdot (-\nabla V) = -\nabla^2 V = \frac{\rho(x)}{\varepsilon_0} \Rightarrow \nabla^2 V = -\frac{\rho(x)}{\varepsilon_0}$

**POISSON'S EQUATION.**
(V) Poisson's equation is the subject of some very turgid books. here we will derive a "formal" solution as an integral.

(Vii) For a point charge, Coulomb's Law gives:

\[ V = \frac{q}{4\pi \varepsilon_0 d} \]

d = distance to the charge.

(Viii) For a point charge at \( \mathbf{r}' \) distance to us at \( \mathbf{r} \) is \( d = |\mathbf{r} - \mathbf{r}'| \)

\[ V(\mathbf{r}) = \frac{q}{4\pi \varepsilon_0 |\mathbf{r} - \mathbf{r}'|} \]

**How about a distribution of charge?** \( \rho(\mathbf{r}) \)

\[ \delta Q' = \text{charge in box } d^3V' = \rho(\mathbf{r}') d^3V' \text{ at } \mathbf{r}' \]

\( \delta V = \text{potential just from charge in } d^3V' = \frac{\delta Q'}{4\pi \varepsilon_0 d} = \frac{\delta Q'}{4\pi \varepsilon_0 |\mathbf{r} - \mathbf{r}'|} \)

(ix) Now divide all of space into little boxes. The potential is the sum of all the \( \delta V \) from all of the boxes. We can superpose solutions because Poisson's equation is linear (Proof?)

\[ V(\mathbf{r}) = \sum_{\text{ALL BOXES}} \delta V = \sum_i \frac{\delta Q_i}{4\pi \varepsilon_0 |\mathbf{r} - \mathbf{r}'|} = \int \frac{\rho(\mathbf{r}') d^3V'}{4\pi \varepsilon_0 |\mathbf{r} - \mathbf{r}'|} \]

take limit of infinitesimal boxes.

Formal solution to Poisson's equation, often not very useful because we don't know where the charge actually is a priori.
Lecture 10: Vector Potential

Last time, Stationary Solutions

\[ \nabla \cdot E = \frac{\rho}{\varepsilon_0}, \nabla \times E = 0 \]

\[ E = -\nabla V, \quad \nabla \times V = -\rho / \varepsilon_0, \quad V = \int \frac{\rho(s') \, ds'}{4\pi \varepsilon_0 |s - s'|} \]

(i) TODAY: MAGNETOSTATICS \rightarrow VECTOR POTENTIAL - GAUGE.

\[ \nabla \cdot B = 0 \quad \nabla \times B = \mu_0 J \]

(ii) We use another very useful mathematical result:

If \( \nabla \cdot \mathbf{E} = 0 \) then we can always find a \( \mathbf{E} \) such that \( \mathbf{E} = \nabla \times \mathbf{G} \)

Converse if \( \mathbf{G} = \nabla \times \mathbf{E} \), \( \nabla \cdot \mathbf{G} = 0 \) vector identity.

(iii) We introduce the vector potential \( \mathbf{A} \)

\[ \nabla \cdot B = 0 \Rightarrow \mathbf{B} = \nabla \times \mathbf{A} \]

(iv) But \( \mathbf{A} \) is not unique. Because:

Let \( \mathbf{A}' = \mathbf{A} + \nabla \chi \)

\[ \Rightarrow \nabla \chi \mathbf{A}' = \nabla \chi \mathbf{A} = \mathbf{B} \]

WE HAVE A FREEDOM TO CHOOSE THE "GAUGE".

MOSTLY WE CHOOSE TO IMPOSE ANOTHER CONDITION ON \( \mathbf{A} \)

Coulomb Gauge: \( \nabla \cdot \mathbf{A} = 0 \) - used for time independent problems.

Lorentz Gauge: \( \nabla \cdot \mathbf{A} = -\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \) - used for time dependent problems.

(v) Transforming Gauge. Suppose we have a vector potential \( \mathbf{A}' \) such that \( \mathbf{B} = \nabla \times \mathbf{A}' \)

But \( \nabla \cdot \mathbf{A}' \neq 0 \) we change/transform gauge so we can make \( \mathbf{B} = \nabla \times \mathbf{A} \) & \( \nabla \cdot \mathbf{A} = 0 \).

\[ \text{Proof:} \quad \text{Let} \quad \mathbf{A} = \mathbf{A}' - \nabla \chi \quad \Rightarrow \quad \nabla \times \mathbf{A} = \nabla \times \mathbf{A}' - \nabla \times \nabla \chi = 0 \]

\[ \Rightarrow \quad \nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{A}' - \nabla \cdot \nabla \chi = 0 \]

Solve \( \nabla^2 \chi = \nabla \cdot \mathbf{A}' \)

\[ \text{to find} \quad \chi \rightarrow \mathbf{A} \]

(vi) Back to magnetostatics:

\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \Rightarrow \nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} \]

Vector identity see formulae.

\[ \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \quad \text{X component} \]

Now apply Coulomb gauge

\[ \nabla \cdot \mathbf{A} = 0 \Rightarrow \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \]
(vii) This is just like potential \[ \nabla^2 V = -\frac{\rho}{\varepsilon_0} : \nabla^2 A_x = -\mu_0 J_x \]
\[ V(x) = \int \rho(s') d^3 s' \quad \text{at} \frac{\partial}{\partial t} \int \frac{\mu_0 J_x(s') d^3 s'}{4\pi |r-s'|} \]
\[ \Rightarrow A(x) = \int \frac{\mu_0 J_x(s') d^3 s'}{4\pi |r-s'|} \]

Formal solution of magnetostatics. Never an easy integral.

(viii) Potentials are a lot of help in stationary case - what about when \( E \) and \( B \) are time-dependent?
\[ \nabla \cdot B = 0 \] is still true so \[ B = \nabla \times A \]

Faraday's law\[ \frac{\partial B}{\partial t} = -\nabla \times E = \frac{\partial}{\partial t} (\nabla \times A) = \frac{\partial A}{\partial t} \]

\[ \Rightarrow \nabla \times \left( \frac{\partial A}{\partial t} + E \right) = 0 \quad \text{let} \quad \xi = \frac{\partial A}{\partial t} + E \]
\[ \nabla \times \xi = 0 \Rightarrow \xi = -\nabla V = \frac{\partial A}{\partial t} + E \]

As in electrostatics.

\[ V \equiv \text{SCALAR POTENTIAL} \quad \xi = \text{VECTOR POTENTIAL} \]

(ix) Again the potentials are not unique given \( E \) and \( B \)
\[ A' = A + \nabla \chi \quad \Rightarrow \quad B = \nabla \times A = \nabla \times A' \]
\[ E = -\frac{\partial A}{\partial t} - \nabla V = -\frac{\partial A'}{\partial t} + \frac{\bar{j}(\nabla \chi)}{\partial t} - \nabla V = -\frac{\partial A'}{\partial t} - \nabla V' \quad \text{where} \]
\[ V = V - \frac{\partial \chi}{\partial t} \]
\[ \text{(8)} \quad \text{Transformation:} \quad A' = A + \nabla \chi \quad \text{AND} \quad V' = V - \frac{\partial \chi}{\partial t} \quad \text{yields unchanged} \]
\[ E \text{ and } B. \quad B = \nabla \times A' \text{ and } E = -\frac{\partial A'}{\partial t} - \nabla V' \]
Lecture #11: Solving Maxwell's Equations: Part I.

Last Time:
- \( \mathbf{B} = \nabla \times \mathbf{A} \)
- Stationary solution: \( \nabla \times \mathbf{B} = \mathbf{0} \)

TODAY:
(i) By defining the potentials we automatically satisfy 2 of Maxwell's equations:
- \( \nabla \cdot \mathbf{B} = 0 \)
- \( \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \)

Now we substitute our potential solution into the other 2 Maxwell equations:

\[
\nabla \cdot \mathbf{E} = \frac{\rho(t, \mathbf{r})}{\varepsilon_0} \quad \Rightarrow \quad -\nabla^2 V - \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}) = \frac{\rho(t, \mathbf{r})}{\varepsilon_0} \quad \text{using} \quad \mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}
\]

\[
\text{AND} \quad \mathbf{B} = \nabla \times \mathbf{A} \quad \Rightarrow \quad \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B} - \mu_0 \mathbf{J} \quad \Rightarrow \quad \mu_0 \varepsilon_0 \left\{ -\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{V} \right\} = \nabla \times (\nabla \times \mathbf{A}) - \mu_0 \mathbf{J}
\]

\[

\nabla \times (\nabla \times \mathbf{A}) - \nabla^2 \mathbf{A} = \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \frac{1}{c^2} \frac{\partial \mathbf{V}}{\partial t} + \nabla \cdot \mathbf{A} = 0
\]

(ii) But remember we have freedom to choose a Gauge:

Choose Lorentz Gauge: \( \nabla \cdot \mathbf{A} = \frac{1}{c^2} \frac{\partial \mathbf{V}}{\partial t} + \nabla \cdot \mathbf{A} = 0 \)

This gives us Maxwell's Equations as 4 Beautiful Equations.

\[
\frac{1}{c^2} \frac{\partial \mathbf{V}}{\partial t} - \nabla^2 \mathbf{V} = \frac{\rho}{\varepsilon_0}
\]

\[
\frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \mathbf{M_0} \mathbf{J}
\]

(III)

3D Wave Equation:

For \( \mathbf{V} \) and \( \mathbf{A} \) with sources \( \rho \) and \( \mathbf{J} \).

\[
\frac{D^2 \mathbf{V}}{D t^2} + \nabla^2 \mathbf{V} = 0
\]

NOTE: Conservation of Charge \( \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \) is guaranteed by Gauge, or vice versa.
Following derivation of the solution is hard - you will not be asked to reproduce it. Enjoy!!

(v) We now want to solve the wave equation with an arbitrary source - we focus on equation for \( V \).

\[
\delta q_i(t) = \rho(\xi_i, t) d^3 \xi_i
\]

charge in the box at \( \xi \);

Let \( V_i \) be potential due to charge in \( i'th \) box only. \[
\frac{1}{c^2} \frac{\partial^2 V_i}{\partial t^2} = \frac{\rho}{\varepsilon_0 \text{box}} \quad \text{for \( \text{in box} \)}
\]

\[
= 0 \quad \text{elsewhere}
\]

\[\sum V_i \Rightarrow \sum \left\{ \frac{1}{c^2} \frac{\partial^2 V_i}{\partial t^2} - \nabla^2 V_i \right\} = \frac{\rho}{\varepsilon_0} \quad \text{everywhere.}\]

So we can solve for solution due to one box at a time then sum.

(vi) Now we can consider a box at the origin and then just shift origins.

\[
\frac{1}{c^2} \frac{\partial^2 V_0}{\partial t^2} - \nabla^2 V_0 = \frac{\rho}{\varepsilon_0} \quad \text{in box} \]

\[
= 0 \quad \text{outside box.}\]

Source is effectively a point at the origin so that all directions must be the same.

\[
\Rightarrow V_0 = V_0(r, t)
\]

Does not depend on angle.

\[
\frac{1}{c^2} \frac{\partial^2 V_0}{\partial t^2} - \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dV_0}{dr} \right) = \frac{\rho}{\varepsilon_0} \quad \text{inside box}
\]

\[
= 0 \quad \text{outside box.}\]
Lecture #12: Solving Maxwell's Equations: Part II.

MAXWELL EQUATIONS \( \begin{align*}
B &= \nabla \times A \\
E &= -\nabla V - \frac{\partial A}{\partial t}
\end{align*} \)

REDUCE TO

LORENTZ GAUGE \( \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} + \nabla \cdot \nabla V = 0 \)

WAVE EQUATIONS WITH SOURCES.

(i) WE ARE MIDWAY THROUGH DERIVATION OF SOLUTION.

a) BREAK SOURCE INTO LITTLE BOXES - \( V_i \) = POTENTIAL FROM \( i \)TH BOX CHARGE.

b) MOVE BOX TO ORIGIN BY SHIFTING COORDS.

c) SOLUTION MUST BE SYMMETRIC BECAUSE SOURCE IS LIKE A POINT.

\( \Rightarrow V_0 = V_0(r,t) \) potential from box at origin.

\[
\frac{1}{c^2} \frac{\partial^2 V_0}{\partial t^2} - \frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^2 \frac{\partial V_0}{\partial r} \right] = \frac{\rho}{\varepsilon_0} \quad \text{in box}
\]

\( = 0 \quad \text{outside box} \).

(ii) HOMEWORK "TRICK" - LET \( V_0 = \frac{U(r,t)}{r} \)

\( \Rightarrow \frac{1}{r} \left\{ \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial r^2} \right\} = \frac{\rho}{\varepsilon_0} \)

\( = 0 \quad \text{outside box} \).

(iii) This is the 1D wave equation and you know from homework that.

\[
U = f(r-ct) + g(r+ct)
\]

\( \Rightarrow \) OUTGOING WAVE

\( \Rightarrow \) INCOMING WAVE

Throw this away it isn't causal i.e. stuff comes from infinity.

(iv) \( V_0(r,t) = \frac{f(r-ct)}{r} \) OUTGOING WAVES THAT DROP IN AMPLITUDE AS THEY EXPAND.
(v) Near origin \( V_0(r,t) = \frac{f(-ct)}{r} \)

but since it takes light no time to travel this short distance the solution must be same as solution for a stationary charge distribution

i.e. convolve \( V_0 = \frac{q_0(t)}{4\pi \varepsilon_0 r} = \frac{\rho(0,t) d^3r}{4\pi \varepsilon_0 r} \) so \( f(-ct) = \frac{\rho(0,t) d^3r}{4\pi \varepsilon_0} \)

\[ \Rightarrow \quad V_0 = \frac{\rho(r=0,t-r/c) d^3r}{4\pi \varepsilon_0 r} = V_0(r,t) \]

(Solution called "Retarded Solution"

\( r-r/c = \) Retarded time.

(vi) Shift Coordinates back (reverse b))

\[ V_i(r,t) = \frac{\rho(\mathbf{r}_i, t-\frac{|\mathbf{r}-\mathbf{r}_i|}{c}) d^3r_i}{4\pi \varepsilon_0 |\mathbf{r}-\mathbf{r}_i|} \]

Note changes at \( \mathbf{r}_i \) in \( \rho \) don't propagate instantly to \( r \) they take a time \( |\mathbf{r}-\mathbf{r}_i|/c \) to get there!

(vii) Add boxes - integrate \( \mathbf{r}_i \rightarrow \mathbf{r}' \)

\[ V(\mathbf{r},t) = \int \frac{\rho(\mathbf{r}', t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}) d^3r'}{4\pi \varepsilon_0 |\mathbf{r}-\mathbf{r}'|} \]

AND BY ANALOGY:

\[ A(\mathbf{r},t) = \int \frac{\mathbf{j}(\mathbf{r}', t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}) d^3r'}{4\pi \varepsilon_0 |\mathbf{r}-\mathbf{r}'|} \]

(viii) So if you know the sources \( \mathbf{j}(\mathbf{r}',t) \& \rho(\mathbf{r}',t) \) you can in principle find \( V(\mathbf{r},t) \& A(\mathbf{r},t) \). Then by differentiating

\[ B = \nabla \times A \quad \mathbf{E} = -\frac{\partial A}{\partial t} - \nabla V \]

MAXWELL'S EQUATIONS SOLVED!
1. Electromagnetic Waves

1.1 A brief history of light

- Electromagnetic (em) radiation decoupled from matter about 400,000 years after the big bang (the end of the plasma period).
- It was assumed light was corpuscular in nature, since it travelled in straight lines.
- Young, Huygens etc. demonstrated that light exhibited wave nature (diffraction, interference...)
- Maxwell shows em radiation natural solutions of Maxwell’s equations, no frame required, but Maxwell thought their must me a medium – the æther.
- Michelson & Morley searched for the æther, but could not find a motion relative to it.
- Planck explained blackbody radiation as being due to quantisation of EM energy; Photoelectric effect (Einstein), x-rays explained by particle picture of light.

1.2 Maxwell’s equations, again!

In differential form (including the displacement current), Maxwell’s Equations (ME) are:

\[
\begin{align*}
(1) \quad \epsilon \nabla \cdot E &= \rho \\
(2) \quad \nabla \cdot B &= 0 \\
(3) \quad \nabla \times E &= -\frac{\partial B}{\partial t} \\
(4) \quad \nabla \times B &= \mu_0 j + \mu \epsilon \frac{\partial E}{\partial t}
\end{align*}
\]

In vacuum,

\[
\begin{align*}
(1) \quad \nabla \cdot E &= 0 \\
(2) \quad \nabla \cdot B &= 0 \\
(3) \quad \nabla \times E &= -\frac{\partial B}{\partial t} \\
(4) \quad \nabla \times B &= \mu_0 \epsilon_0 \frac{\partial E}{\partial t}
\end{align*}
\]

Take \( \nabla \times \text{Faraday} = -\frac{\partial (\nabla \times B)}{\partial t} \)

where we have reversed the order of the time and spatial derivatives on the RHS. We can use the vector identity \( \nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A \).

\[
\nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \nabla^2 E = -\frac{\partial}{\partial t} \left( \mu_0 \epsilon_0 \frac{\partial E}{\partial t} \right)
\]

where we have made use of Gauss’ law \( \nabla \cdot E = 0 \) term and Ampère-Maxwell to substitute for \( \nabla \times B \). Finally we obtain the following wave equation,

\[
\nabla^2 E = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2}
\]

where \( c = (\epsilon_0 \mu_0)^{-1/2} \).

Searching for solutions of the form \( E = E_0 e^{i(kr - \omega t)} \), we finally obtain \( c^2 k^2 E = \omega^2 E \), which has solutions provided

\[
\omega = ck
\]

1.3 Electromagnetic Waves

The above equation does not set any limits on the wavelength of the radiation, and indeed is applicable over a range of wavelengths scaling from \( 10^{-13} \) m to 100’s of m in length. This electromagnetic spectrum has different names over different ranges, as shown in the table below,
1.4 Properties of EM Waves

Wave completes one cycle when \( kr \) goes from 0 to \( 2\pi \), so \( k\lambda = 2\pi \). Also \( \omega t_{\text{per}} = 2\pi \) is one time period, so \( f = 1/t_{\text{per}} = 2\pi/\omega \), and \( c = (\varepsilon_0\mu_0)^{-1/2} = 2.998 \times 10^8 \text{ m/s} \)

\[ \therefore \text{Solution is a plane non-dispersive wave with transverse (and mutually orthogonal) electric and magnetic fields travelling at the speed of light. The direction in which } E \text{ point is called the polarization.} \]

If two orthogonally polarised beams are made to be \( \pi/2 \) out of phase with one another, then the polarisation vector rotates around the direction of propagation. This is called circularly polarised light and can be clockwise or anticlockwise rotating, depending on whether the phase lag is \( \pm\pi/2 \).
2. Radiation

2.1 A moving charge

For a particle moving with constant velocity, the electric field of a moving charge is “shrunk” in the direction in which it is moving as a result of Lorentz contraction, (so angular spread of $\mathbf{E}$ something like $1/\gamma$).

The field transverse to its motion is increased (as electromagnetic energy must be conserved), and a magnetic field is formed as well due to the now non-radial fields.

but the $\mathbf{E}$ field lines still emanate directly from the charge, with no transverse component, and therefore the travelling charge does not generate electromagnetic waves.

![Figure 1: E stationary charge](image1)
![Figure 2: E moving charge](image2)
![Figure 3: B moving charge](image3)

2.2 An accelerating charge

An accelerating charge produces radiation.

![Figure 4: Oscillating charge](image4)
![Figure 5: Accelerated charge](image5)

![Figure 6: Angles involved](image6)

Consider the case of a charge $Q$ initially at rest at $O$, which is accelerated the for a time $\delta t$ with constant acceleration $a$, and then travelling with constant velocity $v(\ll c)$ as in fig. 5. One can see that the electric field lines must be kinked to ensure continuity. This means that within the shaded volume, they must have some transverse component. Consider the flux emanating from the 2 caps...
of the shaded volume;
\[
\int \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0} \int_{\theta = 0}^{\theta_1} \int_{\phi = 0}^{2\pi} \frac{Q}{4\pi \epsilon_0} \sin \theta \, d\theta \, d\phi = Q(1 - \cos \theta_1)/2\epsilon_0
\]
A similar expression can be found for the inner surface, and so the difference in flux is given by \(Q(\cos \theta_2 - \cos \theta_1)/2\epsilon_0\).

This must be equal to the flux out of the ends of the surface (since we are in vacuum); \(E_t \cdot 2\pi r \sin \theta \, c \, \delta t\), where here we have approximated \(\theta_1, \theta_2\) as \(\theta\).

Hence, \(E_t = \frac{Q(\cos \theta_2 - \cos \theta_1)}{4\pi \epsilon_0 \, rc \sin \theta \, \delta t}\)

A little bit of trigonometry: \(\cos \theta_1 - \cos \theta_2 = 2 \sin \frac{1}{2} (\theta_1 - \theta_2) \sin \frac{1}{2} (\theta_1 + \theta_2) \approx (\theta_1 - \theta_2) \sin \theta\)

But consider the figure on the right, which shows that \(\frac{\sin(\theta_1 - \theta_2)}{vt} = \frac{\sin \theta_1}{ct}\)
so \(\theta_1 - \theta_2 \approx \sin \theta(v/c)\)

and so \(\cos \theta_1 - \cos \theta_2 \approx \sin^2 \theta(v/c)\). With this and \(v/\delta t = a\), we finally obtain,

\[
E_t = \frac{Q[a] \sin \theta}{4\pi \epsilon_0 \, rc^2}; \quad B_t = \frac{Q[a] \sin \theta}{4\pi \epsilon_0 \, rc^3}
\]

where we have used \(\omega B = kE \rightarrow |B| = |E|/c\) for an em wave.

Note that the acceleration referred to here is the acceleration that the particle felt at a time \(r/c\) before, and so when we calculate the acceleration we must use the retarded time \(z(t - r/c)\).

2.3 Radiated power
Consider the case of an electron oscillating along the \(z\)-direction with motion defined by \(z = z_0 \sin(\omega t)\),

so \([z] = z_0 \sin(\omega(t - r/c))\) and \([\dot{z}] = \omega^2 z_0 \sin(\omega(t - r/c))\), and the expressions for \(E\) and \(B\) become

\[
\text{So, } E_{\text{rad}} = \frac{q}{4\pi \epsilon_0} \frac{z_0 \omega^2}{rc^2} \sin \theta \sin(\omega(t - r/c)) \, \hat{\theta}; \quad B_{\text{rad}} = \frac{q}{4\pi \epsilon_0} \frac{z_0 \omega^2}{rc^3} \sin \theta \sin(\omega(t - r/c)) \, \hat{\phi}
\]

We can thus obtain for the Poynting Vector

\[
\mathbf{N} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} = \frac{1}{\mu_0} \frac{q^2}{16\pi^2 \epsilon_0^2} \frac{z_0^2 \omega^4}{r^2 c^5} \sin^2 \theta \sin^2(\omega(t - r/c)) \, \hat{r}
\]

This is the behaviour of the flux from a dipole, and has the typical lobed pattern, see fig. 7. Many radiating systems can be considered as dipole oscillators, from many atomic transitions to simple antennas.

We can find the total (time-averaged) power emitted by integrate over a surface at radius \(r\) (and time averaging the oscillating term).

\[
P = \frac{1}{\mu_0} \frac{q^2}{16\pi^2 \epsilon_0} \left[ \frac{z_0^2 \omega^4}{r^2 c^5} \sin^2 \theta \sin \theta \, d\theta \, d\phi \right] \langle \sin^2(\omega(t - r/c)) \rangle
\]
Using \( \langle \sin^2(t - r/c) \rangle = \frac{1}{2} \) and \( \oint \sin^3 \theta \, d\Omega = \frac{8}{3} \pi \), where the latter integral is over all solid angle.

\[
P = \frac{\mu_0 \varepsilon_0}{\varepsilon_0} \frac{q^2}{16 \pi^2 c_0^2} \frac{z_0^2 \omega^4}{c^3} \frac{8}{3} \pi \frac{1}{2}
\]

\[
P = \frac{q^2 z_0^2 \omega^4}{12 \pi \varepsilon_0 c^3}
\]

Larmor formula

Figure 7: Dipole radiation
3. Antennas

3.1 Antenna Potential

In Prof Cowley’s lectures 10 and 11, you were shown that the time-dependent solutions to Maxwell’s equations were:

\[ V = \frac{1}{4\pi\varepsilon_0} \int \frac{[\rho]}{r} \, d\tau \quad \text{and} \quad A = \frac{\mu_0}{4\pi} \int \frac{[J]}{r} \, d\tau \]

where the \([\ ]\) brackets denote that the sources must be considered at the retarded times. We already know oscillating currents produce radiation in 1D. Last lecture, we showed that accelerating charges can produce radiation using Maxwell’s equations with simple assumptions. Let’s now generalise to a 3D problem using potentials.

Consider the small antenna formed by two branching wires (fig. 1), which carry a current \( I = I_0 \cos \omega t \).

Since \( \mathbf{J} \cdot \mathbf{A} = I \), for a short wire (we’ll define short later) of length \( dl \), \( I \, dl = (\mathbf{J} \cdot \mathbf{A}) \, dl = J \, d\tau \). This is just the term in the integrand in the retarded solution above, so we can rewrite,

\[ A = \frac{\mu_0}{4\pi} \int \frac{[I]}{r} \, dl \]

Now if \( dl \ll r \) then there is no retardation between different parts of the antenna (in other words all parts of the antenna radiate with the same phase), and the \( \int dl = dl \). So,

\[ A = \frac{\mu_0}{4\pi} \frac{[I]}{r} \, dl \]

Taking \([I]\) to be lined along the \( z \)-axis, then one can see in fig. 1b that the components of \( A \) are,

\[ A_r = \frac{\mu_0}{4\pi} \frac{[I]}{r} \cos \theta \quad \text{and} \quad A_\theta = -\frac{\mu_0}{4\pi} \frac{[I]}{r} \sin \theta \]

We could use the Lorentz gauge condition to find \( V \) and then \( E \), though this produces 5 terms for the \( E \), so we’ll worry about electric fields later.

3.2 Radiated Fields

We can find the \( \mathbf{B} \) field from \( \mathbf{B} = \nabla \times \mathbf{A} \), note that since \( \mathbf{A} \) has no \( \phi \) component and does not vary with \( \phi \) either, this considerably simplifies the calculation. In spherical coordinates:

\[
\nabla \times \mathbf{A} = \begin{vmatrix}
\frac{1}{r^2 \sin \theta} & \frac{1}{r \sin \theta} & \frac{1}{r} \\
\frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\
A_r & r A_\theta & r \sin \theta A_\phi
\end{vmatrix} = \begin{vmatrix}
\frac{1}{r^2 \sin \theta} & \frac{1}{r \sin \theta} & \frac{1}{r} \\
\frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & 0 \\
A_r & r A_\theta & 0
\end{vmatrix}
\]

which luckily tells us that only the \( \hat{\phi} \) component will survive. So,

\[
\nabla \times \mathbf{A} = \frac{1}{r} \hat{\phi} \left( \frac{\partial (r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right)
\]
\[\nabla \times \mathbf{A} = \frac{1}{r} \hat{\phi} \left( \frac{\mu_0}{4\pi} \frac{[I']}{c} \sin \theta + \frac{\mu_0}{4\pi} \frac{[I]}{r} \sin \theta \right)\]

where \([I']\) is the derivative with respect to time of the retarded current, since

\[
\frac{\partial I(t - r/c)}{\partial r} = \frac{\partial I(\xi)}{\partial r} = \frac{\partial \xi}{\partial r} \frac{\partial I(\xi)}{\partial \xi} = -\frac{1}{c} \frac{\partial I(\xi)}{\partial t} = -\frac{1}{c} \frac{\partial}{\partial t} I(t - r/c) = [I']
\]

(where we used \(\xi = t - r/c\) and \(\frac{\partial \xi}{\partial r} = -\frac{1}{c}, \frac{\partial \xi}{\partial t} = 1\). NB that time and space are linked in the retarded current).

So finally,

\[\mathbf{B} = \left( \frac{\mu_0}{4\pi} \frac{[I']}{rc} \sin \theta + \frac{\mu_0}{4\pi} \frac{[I]}{r^2} \right) \hat{\phi}\]

([I']) is the time derivative of the retarded current.) A little bit of trig shows that since \(\hat{\phi}\) is just the direction orthogonal to \(I d\mathbf{l}\) and \(r\) (again see fig. 1a), then

\[\mathbf{B} = \left( \frac{\mu_0}{4\pi} \frac{[I']}{rc} \times \hat{\mathbf{r}} + \frac{\mu_0}{4\pi} \frac{[I]}{r^2} \hat{\mathbf{r}} \right)\]

The second term can be identified as the Biot-Savart field. It does not depend on the time variation of \(\hat{\mathbf{r}}\) just its momentary (retarded) value, and dominates at small distances since the first term is \(\approx r\omega/c\) smaller. But at large \(r\), the Biot-Savart term drops as \(r^{-2}\) whilst the first term only drops as \(r^{-1}\). This term is called the radiation field, and can exist far away from a source (even in vacuum) with a field in \(\hat{\mathbf{\phi}}\) transverse to its direction of propagation \(\hat{\mathbf{r}}\) and the direction of the current generating it in \(\hat{\theta}\).

Since this is a vacuum field, we can immediately write the corresponding \(\mathbf{E}\) field, so both radiation fields are,

\[\mathbf{B} = \frac{1}{4\pi \varepsilon_0} \frac{[I']}{rc^3} \sin \theta \hat{\phi}\quad \text{and} \quad \mathbf{E} = \frac{1}{4\pi \varepsilon_0} \frac{[I']}{r c^2} \sin \theta \hat{\theta}\]

(Note we used \(\mathbf{E} = -c \hat{\mathbf{k}} \times \mathbf{B}\).

### 3.4 Dipole radiation, again!

Looking at fig. 1, one can see that the current could have been written as an oscillating dipole with

\[I d\mathbf{l} = \frac{dQ}{dt} d\mathbf{l} = \frac{d}{dt} (Q d\mathbf{l}) = \frac{d\mathbf{p}}{dt} = \dot{\mathbf{p}}, \text{ where we identify the dipole moment } \mathbf{p} = Q d\mathbf{l}. \text{ So,} \]

\[\mathbf{B} = \frac{1}{4\pi \varepsilon_0} \frac{[\dot{p}]}{rc^3} \sin \theta \hat{\phi}\quad \text{and} \quad \mathbf{E} = \frac{1}{4\pi \varepsilon_0} \frac{[\dot{p}]}{r c^2} \sin \theta \hat{\theta}\]

As we had last lecture. Once more the Poynting flux is

\[\mathbf{N} = \frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B}) = \frac{1}{16\pi^2 \varepsilon_0} \frac{[\dot{p}]^2 \sin^2 \theta}{r^2 c^3} \hat{\mathbf{r}} = \frac{1}{16\pi^2 \varepsilon_0} \frac{[I']^2 d\mathbf{l}^2 \sin^2 \theta}{r^2 c^3} \hat{\mathbf{r}}\]

### 3.5 Radiation resistance

If \(I = I_0 \cos \omega t\), then \([I'] = -\omega I_0 \sin \omega (t - r/c)\), and in terms of the rms, and the total power radiated is,

\[\langle P \rangle = \frac{2\pi}{3\varepsilon_0 c} \left( \frac{d\mathbf{l}}{\lambda} \right)^2 I_{rms}^2 = R_{rad} I_{rms}^2\]
with $R_{\text{rad}} \approx 800 \left( \frac{dl}{\lambda} \right)^2 \Omega$. Not only are small antennas not very efficient, they have an effective resistance which makes them very difficult to match to an electronic circuit. For longer dipoles $dl \approx \lambda$, phase considerations between the emission must be considered and the quadratic increase in power (and $R_{\text{rad}}$) does not apply (too long and you can get destructive interference between different parts of the antenna). For example a half-wave dipole, has $R_{\text{rad}} = 73 \Omega$, which is perfectly suited to normal co-ax $Z_{\text{coax}} = 75 \Omega$. 

4. Synchrotron radiation

4.1 Synchrotron radiation

We derived the power emitted by an oscillating dipole, the Larmor formula: \( P = \frac{q^2 \langle |a|^2 \rangle}{6 \pi \epsilon_0 c^3} \).

When a particle performs circular motion, the combined motion is the sum of two oscillating motions:
\[
\mathbf{r} = r_0 (\cos(\omega t) \hat{x} + \sin(\omega t) \hat{y}) \quad \rightarrow \quad \mathbf{a} = \frac{\partial^2 \mathbf{r}}{\partial t^2} = -r_0 \omega^2 (\cos(\omega t) \hat{x} + \sin(\omega t) \hat{y})
\]

Or alternatively, \( \mathbf{a} = -\frac{v^2}{r_0} \hat{r} = -r_0 \omega^2 \hat{r} \)

Either way, one can see that the total power emitted:
\[
P = \frac{q^2 r_0^2 \omega^4}{6 \pi \epsilon_0 c^3}
\]

(NB this is twice the value for simple harmonic value found in lecture 3, because now it is doing shm in two direction simultaneously!)

This is called synchrotron radiation, and the radiation travels in the instantaneous direction of the emitting charge at any time, (hence along the direction of the tangent to the circle).

4.2 Liénard-Wiechart potentials

So far we made the assumption that the radiating charge has \( v \ll c \), however as we will see that is not the most interesting case for radiation emission. The potentials we derived before:
\[
V = \frac{1}{4 \pi \epsilon_0} \int \frac{[\rho]}{r} \, d\tau \quad \text{and} \quad A = \frac{\mu_0}{4 \pi} \int \frac{[J]}{r} \, d\tau
\]

are not valid if the charge is moving. In particular, both the charge and current density will look different due to the relative motion. This can be accounted for by a Doppler factor.

Consider the train above. Light from the back of the train, which reaches the eye simultaneously with that from the front, had to originate from a position further than \( L \) from the front of the train - due to the extra time it would take light to pass from back to front of the train. So in the time light travels \( L' \), the train itself moves a distance \( L' - L \). Hence,
\[
\frac{L'}{c} = \frac{L' - L}{v} \quad \rightarrow \quad L' = \frac{L}{1 - v/c}
\]
The second term is then the radiation field, and as before can be seen to fall as \(1/R^2\) its new position is \(\dot{\mathbf{r}}\) (drew last time), and is the only term if there is no acceleration \((1/R)\) ct the retarded time has travelled \(\hat{R}\) always points to the position of the particle at the time of observation \((\hat{R} - \beta)\). Note that both position and velocity need to be calculated at the retarded time. These can be differentiated as before (but lengthly - see Griffiths 10.3.2 for example!) to give:

\[
A(r, t) = \frac{\mu_0 q |\mathbf{v}|}{4\pi c (1 - \hat{\mathbf{r}} \cdot \mathbf{v})} = \frac{\mathbf{v}}{c^2} V(r, t) \tag{Liénard-Wiechart potentials}
\]

Note that both position and velocity need to be calculated at the retarded time. These can be differentiated as before (but lengthly - see Griffiths 10.3.2 for example!) to give:

\[
E(r, t) = \frac{q}{4\pi \epsilon_0} \left\{ \frac{(\hat{\mathbf{R}} - \beta)(1 - \beta^2)}{(1 - \hat{\mathbf{R}} \cdot \beta)^3 R^2} + \frac{\hat{\mathbf{R}}}{(1 - \hat{\mathbf{R}} \cdot \beta)^3 R} \times \frac{1}{c} \left( (\hat{\mathbf{R}} - \beta) \times \dot{\beta} \right) \right\}
\]

\[
B(r, t) = \frac{1}{c} \hat{\mathbf{R}} \times E(r, t)
\]

where we have used \(\mathbf{R} = r - r_1\) to denote the fact that the charge is at \(r_1\) and not the origin, and \(\beta = \mathbf{v}/c\), and \(\dot{\beta} = \dot{\mathbf{v}}/c\). The first term represents the Doppler shifted Coulomb field (as we drew last time), and is the only term if there is no acceleration \((\dot{\beta} = 0)\). Note that remarkably it always points to the position of the particle at the time of observation \((\hat{\mathbf{R}} - \beta)\), since if light from the retarded time has travelled \(ct\) along \(\hat{\mathbf{R}}\), then in that time the particle has moved \(\beta ct\), and so its new position is \((\hat{\mathbf{R}} - \beta)ct\).

The second term is then the radiation field, and as before can be seen to fall as \(1/R\) rather than \(1/R^2\) as the Coulomb term does, and so dominates at large distance. The radiation term is then:

\[
E(r, t) = \frac{q}{4\pi \epsilon_0} \frac{\hat{\mathbf{R}}}{(1 - \hat{\mathbf{R}} \cdot \beta)^3 R} \times \frac{1}{c} \left( (\hat{\mathbf{R}} - \beta) \times \dot{\beta} \right) \tag{1}
\]

### 4.3 Power radiated

We can calculate the power emitted generally by the same technique as before, but it is quite complicated. Hence we look at two particular cases, which simplify the maths.

\(\beta \parallel \dot{\beta}\): In this case \(\beta \times \dot{\beta} = 0\), and the radiation field can be written,

\[
E(r, t) = \frac{q}{4\pi \epsilon_0} \frac{\hat{\mathbf{R}} \times (\hat{\mathbf{R}} \times \dot{\beta})}{(1 - \hat{\mathbf{R}} \cdot \beta)^3 Rc}
\]

If \(\theta\) is the viewing angle i.e. the angle between \(\dot{\beta}\) and \(\hat{\mathbf{R}}\), then the triple product \(|\hat{\mathbf{R}} \times (\hat{\mathbf{R}} \times \dot{\beta})| = \dot{\beta} \sin \theta\) and \((1 - \hat{\mathbf{R}} \cdot \beta) = (1 - \beta \cos \theta)\),

So one might think the emitted power \(\frac{dP}{d\Omega} \propto \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^6}\).

However, we must also account for the fact that the emitter is moving towards the observer, and so the energy emitted is arrives at a faster rate than the emitter thinks it emitted it. Consider the case of a machine-gun being fired out of a moving gun, you can see that the rate at which they hit the wall is increased by the same Doppler factor as before. Hence the power is conserved with another \((1 - \hat{\mathbf{R}} \cdot \beta)\) multiplied to the above answer.

So using \(dP = \frac{1}{2} \epsilon_0 |E_c|^2 c (1 - \beta \cos \theta) R^2 d\Omega\) finally one arrives at,

\[
\frac{dP}{d\Omega} = \frac{1}{4\pi \epsilon_0} \frac{q^2 \beta^2 \sin^2 \theta}{4\pi c (1 - \beta \cos \theta)^{10}}
\]

This is highly peaked in the direction of motion of the acceleration \((\theta = 0)\) for \(\beta \to 1\), as can be seen in the figures.
Figure 2: Bullets reach the target at a faster rate than emitted at the “source”.

Figure 3: $\beta = 0.1$  
Figure 4: $\beta = 0.5$  
Figure 5: $\beta = 0.9$

$\beta \perp \hat{\beta}$: This case is more important but even more tedious to derive. We quote the result for completeness.

$$\frac{dP}{d\Omega} = \frac{1}{4\pi\epsilon_0} \frac{q^2}{4\pi c} \frac{\hat{\beta}^2}{(1 - \beta \cos \theta)^3} \left[ 1 - \frac{\sin^2 \theta \cos^2 \phi (1 - \beta^2)}{(1 - \beta \cos \theta)^2} \right]$$

The angular variations are given in the figures below:

Figure 6: $\beta = 0.1$  
Figure 7: $\beta = 0.5$  
Figure 8: $\beta = 0.9$

Eqn.(1) can be directly integrated over all angles to give the total power;

$$P_{obs} = \frac{q^2}{4\pi\epsilon_0} \frac{2}{3c} \gamma^6 \left[ \hat{\beta}^2 - (\hat{\beta} \times \beta)^2 \right] = \frac{q^2}{4\pi\epsilon_0} \frac{2}{3c} \gamma^4 \left[ \hat{\beta}^2 - (\hat{\beta} \cdot \beta)^2 \right]$$
where \( \gamma = (1 - \beta^2)^{-1/2} \) is the usual relativistic factor. The two forms on the RHS are identical but help us identify that when \( \beta \parallel \dot{\beta}, \)

\[
P_{\text{obs}} = \frac{q^2}{4\pi\epsilon_0} \frac{2}{3c} \gamma^6 \dot{\beta}^2
\]

Note that the closer to \( c \) a particle’s velocity reaches, the more it will radiate, such that they can never actually reach \( c \). Lienard derived these equations in 1898, but did not seem to have realised that it implied nothing could exceed the speed of light.

When \( \beta \perp \dot{\beta}, \)

\[
P_{\text{obs}} = \frac{q^2}{4\pi\epsilon_0} \frac{2}{3c} \gamma^4 \dot{\beta}^2
\]

This type of radiation is what is emitted when a particle travels in a circular orbit, and was first seen in synchrotron (ring) accelerators, hence the name

### 4.4 Accelerator rings

An important consideration for high energy physics is the power radiated in accelerating the particles. Acceleration is usually limited by electrical breakdown (to \( \sim 10 \text{ MV/m} \)), so to reach LHC energies (\( \sim 1 \text{ TeV} = 10^{12} \text{ eV} \)) would take 100 km, too expensive. So instead particles are sent many times round a circular loop (\( r = 4.3 \text{ km} \)). But of course, particles in a circular orbit are being accelerated, and so radiate all the way round.

In this case \( a = v^2/r \rightarrow \dot{\beta} = \beta^2 c/r, \) then

\[
P_{\text{obs}} = \frac{q^2}{4\pi\epsilon_0} \frac{2c}{3} \frac{\beta^4 \gamma^4}{r^2}
\]

For high energy physics \( \beta \approx 1, \) so the power radiated increases by the forth power of energy (\( \gamma^4 \)). For LHC, \( 10^{14} \) protons circulate with a design energy of 7 TeV, (\( \gamma \approx 7.5 \times 10^3 \)), so energy loss is \( \sim 400 \text{ W}. \) However since \( \gamma = E/m \) a similar electron accelerator would have power loss \( 1836^4 \approx 10^{13} \) worse. Hence the next electron-positron collider has to be a linear one.
5. Materials

5.1 Maxwell’s equations in matter

Including the effects of materials, Maxwell’s equations are,

\[ \varepsilon \nabla \cdot \mathbf{E} = \rho_f \]
\[ \nabla \cdot \mathbf{B} = 0 \]
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]
\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{j}_c + \mu_\varepsilon \frac{\partial \mathbf{E}}{\partial t} \]

When a dielectric material is placed in an electric field, \( \mathbf{E}_0 \), the field is reduced to \( \mathbf{E} = \mathbf{E}_0 / \varepsilon_r \). An opposing electric field due to the ‘polarisability’ of the medium is created \( \mathbf{E}_P = -\mathbf{P} / \varepsilon_0 \). So reduced field strength is \( \mathbf{E} = \mathbf{E}_0 - \mathbf{P} / \varepsilon_0 = \mathbf{E}_0 / \varepsilon_r \), provided the material is linear, isotropic, homogenous (sometimes abbreviated LIH). Rearranging for \( \mathbf{P} \), we obtain,

\[ \mathbf{P} = \varepsilon_0 (\varepsilon_r - 1) \mathbf{E} = \chi \mathbf{E} \]

where \( \chi \), the susceptibility, is a constant only in LIH media.

So in Maxwell’s equations, \( \varepsilon = \varepsilon_r \varepsilon_0 \), where \( \varepsilon_r > 1 \) and accounts for the fact that dipoles within the material line up along the \( \mathbf{E} \) field and reduce its value within the dielectric.

![Figure 1](imag1.png)

**Figure 1:** A polar material, showing alignment of internal dipoles, and resulting polarisation field.

\( \mathbf{P} \) can be calculated from the fact that \( \varepsilon_0 \nabla \cdot \mathbf{E} = (\rho_f + \rho_p) \), where we now account for the polarisation charge, \( \rho_p \), created which reduces \( \mathbf{E} \). But we know that without the dielectric \( \varepsilon_0 \nabla \cdot \mathbf{E}_0 = \varepsilon_0 \nabla \cdot (\mathbf{E} + \mathbf{P} / \varepsilon_0) = \rho_f \). Subtracting the two expressions gives,

\[ \nabla \cdot \mathbf{P} = -\rho_p \]

which allows us to calculate the polarisation charge density. (Note the negative implies it opposes \( \mathbf{E} \)).

Sometimes \( \varepsilon \mathbf{E} = \varepsilon_0 \mathbf{E} + \mathbf{P} \) is called \( \mathbf{D} \), the electric displacement, so that Gauss’ Law (for example) becomes \( \nabla \cdot \mathbf{D} = \rho_f \).

Similarly, in the presence of a magnetic material, an applied \( \mathbf{B} \) can be increased (for paramagnetic or ferromagnetism) or decreased (in the case of diamagnetism); \( \mathbf{B} = \mu_r \mathbf{B}_0 = \mathbf{B}_0 + \mu_0 \mathbf{M} \), where \( \mu_0 \mathbf{M} \) is the increase (or decrease) in \( \mathbf{B} \) due to the material.

Since it is \( \mathbf{B}_0 \) which satisfies Maxwell’s equations for the external currents, we can define \( \mu_0 \mathbf{H} = \mathbf{B}_0 = \mathbf{B} - \mu_0 \mathbf{M} \). So \( \nabla \times \mathbf{H} = \mathbf{j}_c \). Confusingly, \( \mathbf{H} \) is also called the magnetic field.
Then $\nabla \times \mathbf{B} = \mu_0 (\mathbf{j}_c + j_m)$, which gives $\nabla \times \mathbf{M} = j_m$, where $j_m$ is the magnetisation current. Just as the effect of polarisation can be accounted for by a relative permittivity, the effect of magnetisation can be accounted for by a relative permeability, so that $\mu = \mu_r \mu_0$, and instead of $\nabla \times \mathbf{H} = \mathbf{j}_c$, we can use $\nabla \times \mathbf{B} = \mu_0 \mu_r j_c$. $\mathbf{D}$ and $\mathbf{H}$ come from a time prior to the atomistic view of matter (giving the fields due to external charges and currents only), and it can be argued how useful they are. We will always use $\epsilon \mathbf{E}$ instead of $\mathbf{D}$ and $\mathbf{B}/\mu$ instead of $\mathbf{H}$ wherever required.

5.2 Boundary conditions

Consider the boundary between two materials with permittivities $\epsilon_1$ and $\epsilon_2$. We can examine the components of the fields either side of the boundary, as shown in the figures below;

![Figure 3: E and B fields at a boundary](image)

First consider Gauss’ Law in the left-hand figure, in integral form

$$\int \nabla \cdot \epsilon \mathbf{E} = \oint_S \epsilon \mathbf{E} \cdot d\mathbf{S} = \int \rho_f d\tau.$$  

$\rho_f$ can be given by $\sigma_c \delta (0)$, so $\int \rho_f d\tau = \int \sigma_c dS$. Hence,

$$\epsilon_1 \mathbf{E}_{1\perp} \cdot d\mathbf{S}_1 - \epsilon_2 \mathbf{E}_{2\perp} \cdot d\mathbf{S}_2 + \epsilon \mathbf{E}_\parallel \cdot l dx \simeq \sigma_c dS$$

In the limit, $dx \to 0$, $dS_1 \simeq dS_2 \simeq dS$, then $\epsilon_1 \mathbf{E}_{1\perp} - \epsilon_2 \mathbf{E}_{2\perp} = \sigma_c$, or $\epsilon \mathbf{E}_\perp$ is discontinuous by $\sigma_c$. Similarly $\nabla \cdot \mathbf{B} = 0$, over a similar integral implies $\mathbf{B}_\perp$ is continuous.

Now consider Faraday’s law integrated over the surface in the second part of fig. 3.

$$\int \nabla \times \mathbf{E} \cdot d\mathbf{S} = - \int \frac{\partial \mathbf{B}}{\partial t} d\mathbf{S}$$

By Stoke’s theorem, this becomes

$$\oint \mathbf{E} \cdot d\mathbf{l} = - \int \frac{\partial \mathbf{B}}{\partial t} d\mathbf{S}.$$  

Integrating over the path shown in the figure,

$$E_{1\parallel} \ell + E_{1\perp} dx - E_{2\parallel} \ell - E_{2\perp} dx = \frac{\partial \mathbf{B}}{\partial t} \ell dx$$

Since $\frac{\partial \mathbf{B}}{\partial t}$ must be finite, then all the terms with $dx$ disappear as $dx \to 0$. Then $E_{1\parallel} = E_{2\parallel}$. Hence $E_\parallel$ is continuous across the boundary.

Finally, one can show

$$\oint \mathbf{B}/\mu \cdot d\mathbf{l} = \int \left( \mathbf{j}_c + \epsilon \frac{\partial \mathbf{E}}{\partial t} \right) \cdot d\mathbf{S}$$

or

$$B_{1\parallel} \ell/\mu_1 + B_{1\perp} dx/\mu_1 - B_{2\parallel} \ell/\mu_2 - B_{2\perp} dx/\mu_2 = \int j_c \ell dx + \epsilon \frac{\partial \mathbf{E}}{\partial t} \ell dx$$
If the current is a surface current \( j_c = j'_c \delta(0) \), then the integral just becomes
\[
\int j_c \, d\tau = \int j'_c \delta(0) \, \ell \, dx = j'_c \ell
\]
(since \( \int \delta(0) \, dx = 1 \)).

So with \( \frac{\partial \vec{E}}{\partial t} \) finite, we find as \( dx \to 0 \),
\[
B_1/\mu_1 - B_2/\mu_2 = j'_c \text{ (current per unit length).}
\]

\( B_\parallel/\mu \) is discontinuous across the boundary by \( j'_c \).

In summary, across a boundary;
\[
\begin{align*}
\epsilon E_\perp & \text{ is discontinuous by } \sigma_c \\
B_\perp & \text{ is continuous} \\
B_\parallel/\mu & \text{ is discontinuous across the boundary by } j'_c
\end{align*}
\]

### 5.3 EM waves in dielectrics

Searching for solutions of the form \( \vec{\psi} = \vec{\psi}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \), ME reduce in a dielectric (no \( \rho_f \) or \( j_c \)) to,
\[
\begin{align*}
(1) & \quad \vec{k} \cdot \vec{E} = 0 \\
(2) & \quad \vec{k} \cdot \vec{B} = 0 \\
(3) & \quad \vec{k} \times \vec{E} = i\omega \vec{B} \\
(4) & \quad \vec{k} \times \vec{B} = -i\omega \mu_0 \epsilon \vec{E}
\end{align*}
\]

Almost the same as in vacuum except with \( \epsilon_0 \to \epsilon \), (note \( \mu_r \approx 1 \) except for ferromagnets.)

As before (1), (2) imply \( \vec{E} \perp \vec{k} \), and (3), (4) state that \( \vec{E} \perp \vec{B} \).

taking \( i\vec{k} \cdot \) of 3\textsuperscript{rd} eqn
\[
\vec{k} \times i(\vec{k} \times \vec{E}) = -\vec{k} \times (\vec{k} \times \vec{E}) = -[\vec{k}(\vec{k} \cdot \vec{E}) - k^2 \vec{E}] = i\vec{k} \times i\omega \vec{B} = -(-\mu_0 \omega^2 \vec{E})
\]

since \( \vec{k} \cdot \vec{E} = 0 \), we have,
\[
k^2 = \epsilon \mu_0 \omega^2 \quad \Rightarrow \quad \omega = c'k
\]
dispersion relation

where \( c' = (\epsilon \mu_0)^{-1/2} = (\epsilon_r \epsilon_0 \mu_0)^{-1/2} = c/\eta \)

where \( \eta = \epsilon_r^{1/2} \) is the refractive index.

Note that from (3), we immediately obtain \( |\vec{E}| = (\omega/k)|\vec{B}| = c'|\vec{B}| \). Hence \( B \) increases from its vacuum value, \( B = \eta E/c = (\epsilon_r)^{1/2} E/c \).

Since for most materials \( \epsilon_r > 1 \), so \( \eta > 1 \) in dielectrics (e.g. \( \eta \approx 1.5 \) for most glasses and \( c' = 2.4 \times 10^8 \text{ ms}^{-1} \)).
6. Reflections

6.1 Boundary conditions

Across a dielectric boundary ($\mu_r \approx 1; \rho_f = 0; j_c = 0$):

$\epsilon E_\perp$ is continuous $B_\perp$ is continuous.

$E_\parallel$ is continuous $B_\parallel$ is continuous.

6.2 Reflection at a boundary

An em wave is incident normally on a dielectric boundary, as in figure 1. The incident, reflected and transmitted waves can be represented as:

$E_i = E_{i0}e^{i(kz-\omega_i t)}$; $E_r = E_{r0}e^{i(kz-\omega_r t)}$; $E_t = E_{t0}e^{i(kz-\omega_t t)}$.

![Figure 1: EM wave incident normally on dielectric boundary.](image1)

At the boundary $z = 0$, so for the three waves to be “phase-matched” at all times, then $\omega_i = \omega_r = \omega_t$. (NB $E_{i0}$, $E_{t0}$, $E_{r0}$ can include a constant phase offset by being complex).

Continuity of $E_\parallel$,

$E_{i0} + E_{r0} = E_{t0}$  \hspace{1cm} (6.1)

Continuity of $B_\parallel$,

$B_{i0} - B_{r0} = B_{t0}$

$\rightarrow (\eta_1 E_{i0}/\phi) - (\eta_1 E_{r0}/\phi) = (\eta_2 E_{t0}/\phi)$  \hspace{1cm} (6.2)

Adding $\eta_1 \times (6.1)$ and (6.2) gives,

$t = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_1}{\eta_1 + \eta_2}$

Subtracting (6.2) from $\eta \times (6.1)$ gives,

$r = \frac{E_{r0}}{E_{i0}} = \frac{\eta_1 - \eta_2}{2\eta_1} \frac{2\eta_1}{\eta_2 + \eta_1} = \frac{\eta_1 - \eta_2}{\eta_2 + \eta_1}$

![Figure 2: r, t versus $\eta_1/\eta_2$](image2)

Figure 2 plots the reflection and transmission coefficients as a function of $\eta_1/\eta_2$.

If $\eta_1 > \eta_2$, (e.g. glass to air); $r > 0$, $E_{r0}$ and $E_{t0}$ are in phase at the boundary, (so $B_{i0}$ and $B_{r0}$ out of phase by $\pi$)
if $\eta_1 < \eta_2$, (e.g. air to glass); $r < 0$, $E_{i0}$ and $E_{e0}$ out of phase by $\pi$, (so $B_{i0}$ and $B_{e0}$ are in phase at the boundary).

6.3 Intensity

Note $r^2 + t^2 \neq 1$. But Poyting vector also depends on $c'$, $N = \epsilon_r \epsilon_0 E^2 c' = \epsilon_r^{1/2} \epsilon_0 E^2 c = \eta \epsilon_0 E^2 c$.

So ratios of energy reflected and energy transmitted to energy incident, $R$ and $T$ respectively, are given by,

$$R = \frac{c_1 E_{r0}^2}{c_1 E_{i0}^2} = |r|^2 = \left| \frac{\eta_1 - \eta_2}{\eta_2 + \eta_1} \right|^2; \quad T = \frac{\eta_2 E_{t0}^2}{\eta_1 E_{i0}^2} = \left( \frac{\eta_2}{\eta_1} \right) |t|^2 = \left| \frac{4\eta_2 \eta_1}{(\eta_2 + \eta_1)^2} \right|$$

where the $\left( \frac{\eta_2}{\eta_1} \right)$ accounts for the difference in velocities and permittivities. Thankfully, $R + T = 1$ (as it should to conserve energy). For glass $\eta \approx 1.5$, so $r = 0.2$ and $R = 0.04 = 4\%$ reflection.

NB, $E$ has units volts/metre, and we said $B/\mu$ can be generated by amps/metre, so $\mu E/B$ has units of impedance. Hence reflection off a dielectric surface can be considered to be due to impedance mismatch. The impedance of free air (to EM radiation), $Z_0 = \mu_0 E/B = \mu_0 c = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377 \Omega$. 

2
7. Fresnel’s Equations

7.1 Reflection at an angle

Consider incidence of an em wave on to the boundary at an angle of incidence \( \theta \).

The electric field of the waves are:

\[
\mathbf{E}_i = E_i e^{i(k_i \cdot r - \omega t)} \hat{y}; \quad \mathbf{E}_r = E_r e^{i(k_r \cdot r - \omega t)} \hat{y}; \quad \mathbf{E}_t = E_t e^{i(k_t \cdot r - \omega t)} \hat{y}
\]

Note again \( \omega \) same through-out. For phase-matching at \( z = 0 \) for all \( x \), so, \( k_{xi} = k_{xr} = k_{xt} \).

\[
\rightarrow k_i \sin \theta_i = k_r \sin \theta_r = k_t \sin \theta_t
\]

\[
\rightarrow \eta_1 (\omega/c) \sin \theta_i = \eta_1 (\omega/c) \sin \theta_r = \eta_2 (\omega/c) \sin \theta_t
\]

So \( \theta_i = \theta_r \quad \text{(reflection, angle of incidence = angle of reflection)} \)

and \( \eta_1 \sin \theta_i = \eta_2 \sin \theta_t \quad \text{(Snell’s Law)} \).

7.2 Fresnel coefficients

**E∥ surface – s-polarisation:**

If all \( \mathbf{E} \) are in \( y \)-direction, and \( \mathbf{B} \) as given in diagram 1a with magnitude \( B = E/\nu_\phi = \eta E/c \).

Continuity of \( E_\parallel \) gives,

\[
\rightarrow E_i + E_r = E_t \quad (7.1)
\]

and continuity of \( B_\parallel \),

\[
B_i \cos \theta_i - B_r \cos \theta_r = B_t \cos \theta_t \quad \text{where we used } \theta_i = \theta_r
\]

\[
\rightarrow \eta_1 E_i \cos \theta_i - \eta_1 E_r \cos \theta_r = \eta_2 E_t \cos \theta_t \quad (7.2)
\]

and multiplying (7.1) by \( \eta_1 \cos \theta_i \),

\[
\eta_1 E_i \cos \theta_i + \eta_1 E_r \cos \theta_r = \eta_1 E_t \cos \theta_i \quad (7.3)
\]

Adding (7.2) and (7.3) gives

\[
t = \frac{E_t / E_i}{\eta_1 \cos \theta_i + \eta_2 \cos \theta_t}
\]
Subtracting (7.2) from (7.3) gives
\[ r = \frac{E_r}{E_i} = \frac{\eta_1 \cos \theta_i - \eta_2 \cos \theta_t}{2\eta_1 \cos \theta_i} \cdot t = \frac{\eta_1 \cos \theta_i - \eta_2 \cos \theta_t}{\eta_1 \cos \theta_i + \eta_2 \cos \theta_t} \]

**B III surface – p-polarisation:**
Now all \( B \) in the \( y \)-direction, and \( E \) as shown in diagram 1b.

Take continuity of \( E_\parallel (= E_x) \), \[ E_i \cos \theta_i + E_r \cos \theta_r = E_t \cos \theta_t \] (7.4)

Continuity of \( B_\parallel \) gives, \[ B_i - B_r = B_t \rightarrow \eta_1 E_i/c - \eta_1 E_r/c = \eta_2 E_t/c \] (7.5)
multiply (7.4) by \( \eta_1 \) and (7.5) by \( \cos \theta_i \) through-out and using \( \theta_r = \theta_i \)

\[ \eta_1 E_i \cos \theta_i + \eta_1 E_r \cos \theta_i = \eta_1 E_t \cos \theta_t \]
\[ \eta_1 E_i \cos \theta_i - \eta_1 E_r \cos \theta_i = \eta_2 E_t \cos \theta_i \]

Adding gives,
\[ t = \frac{E_t / E_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_i} \]

Subtracting gives \( r = \frac{E_r}{E_i} = \frac{\eta_1 \cos \theta_i - \eta_2 \cos \theta_t}{2\eta_1 \cos \theta_i} \cdot t = \frac{\eta_1 \cos \theta_i - \eta_2 \cos \theta_t}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t} \)

### 7.3 Brewster’s Angle

![Figure 2: Fresnel’s coefficients as a function of angle for transition from air to glass.](image)

Consider when light travels from low to high refractive index, say air to glass, as in figure 2. For \( s \)-polarisation (polarisation perpendicular to the plane of the reflection), \( r < 0 \) always. For \( p \)-polarisation, at \( \theta = 0 \), \( r_p = r_s = 0.2 \) as they should, corresponding to 4% reflection. But they diverge, so that \( r_p \), passes through \( r_p = 0 \), implying for a given angle - Brewster’s angle - there is no reflection for this polarisation.

This happens when \( \eta_2 \cos \theta_i - \eta_1 \cos \theta_t = 0 \). Using Snell’s law \( \eta_1 \sin \theta_i = \eta_2 \sin \theta_t \),
\[ \rightarrow \sin \theta_i \cos \theta_i = \sin \theta_i \cos \theta_t \rightarrow \sin 2\theta_i = \sin 2\theta_t. \]

Since \( 0 < \theta_i, \theta_t < \pi/2 \), then implies \( \theta_t = \pi/2 - \theta_i \), so \( \cos \theta_t = \sin \theta_i \).
Putting this finally into initial condition, \( \eta_2 \cos \theta_i - \eta_1 \sin \theta_i = 0; \)

\[ \Rightarrow \tan \theta_B = \frac{\eta_2}{\eta_1} \]  

(Brewster’s Angle)

For air-glass interface \( \theta_B = 56.3^\circ \). Hence the reflection off a plate at Brewster’s angle can act as a polariser (completely cuts \( p \)-polarisation). Alternatively transmission through multiple plates at Brewster’s angle preferentially reflects \( s \), so only leaving the \( p \)-polarisation.

At large angle of incidence (grazing incidence) the reflectivity of both \( s \) and \( p \) polarisation tend to 1, though in the case of \( p \) there is a phase change of \( \pi \).

### 7.4 Total Internal Reflection

Figure 3 shows the corresponding plots of \( r \) and \( t \) for propagation from high to low refractive index (e.g. glass to air) as a function of angle. The curves are similar (with \( p \)-polarisation having a Brewster angle once more). One can see that above a certain critical angle \( \theta_{cr} \), \( t = 0, r = 1 \). This is total internal reflection. Snell’s law fails for \( \sin \theta_i = (\eta_1/\eta_2) \sin \theta_i > 1 \), since \( \eta_1 > \eta_2 \). Hence;

\[ \Rightarrow \sin \theta_{cr} = \frac{\eta_2}{\eta_1}. \]  

(Critical Angle)

\[ \text{Figure 3: Fresnel’s coefficients as a function of angle for transition from glass to air.} \]

### 7.5 Evanescence

For \( \theta > \theta_{cr} \), we can write \( \cos \theta_i = (1 - \sin^2 \theta_i)^{1/2} = (1 - (\eta_1/\eta_2)^2 \sin^2 \theta_i)^{1/2} = ib \). So both the reflected coefficients can be written as \( r = (a - ib)/(a + ib) \), i.e. complex numbers with amplitude 1 (\( |r| = 1 \)) and phase between \( \phi \) between 0 and \( \pi \).

Writing them in complex form, the reflected waves become \( E_r = E_i e^{i\phi} = E_{i0} e^{i(k \cdot r - \omega t + \phi)} \).

Now considering the transmitted wave, it becomes \( E_t = E_{i0} e^{i(k \sin \theta_t \cdot x + k \cos \theta_t \cdot z - \omega t)} = E_{t0} e^{i(k_m \cdot x - \omega t)} e^{-\omega bx/c} \),

where we have used \( k = \omega/c \) and \( k_m = k \sin \theta_t = (\omega/c) \cdot (\eta_1/\eta_2) \sin \theta_i \). The solution is a surface wave along with a field that decays on the order of \( (c/\omega b) \).

The evanescent region can be detected by bringing a second high refractive index object close to the other one (frustrated internal reflection!).

3
8. Plasmas

8.1 Plasmas

Plasmas are materials where the electrons can move “free” of their parent atoms, which usually means they have a significant fraction of ionised atoms.

Though not common on earth, plasma make up > 99% of the “observable” universe. Some common examples of plasmas include stars, tokamaks, the northern lights, lightning... even the electrons in a metal can be considered as a plasma.

8.2 EM waves in a plasma

For a plasma, there may be both currents and variations in charge density, but we can ignore the effect of polarisation and magnetisation, so \( \epsilon = \epsilon_0 \) and \( \mu = \mu_0 \). So Maxwell’s Equations can be written as:

\[
\begin{align*}
(1) \quad & \epsilon_0 \nabla \cdot E = \rho \\
(2) \quad & \nabla \cdot B = 0 \\
(3) \quad & \nabla \times E = -\frac{\partial B}{\partial t} \\
(4) \quad & \nabla \times B = \mu_0 j + \mu_0\epsilon_0 \frac{\partial E}{\partial t}
\end{align*}
\]

We investigate wave solutions of the form \( \psi = \psi_0 e^{i(k \cdot r - \omega t)} \). ME reduce to,

\[
\begin{align*}
(1) \quad & ik \cdot E = \rho \\
(2) \quad & ik \cdot B = 0 \\
(3) \quad & ik \times E = i\omega B \\
(4) \quad & ik \times B = \mu_0 j - i\omega \mu_0\epsilon_0 E
\end{align*}
\]

Since the electrons are free in the plasma, they can respond to any external fields. Their equation of motion is given by the Drude model:

\[
m \frac{\partial v}{\partial t} = -e(E + v \times B) - \frac{mv}{\tau}
\]

where the first term is the Lorentz force, and the second is a “drag” term due to collisions occurring on an average time \( \tau \). Note we can ignore the effect of the ions since they will move \( (M/m) \) times slower in response to the same fields. Also the effect of the magnetic field can be ignored for \( v \ll c \), since for em waves \( |F_B| = evB = e(v/c)E = (v/c)F_E \ll F_E \).

Assuming \( v = v_0 e^{i(k \cdot r - \omega t)} \), \( \frac{\partial v}{\partial t} = (-i\omega)v_0 e^{i(k \cdot r - \omega t)} = -i\omega v \), so eqn. of motion becomes,

\[
-im\omega v = -eE - \frac{mv}{\tau} \quad \rightarrow \quad v = \frac{-eE}{m\left(\frac{1}{\tau} - i\omega\right)}
\]

We’ll consider the case that the collision rate is greater than the angular frequency next lecture, but in the ‘collisionless’ case \( (1/\tau < \omega) \), this implies there the current density \( j = -nev \) is given by;

\[
j = \frac{ne^2 E}{im\omega} = \frac{im^2 e^2 E}{m\omega} \quad \text{where } n \text{ is the electron density.}
\]

And Ampère-Maxwell becomes: \( k \times B = \frac{\mu_0 ne^2 E}{m\omega} - \omega \mu_0\epsilon_0 E \).
Now taking \( \mathbf{k} \times \) Faraday’s law,

\[
\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) = \mathbf{k}(\mathbf{k} \cdot \mathbf{E}) - k^2 \mathbf{E} = \mathbf{k} \times \mathbf{B} = \frac{\mu_0 ne^2 \mathbf{E}}{m} - \epsilon_0 \mu_0 \omega^2 \mathbf{E}
\]

Now \( \mathbf{k} \cdot \mathbf{E} = 0 \) is not necessarily true. But if we insist on looking at transverse waves only, i.e. \( \mathbf{k} \cdot \mathbf{E} = 0 \) (which implies the charge density \( \rho = 0 \)), then:

\[
k^2 \mathbf{E} = -\frac{\mu_0 ne^2 \mathbf{E}}{m} + \epsilon_0 \mu_0 \omega^2 \mathbf{E}
\]

multiply through-out by \( c^2 = (1/\epsilon_0 \mu_0) \) gives,

\[
c^2 k^2 = -\frac{ne^2}{\epsilon_0 m} + \omega^2 \quad \text{or} \quad c^2 k^2 = \omega^2 - \omega_p^2
\]

plasma dispersion relation (8.1)

where we have defined the plasma frequency \( \omega_p = \left( \frac{ne^2}{\epsilon_0 m} \right)^{1/2} \).

Figure 1: Dispersion relation for em waves in a plasma.

8.3 Dispersion relations

We plot \( \omega \) as a function of \( k \) in figure 1. One can see that for \( \omega < \omega_p \) there are no corresponding value of \( k \). Indeed \( k \) goes to zero for \( \omega = \omega_p \).

We can consider the wave velocities to investigate what is happening, first to find \( v_{ph} = \omega/k \). Take (8.1) and divide through by \( \omega \),

\[
c^2 k^2/\omega^2 = 1 - \omega_p^2/\omega^2
\]

Inverting, \( \omega^2/c^2 k^2 = (1 - \omega_p^2/\omega^2)^{-1} \)

So, \( \omega/k = c (1 - \omega_p^2/\omega^2)^{-1/2} \)

Note, since \( \omega > \omega_p \), then the refractive index \( \eta = (1 - \omega_p^2/\omega^2)^{1/2} < 1 \) (see figure 2), and so \( v_{ph} > c \). But causality is not violated since no energy or signals travel at \( v_{ph} \).

To find group velocity, differentiate dispersion relation:

\[
2c^2 k dk = 2 \omega d\omega
\]
\[ v_g = \frac{\partial \omega}{\partial k} = c^2 k/\omega = c^2/v_{ph} = \eta c = c(1 - \omega_p^2/\omega^2)^{1/2} \]

Luckily \( v_g < c \), since \( \eta < 1 \) in a plasma.

### 8.4 Cut-offs

Since \( \omega > \omega_p = \left( \frac{m_e^2}{\epsilon_0 m} \right)^{1/2} \), this implies a maximum density to which radiation of angular frequency \( \omega_p \) can propagate. Inverting this expression, this critical density is given by;

\[ n_{cr} = \frac{\epsilon_0 m \omega^2}{e^2} \]

If a ray falls normally on a plasma surface, then this is indeed the maximum density it will propagate to. However consider a ray falling on a linear plasma ramp, as in the figure 4. The ray will be deflected by the ramp. The condition for the wave to no longer propagate up the ramp is \( k_\parallel = 0 \) (= \( k_z \) in the diagram).

If light is initially incident on a density ramp (in the \( x \) direction) with angle \( \theta \), then we can write the dispersion relation for a wave \((\omega_0, k_0)\) where the subscript 0, refer to their values in vacuum,

\[ \omega_0^2 = \omega_p^2(z) + c^2(k_x^2 + k_z^2) \]

as

\[ \omega_0^2 = \omega_p^2(z) + c^2(k_x^2 + k_0^2 \sin^2 \theta) \]

where the wavenumber is unaffected in the direction perpendicular to the density ramp. Using \( \omega_0 = c k_0 \) and noting that propagation stops when \( k_z = 0 \), and that \( k_x = k_0 \sin \theta \),

\[ \omega_p^2(z) = c^2 k_0^2 (1 - \sin^2 \theta) = c^2 k_0^2 \cos^2 \theta \]

So the light is reflected when,

\[ \frac{\omega_p^2}{\omega_0^2} = \frac{n_e}{n_{cr}} = \cos^2 \theta \]

If \( \omega_p^* \) is the maximum density of the layer, then at any given angle the maximum frequency that can be reflected is given by \( \omega = \omega_p^*/\cos \theta \). This can be shown to be the same as the condition for total internal reflection at a plasma boundary.

---

Figure 3: Group and phase velocities for electromagnetic waves in a plasma.

Figure 4: Reflection of an electromagnetic wave at a plasma boundary.
9. Conductors

9.1 Ohm’s Law

As we found last time, including the effect of collisions the current density is given by:

\[ j = -nev = \frac{ne^2E}{m \left( \frac{1}{\tau_c} - i\omega \right)} \]

If collisions dominate \((\frac{1}{\tau_c} > \omega)\), then for a material with density of conductors \(n\), the current density is given by,

\[ j = nqv = \frac{ne^2\tau_cE}{m} \]

Defining, \(\sigma = \frac{ne^2\tau_c}{m}\) (note the sign of the charge was unimportant because of the \(e^2\)), then one can write,

\[ j = \sigma E \]  
(Ohm’s Law)

If this doesn’t strike you as being the Ohm’s Law you know, remember that the resistance of an object is written as \(R = \rho l/A\), where \(\rho\) is a material property, resistivity - the resistance increases for longer length and thinner wire, as expected. We can rewrite Ohm’s Law above as,

\[ j \cdot A = \left( \frac{\sigma A}{l} \right) (E \cdot l) \]

where we recognise \(j \cdot A = I\), the current, and \(E \cdot l = V\), the potential difference. So,

\[ I = \frac{\sigma A}{l} V \quad \rightarrow \quad V = IR \]

provided \(\frac{l}{\sigma A} = R\), which comparing above is the same if \(\rho = 1/\sigma\).

9.2 Maxwell’s equations in conducting media

Using Ohm’s Law for \(j_c\), Maxwell’s Equations become in a conductor (which carries currents but has no free charges):

\[
\begin{align*}
(1) \quad \epsilon \nabla \cdot \mathbf{E} &= 0 \\
(2) \quad \nabla \cdot \mathbf{B} &= 0 \\
(3) \quad \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
(4) \quad \nabla \times \mathbf{B} &= \mu \sigma \mathbf{E} + \mu \epsilon \frac{\partial \mathbf{E}}{\partial t}
\end{align*}
\]

From Faraday: \(\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial (\nabla \times \mathbf{B})}{\partial t}\)

But \(\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E}\), since \(\nabla \cdot \mathbf{E} = 0\)

So \(\nabla^2 \mathbf{E} = \mu \sigma \frac{\partial \mathbf{E}}{\partial t} + \mu \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}\)
where we substituted Ampère-Maxwell also multiplied throughout by $-1$.

Searching for transverse solutions of the form $\mathbf{E} = E_0 e^{i(k \cdot r - \omega t)}$,

$$-k^2 \mathbf{E} = -i \omega \mu \sigma \mathbf{E} - \mu \epsilon \omega^2 \mathbf{E}$$

$$k^2 = \epsilon \mu \omega^2 + i \mu \sigma \omega$$

dispersion relation in metal

and using $k = \frac{\omega \eta}{c}$, we also obtain $\eta^2 = \epsilon \mu \epsilon r \left(1 + \frac{i \mu \sigma}{\omega \epsilon}\right)$

Note that the wavenumber and refractive index will now both be complex.

### 9.3 Good conductor, $\sigma \gg \epsilon \omega$

Note that $\sigma/\epsilon \omega$ is the ratio of conduction to displacement current. For good conductors, (high $\sigma$) $\sigma \gg \epsilon \omega$, then the conduction current term dominates in the dispersion relation and,

$$k^2 = i \mu \sigma \omega \rightarrow k = \sqrt{i \mu \sigma \omega} = \frac{1 + i \delta}{\sqrt{2}}$$

where $\delta = \sqrt{2/\mu \sigma \omega}$ is called the skin depth. In this case,

$$\mathbf{E} = E_0 e^{i(k \cdot r - \omega t)} = E_0 e^{-z/\delta} e^{i(k z - \omega t)}$$

For Cu, $\sigma \approx 10^7 \Omega^{-1} \text{m}^{-1}$, $\epsilon \approx \epsilon_0$, so Cu is a good conductor for $\omega \ll 10^{18} \text{ rad/s}$, or $\lambda \gg 2 \times 10^7 \text{ m}$. Anything other than x-rays are strongly attenuated at the surface on a length scale $\sim \delta$.

Choose direction of propagation to be $z$ and $\mathbf{E}$ to be in $x$, 

$$\mathbf{E} = E_0 e^{i(z/\delta - \omega t)} e^{-z/\delta} \mathbf{x}$$

From Faraday’s Law, $-\dot{\mathbf{B}} = \nabla \times \mathbf{E} \rightarrow i \omega \mathbf{B} = i \mathbf{k} \times \mathbf{E}$

$$\omega B_y = k E_x = e^{i\pi/4} \sqrt{\mu \sigma \omega} E_x$$

$$\rightarrow B_y = e^{i\pi/4} \left(\frac{\mu \sigma}{\omega}\right)^{1/2} E_x$$

(9.1)

$\mathbf{B}$ lags $\mathbf{E}$ by $\pi/4$.

### 9.4 Poor Conductor, $\sigma \ll \epsilon \omega$

Try a solution of the type,

$$\mathbf{E} = E_0 e^{i((k + i\alpha) z - \omega t)} \mathbf{x} = E_0 e^{-\alpha z} e^{i(k z - \omega t)} \mathbf{x}$$

So, $(k + i\alpha)^2 = \epsilon \mu \omega^2 + i \mu \sigma \omega$

$$\rightarrow k^2 - \alpha^2 = \epsilon \mu \omega^2$$

and $2k\alpha = \mu \sigma \omega$
Figure 1: $\mathbf{E}$ and $\mathbf{B}$ field of EM wave propagating in a good conductor (magnitudes normalised to one).

Since $\sigma \ll \epsilon \omega$, $k \gg \alpha$

$\therefore \frac{\omega}{k} = (\epsilon \mu)^{-1/2} = c/\eta$ where $\eta = (\epsilon)$ as before for dielectrics.

and $\alpha = \left(\frac{\mu \sigma}{2}\right) \left(\frac{\omega}{k}\right) = \left(\frac{c}{2\eta}\right) \sigma \mu = \left(\frac{\mu}{\epsilon}\right)^{1/2} \frac{\sigma}{2}$

A gently damped wave, whose decay length $\alpha^{-1} = \frac{2\eta}{\sigma \mu}$

(Note since units of $\sigma$ are mhos/metre, this means $Z = (\mu/\epsilon)^{1/2}$ has units of impedance too. For $\epsilon_r \to 1$ and $\mu_r \to 1$, then $Z = Z_0 = (\mu_0/\epsilon_0)^{1/2} \approx 377 \Omega$, the impedance of free space.)

Again from Faraday’s Law, $-\dot{\mathbf{B}} = \nabla \times \mathbf{E} \rightarrow \omega B_y = (k + i\alpha) E_x$

$\rightarrow B_y = (k/\omega)(1 + i\alpha/k)E_x$

$\mathbf{B}$ lags $\mathbf{E}$ by $\tan^{-1}(\alpha/k) = \tan^{-1}\frac{\sigma}{2\omega \epsilon}$.

Again this is dependent on ratio of conduction to displacement term. In the case $\sigma \to 0$, then wave propagates with $\mathbf{E}$ and $\mathbf{B}$ in phase (as in dielectric).

Figure 2: Weak attenuation ($\omega = 2.10^{15} \text{ rad s}^{-1}$, $\sigma = 10^2 \text{ mhos m}^{-1}$)

Figure 3: Stronger attenuation ($\omega = 2.10^{15} \text{ rad s}^{-1}$, $\sigma = 3.10^3 \text{ mhos m}^{-1}$)
10. Metals

10.1 Reflections from metals

For metals, conductivity is high \( \sigma \approx 10^7 \ \Omega^{-1} m^{-1} \). For visible light \( \omega \approx 4 \times 10^{15} \ \text{rads}^{-1} \), so \( \sigma / \varepsilon_0 \omega \approx 250 \). From last lecture we had \( k = \sqrt{\mu \sigma \omega e^{i \pi/4}} = 1 + i \delta \) but \( k = \eta \omega / c \). Hence \( |\eta| \approx (\sigma / \varepsilon_0 \omega)^{1/2} \gtrsim 15 \) is large. This means that the speed of light in a metal is very slow \( c_m = c / \eta \approx 10^7 \ \text{ms}^{-1} \) - for radio waves it can be slower than the speed of sound!

Boundary conditions for a metal have to remain generality,

\[
\varepsilon E_\perp \text{ changes by } \sigma c, \quad E_\parallel, B_\perp \text{ continuous}, \quad B_\parallel / \mu \text{ changes by } j_\parallel.
\]

However for normal conductors, the currents must remain finite (if \( j_\parallel \) was infinite - a delta function - then \( E = j_\parallel / \sigma \) would also be infinite, and the energy in the field \( \propto E^2 dx \) would also be infinite!). This implies that \( B_\parallel \) is continuous (except for superconductors).

Snells’ law applies as before \( \sin \theta_i = \eta \sin \theta_t \), but since \( \eta \) is large, then \( \theta_t \to 0 \) for all \( \theta_i \). Hence the wave flows almost normal to the surface inside the metal, irrespective of incident angle.

Consider normal incidence, as before (see lecture 5);

\[
r = \frac{1 - \eta}{1 + \eta} \quad \text{and} \quad t = \frac{2}{1 + \eta} \quad \text{and} \quad R = \left| \frac{1 - \eta}{1 + \eta} \right|^2
\]

This means that \( r \approx -\frac{\eta}{\eta} = -1 \). Hence the wave is almost perfectly reflected but always with a phase change of \( \pi \). \( t \) is consequently small.

But reflection is not quite perfect. To calculate energy reflected back must remember that \( \eta \) is complex, \( \eta = (\sigma / 2\varepsilon_0 \omega)^{1/2} (1 + i) = \eta_0 (1 + i) \), (taking \( \mu_r \approx 1 \)).

\[
R = \left| \frac{1 - \eta}{1 + \eta} \right|^2 = \left| \frac{(1 - \eta_0) - i \eta_0}{(1 + \eta_0) + i \eta_0} \right|^2 = \frac{(1 - \eta_0)^2 + \eta_0^2}{(1 + \eta_0)^2 + \eta_0^2} = \frac{1 - 2\eta_0 + 2\eta_0^2}{1 + 2\eta_0 + 2\eta_0^2}
\]
For $\eta_0 \gg 1$, 

$$R \approx \frac{I - 2\eta_0 + 2\eta_0^2}{I + 2\eta_0 + 2\eta_0^2} = \frac{\eta_0 - 1}{\eta_0 + 1} = \frac{(\eta_0 - 1)^2}{\eta_0^2 - 1} \approx 1 - \frac{2\eta_0}{\eta_0^2} = 1 - \left(\frac{8\epsilon_0\omega}{\sigma}\right)^{1/2}$$

So using the numbers given above, the fraction not reflected $= \left(\frac{8\epsilon_0\omega}{\sigma}\right)^{1/2} \lesssim 20\%$

Note that reflection losses increase with frequency $\propto \omega^{1/2}$. Hence metals are better reflectors of infra-red, radiowaves (and partly explaining the brownish / golden tint of Cu, Au.)

### 10.2 Energy Flow

Using $N = \frac{1}{\mu}(E \times B)$, so,

$$\langle N \rangle = \left\langle \frac{1}{n} E_x e^{-z/\delta} \cos \left(\frac{z}{\delta} - \omega t\right) (kE_x/\omega) e^{-z/\delta} \cos \left(\frac{z}{\delta} - \omega t + \pi/4\right) \right\rangle$$

remember we had to take the real values of the complex phases before doing the product.

But $\langle \cos(x) \cos(x + \pi/4) \rangle = \left\langle \cos(x) \left(\frac{1}{\sqrt{2}} \cos(x) + \frac{1}{\sqrt{2}} \sin(x)\right) \right\rangle = \frac{1}{2\sqrt{2}}$

$$\langle N \rangle = \frac{1}{2} \left(\frac{\sigma}{2\omega\mu}\right)^{1/2} E_x^2 e^{-2z/\delta}$$

As noted before the energy flow decays rapidly into the metal. Note that the time averaged Ohmic heating (per unit areas) is given by;

$$P_{ohm} = \langle \mathbf{j} \cdot \mathbf{E} \rangle = \langle \sigma \mathbf{E}^2 \rangle = \langle \sigma (E_x e^{-z/\delta} \cos(z/\delta - \omega t))^2 \rangle = \frac{1}{2\sigma} E_x^2 e^{-2z/\delta}$$

but also

$$\frac{\partial \langle N \rangle}{\partial z} = \frac{1}{2} \left(\frac{\sigma}{2\omega\mu}\right)^{1/2} E_x^2 e^{-2z/\delta} - \frac{2}{\delta} = \left(\frac{\sigma}{2\omega\mu\epsilon}\right)^{1/2} \left(\frac{\sigma\mu\epsilon}{2}\right)^{1/2} E_x^2 e^{-2z/\delta} = -P_{ohm}$$

Hence the energy lost by the em wave goes into Ohmic heating.

Note also that:

$$\text{Electric energy} = \frac{\mathbf{j} \times \mathbf{E}}{\mathbf{k}} \frac{\epsilon \mathbf{E}^2}{2} = \frac{\epsilon^2 \mu \mathbf{E}^2}{\eta^2 \mathbf{E}^2} = \frac{\epsilon \mu}{\epsilon_0 \mu_0} \left(\frac{\epsilon_0 \omega}{\sigma}\right) = \epsilon_r \mu_r \left(\frac{\epsilon_0 \omega}{\sigma}\right) \ll 1$$

The term in brackets is the ratio of displacement to conduction current. For metals $\sigma > \epsilon_0 \omega$, so that most of the energy is carried in the magnetic field. This is not surprising as charges will move to try and reduce the $\mathbf{E}$ field and it is the resulting currents that retain most of the energy.

**Figure 3:** At high frequencies, current is carried only on skin at surface of conductor.

Note also this implies that the power in an AC current is carried by the electromagnetic fields. This is important for transmission of power, since the fields are only found within a skin-depth;
\[ E = E_0 e^{-z/\delta} \]. We can see from the figure that current only flows through an area of \( 2\pi a \times \delta \).

Hence at high frequencies, current only flows in the skin and resistance can become a problem, e.g. for Cu,

Table 1: Resistance of \( r = 1 \text{ mm} \) copper wire as a function of frequency

<table>
<thead>
<tr>
<th>( f ) (MHz)</th>
<th>( R ) (( \Omega/m ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.005</td>
</tr>
<tr>
<td>1</td>
<td>0.04</td>
</tr>
<tr>
<td>100</td>
<td>0.4</td>
</tr>
<tr>
<td>1000</td>
<td>1.3</td>
</tr>
</tbody>
</table>

which is why for high frequencies many thin wires are better than one thick wire.

10.3 Radiation pressure

Previously we have ignored the effect of the magnetic field on the motion of electrons within a material. As the electrons move in the electric field of the incident wave, they can interact with the \( B \) field, pushing the whole material in the direction of the incident radiation.

Force per unit volume, \( \frac{dF}{d\tau} = -env \times B = j \times B \)

Force is felt only over \((2/\delta)^{-1}\), so the pressure;

\[ P_{rad} = \frac{dF}{dA} = (\delta/2)(\sigma E \times B) = (\delta^2/2\delta)(\sigma E \times B) = (2\sigma/\omega \delta)(\sigma E \times B) = (1/\omega \delta)(\sigma E \times B) / \mu \]

but \( c_m = \omega \delta \) is just the phase speed of wave in the material, and so \( P_{rad} = \frac{1}{\mu}(\sigma E \times B) / c_m = N / c_m \)

The resulting pressure can be found most simply from a photon picture: if a beam of photons of density \( n_{ph} \) are incident on a surface of area \( A \) in a time \( \Delta t \), then the total energy incident is \( n_{ph} \hbar \omega \Delta t A \). The intensity (energy per unit area per unit time), which by definition is equivalent to the Poynting vector is given by,

\[ N = \frac{n_{ph} \hbar \omega c \Delta t A}{\Delta t A} = n_{ph} \frac{\hbar \omega c}{A} \]

But each photon exchanges \( \Delta p = (1 + R)p_{phot} = (1 + R)h k \) where \( R \) is the reflectivity, so the pressure on the surface due to all the photons reflected is given by;

\[ P_{rad} = \frac{\Delta p_{tot}}{A \Delta t} = \frac{(1 + R)n_{ph} h k c \Delta t A}{\Delta t A} = \frac{(1 + R)N k}{\omega} = \frac{(1 + R)N}{c} \]

Figure 4: A beam of photons incident on a surface.
11. Waveguides

11.1 Waveguides

Until now we only considered unbounded plane waves. However under many circumstances it is necessary to enclose the waves (for example if propagation over long distances is required).

Consider an em wave bounded by a rectangular metallic tube (a waveguide) as in the diagram. The em wave must satisfy the 3D wave equation for both $E$ and $B$:

$$\nabla^2 E - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = 0; \quad \nabla^2 B - \frac{1}{c^2} \frac{\partial^2 B}{\partial t^2} = 0.$$ 

Last time we found for low frequencies $R = 1$. Hence can treat walls as perfect conductors (and thus perfect reflectors) with boundary conditions at wall:

$$E_\parallel = 0 \text{ (so } E \perp \text{ to walls)} \quad \text{and} \quad B_\parallel \text{ surface but } \perp j$$

Evidently a plane wave with both transverse electric and magnetic fields (TEM) cannot propagate in the cavity ($E_\parallel$ must be non-zero on one set of walls!)

Figure 1: A waveguide

Figure 2: $E_{z0}$ for lowest number TM modes

11.2 TM (transverse magnetic) modes

So look for solutions where the fields are not purely transverse: $E_z = E_{z0}(x, y) e^{i(k_g z - \omega t)}$, where $E_{z0}$ will now be a function of $x$ and $y$. First, choose $B$ field to still be transverse to $k$. The wavelength in the cavity $\lambda_g$ will be different from the vacuum solution $\lambda_0$, so we define the wavevectors:

$$k_g = 2\pi/\lambda_g; \quad k_0 = 2\pi/\lambda_0 = \omega/c.$$ 

Each component of $E$ must satisfy the wave equation. Putting trial solution for $E_z$ into the wave-equation:

$$\nabla^2 E_{z0} = \frac{\partial^2 E_{z0}}{\partial x^2} + \frac{\partial^2 E_{z0}}{\partial y^2} + \frac{\partial^2 E_{z0}}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 E_{z0}}{\partial t^2}$$

$$\frac{\partial^2 E_{z0}}{\partial x^2} + \frac{\partial^2 E_{z0}}{\partial y^2} + -k_g^2 E_{z0} = -\frac{\omega^2}{c^2} E_{z0} = -k_0^2 E_{z0}$$

$$\frac{\partial^2 E_{z0}}{\partial x^2} + \frac{\partial^2 E_{z0}}{\partial y^2} = (k_g^2 - k_0^2) E_{z0} \quad (11.1)$$
BUT the $E_{z0}$ (along the waveguide) must be zero at the walls (due to boundary conditions). Hence $E_{z0} = 0$ for $x = 0$, $a$ and $y = 0$, $b$. This is solutions for a string fixed at both ends, so trying solutions of the type $E_{z0} = E_0 \sin(m \pi x/a) \sin(n \pi y/b)$ in eq. (11.1).

$$k_0^2 - k_g^2 = \frac{m^2 \pi^2}{a^2} + \frac{n^2 \pi^2}{b^2}$$

Hence it is possible to have a number of different modes, with different values of $l$ and $m$ (provided both $m,n > 0$). These are called $TM_{mn}$ modes.

The other components of $E$ and $B$ can be found from Maxwell’s equations, with $B_{z0} = 0$:

$$E_{0x} = \frac{iE_0k}{k^2 - k_0^2} \frac{m \pi}{a} \cos \left( \frac{m \pi x}{a} \right) \sin \left( \frac{n \pi y}{b} \right) ; \quad B_{0x} = -\frac{iE_0k}{c(k^2 - k_0^2)} \frac{n \pi}{a} \sin \left( \frac{m \pi x}{a} \right) \cos \left( \frac{n \pi y}{b} \right) ;$$

$$E_{0y} = \frac{iE_0k}{k^2 - k_0^2} \frac{m \pi}{a} \sin \left( \frac{m \pi x}{a} \right) \cos \left( \frac{n \pi y}{b} \right) ; \quad B_{0y} = \frac{iE_0k}{c(k^2 - k_0^2)} \frac{n \pi}{a} \cos \left( \frac{m \pi x}{a} \right) \sin \left( \frac{n \pi y}{b} \right) ;$$

$$E_{0z} = E_0 \sin \left( \frac{m \pi x}{a} \right) \sin \left( \frac{n \pi y}{b} \right) ; \quad B_{0z} = 0 ;$$

Note TM modes produce longitudinal electric fields (see fig. 3), that can be used for accelerating particles along the waveguide. This is the basis of RF linear accelerators.

![Figure 3: Longitudinal variations of $E_z$ and $E_y$ for $TM_{11}$](image)

![Figure 4: Lowest modenumber TM mode](image)

### 11.3 Cut-offs

If we rearrange (11.2)

$$k_g^2 = k_0^2 - \frac{m^2 \pi^2}{a^2} - \frac{n^2 \pi^2}{b^2}$$

we can see that for sufficiently large values of $m,n$, then the RHS becomes negative (or for values of $a,b$ that are too small). This implies $k_g$ becomes imaginary, and as before will mean that the wave will be strongly damped. Note that $a,b$ can be chosen so that only one mode can propagate.

Applying the condition for $k_g$, we can find a cut-off wavelength for the waveguide;

$$k_c^2 = \frac{m^2 \pi^2}{a^2} + \frac{n^2 \pi^2}{b^2} \quad \text{or} \quad 1/\lambda_c^2 = (m/2a)^2 + (n/2b)^2$$
11.4 TE (transverse electric) modes

We could have instead of choosing transverse magnetic modes, chosen instead that the electric field remains transverse, and that it is \( B \) that has a longitudinal variation; \( B_z = B_0 e^{i(k_g z - \omega t)} \), that gives an equivalent equation for the transverse variations;

\[
\frac{\partial^2 B_{z0}}{\partial x^2} + \frac{\partial^2 B_{z0}}{\partial y^2} = (k_g^2 - k_0^2)B_{z0}
\]

The boundary conditions are more complicated and must be derived from \( E_\perp = 0 \) and Maxwell’s equations, which gives that \( \partial B_z / \partial x \) and \( \partial B_z / \partial y \) are respectively zero at the \( x \) and \( y \) boundaries. Applying these conditions, results in a comparable solution for the TE modes.

\[
B_{z0} = B_0 \sin(m \pi x/a) \sin(n \pi y/b)
\]

with similar cut-offs for the TM modes. The only major difference is now TE\(_{10}\) and TE\(_{01}\) modes are allowed, since \( B \neq 0 \) at the boundaries.
12. Electromagnetism and relativity

12.1 Special Relativity

Einstein formulated the special theory of relativity to counter inconsistencies in electrodynamics. In particular, the laws of physics should be the same in every frame. However an observer travelling on a train with a charged ball would say it was stationary and so produced no magnetic field. Yet an observing the train passing would also see the charge moving and so report that it did produce a magnetic field?

Before addressing this problem, we revise some SR. The Lorentz transforms are;

\[
\begin{align*}
ct' &= γ (ct - βx) \\
x'_0 &= γ (x_0 - βx_1) \\
x' &= γ (x - βct) \\
x'_1 &= γ (x_1 - βx_0) \\
y' &= y \\
x'_2 &= x_2 \\
z' &= z \\
x'_3 &= x_3
\end{align*}
\]

where we multiplied all times by \(c\) to give them the dimensions of length, and used \(β = v/c\), \(γ = (1 - β^2)^{-1/2}\). On the RHS we have written \(ct = x_0\) as an extra spatial dimension, to emphasise the symmetry between the first two terms. Indeed including the time term, the spatial terms can be written as a four-vector, now with an extra “dimension” (the time one). The Lorentz transforms can then be written as a matrix multiplication:

\[
\begin{bmatrix}
ict' \\
x' \\
y' \\
z'
\end{bmatrix} =
\begin{bmatrix}
γ & -iβγ & 0 & 0 \\
iβγ & γ & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
ict \\
x \\
y \\
z
\end{bmatrix}
\]

Where primes correspond to a frame \(S'\) moving at \(v\) relative to an unboosted frame \(S\). Note I use the complex time \(ict\) to represent the zeroth component, to emphasise that time is a little different from the rest, and also to make the square of vector to be the interval \(-c^2t^2 + x^2 + y^2 + z^2\) which is Lorentz invariant. This can be seen by noting that the transformation matrix has a determinant of 1 (and so doesn’t change the magnitude). Making \(x_0\) complex also makes the transformations of the \(x_0, x_1\) components look like \((\cos θ - i\sin θ)\), and hence emphasise that it is a rotation in space-time.

Note that two events occurring at the same place (say \(x = 0\)) at different times in an inertial frame separated in time by \(Δt = t_1 - t_2\), appear separated in time in a boosted frame by \(Δt' = γ(t_1 - t_2) = γΔt\). Hence things appear slower in the moving frame (since \(γ > 1\)), this is time dilation.

Similarly two points separated by \(Δx = x_1 - x_2\) in the unprimed frame at the same time, will appear \(Δx' = γ(x_1 - x_2) = γΔx\) further apart in the boosted frame. So at a fixed time (as in the unprimed frame), lengths appear shorter \(Δx = Δx'/γ\), this is length contraction.

Finally we note that if \(F_x, F_y, F_z\) are the components of the force on a particle in a stationary frame \(S\), then in a boosted frame \(S'\) the force components are \(F'_x = F_x, F'_y = F_y/γ, F'_z = F_z/γ\).

12.2 Magnetism as a relativistic effect

How do \(E\) and \(B\) transform? A particle is at rest above a stationary line charge \(λ\) in fig. 1a. The
electric field due to the wire, and the corresponding force on the charge are;

\[ F_0 = \frac{\lambda q}{2\pi\varepsilon_0 d}; \quad E_0 = \frac{\lambda}{2\pi\varepsilon_0 d} \]

Now consider what happens if the line charge is moving with velocity \( v \) (fig. 1b). An important point is that charge is \textit{Lorentz invariant} (or otherwise particles and anti-particles would have different charges depending on what frame they were born in!). But that means \textit{charge density} is not. The line charge density will be Lorentz contracted, and so increase in magnitude to \( \lambda' = \gamma \lambda \). There is a consequent increase in \( F = \gamma F_0 \) and \( E = \gamma E_0 \)

Now consider what happens if the line charge is moving with velocity \( v \) (fig. 1b). An important point is that charge is \textit{Lorentz invariant} (or otherwise particles and anti-particles would have different charges depending on what frame they were born in!). But that means \textit{charge density} is not. The line charge density will be Lorentz contracted, and so increase in magnitude to \( \lambda' = \gamma \lambda \). There is a consequent increase in \( F = \gamma F_0 \) and \( E = \gamma E_0 \)

\[ a) \quad F_0 = \frac{\lambda q}{2\pi\varepsilon_0 d}; \quad E_0 = \frac{\lambda}{2\pi\varepsilon_0 d} \]

\[ b) \quad F = \gamma F_0; \quad E = \gamma E_0 \]

\[ c) \quad F = \frac{F_0}{\gamma} \]

Figure 1: A charged particle in the proximity of a line charge

Now suppose that fig. 1b is observed in a frame where both the line and point charge are moving with \( v \) (fig. 1c). But this is just the boosted version of fig. 1a, and so the force should be reduced by \( F = \frac{F_0}{\gamma} \). Clearly there is a difference in force between b) and c) which arose purely due to the motion of the particle (in the opposite direction to the electric repulsion), given by: \( F_0/\gamma - \gamma F_0 \).

We can thus identify it as the magnetic force, though it was derived purely from electrostatic and relativistic considerations. Equating the extra force to \( qvB \):

\[ qvB = \gamma F_0 \left( \frac{1}{\gamma^2} - 1 \right) = \frac{q\lambda\gamma}{2\pi\varepsilon_0 d} \left( 1 - \beta^2 - 1 \right) \]

\[ B = \frac{q\lambda\gamma}{qv2\pi\varepsilon_0 d} \left( -\beta^2 \right) = -\frac{\lambda'v}{2\pi\varepsilon_0 dc^2} = \frac{\mu_0 I}{2\pi d} \]

where we used \( \lambda\gamma = \lambda' \), and \( I = \lambda'v \). The result is just that expected for a infinite straight current.

\textbf{12.3 Relativistic transformations of E, B}

Clearly \( E \) and \( B \) fields transform from one to another through different inertial frames. Consider as in fig. 2, a charge moving at relativistic speed \( v \) between two parallel sheets of charge, with surface charge densities \( +\sigma_0 \) and \( -\sigma_0 \). The distance between the plates \( d \) is \textit{length contracted}, but the electric field \( E_{\parallel} = \sigma_0/\varepsilon_0 \), does not depend on \( d \) and so is unchanged. i.e.,

\[ E'_{\parallel} = E_{\parallel} \]

Now consider fig. 3, where there is a coil carrying current \( I \). For a charged particle moving along the coil, the coil is length contracted so that the coils per turn \( n' = N'/L' = N/(L/\gamma) = \gamma n \), so the density of windings increases. But the current \( I' = dq'/dt' = dq'/dt' = dq/\gamma dt = I/\gamma \). So \( B'_{\parallel} = \mu_0 n'I' = \mu_0 n I = B_{\parallel} \). Length contraction and time dilation exactly cancel in this case!

\[ B'_{\parallel} = B_{\parallel} \]
What about a particle travelling perpendicular to the fields (as in fig.4)? Now the charge density on the plates is increased by $\sigma = \gamma \sigma_0$ due to length contraction in the boosted frame. Hence the electric field has to increase too: $E'_y = \gamma E_y$. But as we saw above, that is not the whole story, there is a magnetic field generated as well. If $E$ is in $y$, so that $E_y = \gamma \sigma_0 / \epsilon_0$, in the boosted frame there is also a $B_z = -\mu_0 \gamma \sigma_0 v$. So in that frame, 

$$F_y = q(E'_y + vB'_z)$$

$$= q(\gamma \sigma_0 / \epsilon_0 - \mu_0 \gamma \sigma_0 v^2)$$

$$= (q \gamma \sigma_0 / \epsilon_0)(1 - v^2 / c^2)$$

$$= q \epsilon_0 / \gamma$$

as expected. But equating first and last lines, we can see $E_y = \gamma(E'_y + vB'_z)$, or inverting $E'_y = \gamma(E_y - vB_z)$. A similar expression can be obtained for the transformation of $B_\perp$, and writing all the field transformations in short form gives:

$$E'_\parallel = E_\parallel; \quad E'_y = \gamma(E_y - vB_z); \quad E'_z = \gamma(E_z + vB_y);$$

$$B'_\parallel = B_\parallel; \quad B'_y = \gamma(B_y + vE_z / c^2); \quad B'_z = \gamma(B_z - vE_y / c^2);$$

### 12.3 Boiling the vacuum

Julian Schwinger predicted an absolute limit to the maximum intensity of electromagnetic radiation in 1951. He supposed that within a Compton wavelength ($\lambda_c = h / mc$ - which can be considered as the uncertainty in size of the electron during Compton scattering), a virtual electron-positron pair can gain enough energy to become real. i.e.

$$eE_s \lambda_c \approx 2mc^2.$$  

Above this intensity, the laser energy would quickly become attenuated through pair production. We find,

$$E_s = \frac{2m^2c^3}{eh} \geq 4 \times 10^{17} \text{Vm}^{-1}.$$  

This corresponds to an intensity $> 10^{36}$ Wm$^{-2}$ for optical radiation. This is more than 10 orders of magnitude greater than presently available.

However a particle with $\gamma > 10^5$ would see the intensity of radiation $> 10^{10}$ times greater, if travelling $\perp$ to $E$, hence allowing the Schwinger limit to be reached. In a paper published in Physical Review Letters in 1997, Burke et al, were able to demonstrate the creation of a significant numbers of positrons ($\sim 100$), by colliding 46.6 GeV electrons $\gamma = 9.1 \times 10^4$, with a laser of intensity $\sim 10^{25}$ Wm$^{-2}$. 


A list of bugs that have been squashed

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Last up-date: 8/5/12

Apologies for the typos/bugs in the lecture notes - these are the first year that I have given them out, it would be great if you help me improve the quality of the notes by telling me of mistakes or even places where the clarity could be improved. (Latest updates in !).

Lecture 1 Section 1.2 and again in 1.4: c = (\epsilon_0\mu_0)^{-1/2} not c = (\epsilon_0\mu_0)^{1/2}

Lecture 2 Figure 5 was confusing, and has been changed (hopefully to be clearer)

Section 2.2: “\[ \int E \cdot dS = \frac{1}{\epsilon_0} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\theta_1} \frac{Q}{4\pi \epsilon_0} \sin \theta d\theta d\phi = \]” had an extra 1\epsilon_0 and should be “\[ \int E \cdot dS = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\theta_1} \frac{Q}{4\pi \epsilon_0} \sin \theta d\theta d\phi = \]”

Also: The trig function “cos \theta_1 - cos \theta_2 = 2 \sin \frac{1}{2} (\theta_1 - \theta_2) \sin \frac{1}{2} (\theta_1 + \theta_2)” should be cos \theta_2 - cos \theta_1 = 2 \sin \frac{1}{2} (\theta_1 - \theta_2) \sin \frac{1}{2} (\theta_1 + \theta_2), though the rest of the derivation was correct (there is no need for extra minus signs anywhere else).

Also: “the retarded time z(t - r/c).” should be “the retarded time (t - r/c).”

Section 2.3: “[\tilde{z}] = \omega^2 z_0 \sin(\omega(t - r/c)),” should be “[\tilde{z}] = -\omega^2 z_0 \sin(\omega(t - r/c)),” this makes the expressions for E_{rad} and B_{rad} negative too, but the Poynting flux (which is the product of the two) is still in the positive r direction as it should be.

CW IV In (questions) part xii):
E_2 = E_0 e^{i(k_1 \cdot r - \omega_1 t)} should have been E_2 = E_0 e^{i(k_2 \cdot r - \omega_2 t)}

and
E(r, t) = E_0 e^{i(kr - \omega t)} cos(\Delta k \cdot r - \Delta \omega t)

should have been
E(r, t) = 2 E_0 e^{i(kr - \omega t)} cos(\Delta k \cdot r - \Delta \omega t)

Lecture 3 Section 3.1: “Since J \cdot (A) = I,” should have been “Since J \cdot A = I,”

Section 3.2: “where [I'] is the derivative with respect to time of the retarded potential,” should have been “where [I'] is the derivative with respect to time of the retarded current,”

and “and we used, \[ \frac{\partial I(t - r/c)}{\partial r} = \frac{\partial t}{\partial r} \frac{\partial}{\partial t} I(t - r/c) = -\frac{1}{c} \frac{\partial}{\partial t} I(t - r/c) \]” should have been

“since \[ \frac{\partial I(t - r/c)}{\partial r} = \frac{\partial I(\xi)}{\partial r} = \frac{\partial \xi}{\partial r} \frac{\partial}{\partial \xi} I(\xi) = -\frac{1}{c} \frac{\partial t}{\partial \xi} I(\xi) = \frac{1}{c} \frac{\partial t}{\partial r} \frac{\partial}{\partial t} I(\xi) = \frac{1}{c} \frac{\partial t}{\partial r} I(t - r/c) = [I'] \]”

End Section 3.2: E = -c k \times B should have been E = -c \hat{k} \times B

Section 3.5:
\[ \langle P \rangle = \frac{2\pi}{3\epsilon_0 c} \left( \frac{d\lambda}{\lambda} \right)^2 I_{rms}^2 = R_{rad} I^2 \]

had omitted the
\[ \langle P \rangle = \frac{2\pi}{3\epsilon_0 c} \left( \frac{d\lambda}{\lambda} \right)^2 I_{rms}^2 = R_{rad} I_{rms}^2 \]
Lecture 4 Section 4.1: “\( P = \frac{q^2 |a|^2}{12\pi \epsilon_0 c^3} \)” should have been “\( P = \frac{q^2 |a|^2}{6\pi \epsilon_0 c^3}. \)” (see changes in the ensuing discussion in this section too).

and “\( a = \frac{m_r \omega_0^2}{r_0} \) \( \text{and} \) \( \mathbf{r} = m_r \omega_0^2 \mathbf{r} \)” should have been “\( a = -\frac{\omega_0^2}{r_0} \mathbf{r} = -r_0 \omega_0^2 \mathbf{r} \)”

Section 4.3: Figures 3, 4, 5 and 6, 7, 8 were originally in the wrong order.

Section 4.3: \( \mathbf{R} \times \mathbf{R} \times \beta = \hat{\beta} \sin \theta \) should have been \( \mathbf{R} \times (\mathbf{R} \times \beta) = \hat{\beta} \sin \theta \)

Also “the emitted power \( \frac{dP}{dt} \)” should have just been “the emitted power \( \frac{dP}{d\Omega} \)”

Also now defined \( \gamma \) as: where \( \gamma = (1 - \beta^2)^{-1/2} \) is the usual relativistic factor.

Also: “\( P = \frac{1}{2} \epsilon_0 |E_r|^2 (1 - \beta \cos \theta) R^2 d\Omega \)” should have been “\( dP = \frac{1}{2} \epsilon_0 |E_r|^2 c (1 - \beta \cos \theta) R^2 d\Omega \)”

PS 6 Q.4 Intro: It should say “has a time-varying current \( I \)” not “has a time-varying current \( \dot{I} \)”.

Q.4 part (a): It should say “If the current flowing is given by \( I = I_0 \sin \omega t \)” not “If the current flowing is given by \( I = I_0 \cos \omega t \)” (it was correct in the answers).

PS 6a Q.1(d) Answer changed to use \( \hat{x}, \hat{y}, \phi \) instead of \( \hat{y}, \hat{x}, \phi \). Also RHS figure corrected.

Q.3(b) Added \( U_E = \frac{1}{2} \epsilon_0 E^2 = \frac{1}{2} \epsilon_0 (\mathbf{E} \cdot \mathbf{E}) \) to make it clearer.

Q.4(b) “\( c(\mathbf{U}) \hat{k} = \frac{\epsilon_0 \epsilon_0}{2} \hat{k} \) \( \hat{k} \)” is now written more clearly as “\( c(\mathbf{U}) \hat{k} = c \epsilon_0 \epsilon_0 \hat{k} \) \( (\cos^2 \omega (t - r/c)) \mathbf{r} = \)” (it doesn’t change the answer).

Q.4(c) \( I_0 = 2I_{rms} \) should say \( I_0 = \sqrt{2} I_{rms} \) and the answers should have been \( I_{rms} = 2.25 \) A, and \( \langle P \rangle = 111 \) W

Q.6(c) \( \gamma = E/m \) is better written \( \gamma = E/mc^2 \)

and \( \beta \approx c \) should be \( \beta \approx 1 \)

Lecture 5 Section 5.1: \( \epsilon_0 \nabla \cdot \mathbf{P} = -\rho_p \) should be \( \nabla \cdot \mathbf{P} = -\rho_p \).

and “Sometimes \( \epsilon \mathbf{E} = \epsilon_0 (\mathbf{E} + \mathbf{P}) \) is called \( \mathbf{D} \)” should be Sometimes \( \epsilon \mathbf{E} = \epsilon_0 \mathbf{E} + \mathbf{P} \) is called \( \mathbf{D} \)

Section 5.2: “the boundary between two dielectrics” should be more generally “the boundary between two materials”

Should have been \( j_c = j_c' \delta(0), \) not \( j_c' = j_c \delta(0) \), (was correct in lecture).

and also \( B_j/\mu \text{ is discontinuous across the boundary by } j' \) not \( B_j/\mu \text{ is discontinuous across the boundary by } j_c \)

Figure 3: Now improved (thicker lines and symbols agree with the text)

Section 5.3:

taking \( i \mathbf{k} \cdot \) of 3rd eqn

\[
\begin{align*}
\mathbf{i} \mathbf{k} \times \mathbf{i} (\mathbf{k} \times \mathbf{E}) &= -\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) = -\left[ \mathbf{k} (\mathbf{k} \cdot \mathbf{E}) - k^2 \mathbf{E} \right] \\
&= \mathbf{i} \mathbf{k} \times \mathbf{i} \omega \mathbf{B} = -(-\epsilon \mu_0 \omega^2 \mathbf{E})
\end{align*}
\]

CW V In part (vi), In the text I wrote “ratio of power radiated to the power incident” when it should have been “ratio of power radiated to the incident energy (Poynting) flux” (which now correctly has the dimensions of an area).

So \( \sigma_T = P/P_0 \) that should have been \( \sigma_T = P/\langle N_m \rangle \), (but answer for \( \sigma_T \) was ok).

In part (vii), I wrote \( z_0 = -\frac{eE_0}{m\omega_0^2} = -\frac{eE_0}{m\omega^2} \left( \frac{\omega^4}{\omega_0^4} \right) \), should have been \( z_0 = -\frac{eE_0}{m\omega_0^2} = -\frac{eE_0}{m\omega^2} \left( \frac{\omega^2}{\omega_0^2} \right) \) (again answer was ok).
Lecture 6 In part (6.2), \( r = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 - \eta_1}{2\eta_1} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \) should be \( r = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 - \eta_1}{2\eta_1} = \frac{2\eta_1}{\eta_2 + \eta_1} \).

Unfortunately that meant the line for \( r \) in figure 3, is negative of what it should be (should have same slope as \( t \)) - now corrected.

Also in the following discussion it should say;
if \( \eta_1 > \eta_2, \) (e.g. glass to air); \( r > 0 \) not “if \( \eta_1 > \eta_2, \) (e.g. glass to air); \( r > 1 \)”
and if \( \eta_1 < \eta_2, \) (e.g. air to glass); \( r < 0 \), not if \( \eta_1 < \eta_2, \) (e.g. air to glass); \( r > 1 \).

In part (6.3), expression for Poynting vector should be \( N = \epsilon_r \epsilon_0 E^2 c' = \epsilon_r^{1/2} \epsilon_0 E^2 c = \eta \epsilon_0 E^2 c \).

also \( T = \frac{\eta_2 E_{r0}^2}{\eta_1 E_{i0}^2} = \left( \frac{\eta_2}{\eta_1} \right) |t|^2 = \frac{4\eta_2 \eta_1}{(\eta_2 + \eta_1)^2} \) not \( T = \frac{c_2 E_{r0}^2}{c_1 E_{i0}^2} = \left( \frac{\eta_1}{\eta_2} \right) |t|^1 = \frac{4\eta_2 \eta_1}{(\eta_2 + \eta_1)^2} \)

And finally I moved the last section (6.4) to lecture 7, where it belonged (note that the direction of the arrow in the \( p \)-polarisation has been turned around to be consistent with the \( s \)-polarisation case.

CW VI In the useful information: “\( eE_\parallel \) is discontinuous by \( \rho_c \)” should have been “\( \epsilon E_\bot \) is discontinuous by \( \rho_c \)” and “\( B_\parallel \) is continuous; \( B_\bot /\mu \) is discontinuous by \( j_c \)” should have been “\( B_\bot \) is continuous; \( B_\parallel /\mu \) is discontinuous by \( j_c \)” and “\( \eta \simeq \sqrt{\epsilon/\mu} \)” should be “\( \eta \simeq \sqrt{\epsilon/\mu} \)”

Part (i): In the definitions of the \( E \) fields, all of the waves should be moving in \( z \) not \( x \), e.g. “\( E_1 = A_1 e^{i(\omega t - k_1 z)} \)” should be “\( E_1 = A_1 e^{i(\omega t - k_1 x)} \)” Part (ii): “\( 2\eta_1 E_1' = (\eta_1 - \eta_2) E_2 + (\eta_1 + \eta_2) E_2 \)” should be “\( 2\eta_1 E_1' = (\eta_1 - \eta_2) E_2 + (\eta_1 + \eta_2) E_2' \)”
Answers Part (iv) “Where we used \( e^{i\pi} = 0 \)” should have been “Where we used \( e^{i\pi} = -1 \)”

PS 7 Q.1(b) answers: I said: “\( u_E + u_B = \epsilon E^2 = \frac{1}{2} \epsilon E_0^2 \cos^2(kx - \omega t) \)” but should have said “\( u_E + u_B = \epsilon E^2 = \epsilon E_0^2 \cos^2(kx - \omega t) \)” Q.3(c) answer: added an extra step for clarity; “Remembering (from e.g. Q1), \( N = c' U \hat{\mathbf{z}} = (c/\eta) \epsilon E^2 \hat{\mathbf{z}} = (c/\epsilon)^{1/2} \epsilon \epsilon_0 c' E^2 \hat{\mathbf{z}} = \eta \epsilon \epsilon_0 c' E^2 \hat{\mathbf{z}}. \) (NB for dielectric we used \( \eta = \epsilon_\epsilon_\epsilon^{1/2} \))” Q.4(c) answers: “Continuity of \( B_\bot \), \( B_\parallel = B_\parallel \cos \theta_t - B_\perp \cos \theta_t = B_\parallel \cos \theta_t \)” should have been “Continuity of \( B_\parallel \), \( B_\parallel = B_\parallel \cos \theta_t - B_\perp \cos \theta_t = B_\parallel \cos \theta_t \)”

Lecture 7 Section 7.2: “and continuity of \( B_\bot \)” should have been “and continuity of \( B_\parallel \)” Also: “\( B_\parallel \cos \theta_t - B_\perp \cos \theta_t \)” should have been “\( B_\parallel \cos \theta_t - B_\perp \cos \theta_t \)” , and this problem of the subscripts carries-on on the next couple of lines.

Also: For \( p \)-polarisation case, “Take continuity of \( E_\bot = E_x (= E_y) \)” should be “Take continuity of \( E_y \)”.

Section 7.3: “For \( s \)-polarisation (polarisation in the plane of the reflection),” should have been “For \( s \)-polarisation (polarisation perpendicular to the plane of the reflection),” Also: “completely cuts \( s \)-polarisation” should have been “completely cuts \( p \)-polarisation” (yikes!)

Section 7.5 changes suffixes to be consistent with rest of lecture: “And the transmitted wave becomes \( E_t = E_{t0} e^{i(k \sin \theta_t x + k \cos \theta_t z - \omega t)} = E_{t0} e^{i(k \sin \theta_t x - \omega t)} e^{-\omega z/c} \)
Also \( k_m = k \sin \theta_t = (\omega/c) \cdot (\eta_1/\eta_2) \) should be \( k_m = k \sin \theta_t = (\omega/c) \cdot (\eta_1/\eta_2) \sin \theta_t \)”
Lecture 8 Section 8.2: “the ions since they will move \((m/M)\) times slower” should be “the ions since they will move \((M/m)\) times slower”

Section 8.2 “\(\mathbf{k} \cdot \mathbf{E} = \rho\)” should have been “\(i \mathbf{k} \cdot \mathbf{E} = \rho/\epsilon_0\)”

Section 8.4 Subscripts in figure didn’t correspond to those in text, so changed the text to read: “for the wave to no longer propagate up the ramp is \(k_z = 0\) (=\(k_z\) in the diagram).”

\[
\omega_0^2 = \omega_0^2(z) + c^2(k_x^2 + k_y^2)
\]

\[
\omega_0^2 = \omega_0^2(z) + c^2(k_x^2 + k_y^2 \sin^2 \theta)
\]

where the wavenumber is unaffected in the direction perpendicular to the density ramp.

Using \(\omega_0 = \frac{c k_0}{\mu}\) and noting that propagation stops when \(k_z = 0\), and that \(k_x = k_0 \sin \theta\),

\[
\omega_0^2(z) = c^2 k_0^2 (1 - \sin^2 \theta) = c^2 k_0^2 \cos^2 \theta
\]

PS 8 Q.1(a) “\(B_t = (-k_t \hat{x} + i k_t \hat{z})(E_t/\omega)e^{i(k_t r - \omega t)}\)” should have been \(\mathbf{B}_t = (-k_t \hat{x} + i k_t \hat{z})(E_t/\omega)e^{i(k_t r - \omega t)}\)

Q.1(b) answers. There is a missing factor of \(e^{-2k_x x}\) through-out.

Q.2(b) answers: I wrote “\(-i m \omega \mathbf{v} = -e \mathbf{E} \rightarrow i \omega m \mathbf{v} = -e \mathbf{E}\)” should have been “\(-i m \omega \mathbf{v} = -e \mathbf{E} \rightarrow i \omega m \mathbf{v} = e \mathbf{E}\)”.

Q.2(c) “\(\mathbf{B} = B_y e^{i(k_z - \omega t)} \hat{y}\)” should have been “\(\mathbf{B} = B_y e^{i(k_z - \omega t)} \hat{y}\)”.

Q.2(f) answers: “Eliminate \(E_{xp}, (k + ik_p)E_{xi} - (k - ik_p)E_{xr} = 0 \rightarrow |E_{xr}| = \frac{|k + ik_p|}{|k - ik_p|} |E_{xi}|” should have been:

“Eliminate \(E_{xp}, (k - ik_p)E_{xi} - (k + ik_p)E_{xr} = 0 \rightarrow |E_{xr}| = \frac{|k - ik_p|}{|k + ik_p|} |E_{xi}|\”

Q.3: “the conduction current density, \(\mathbf{j}_c\)” should have been “the conduction current density, \(\mathbf{j}_c\)”.

Q.4(c) answers: The answer should have been “\(3.1 \times 10^{19} \text{ m}^{-3}\)” not “\(3.1 \times 10^{19} \text{ cm}^{-3}\)”.

CW VII Part (iii) The answer “So \(\eta = (1 - \omega_p^2/\omega^2)^{-1/2}\)” should have been “So \(\eta = (1 - \omega_p^2/\omega^2)^{1/2}\)”.

Lecture 9 Section 9.3: \(\mathbf{E} = E_0 e^{((1+i)z/\delta - \omega t)} e^{-z/\delta} \hat{x}\). was rewritten correctly as \(\mathbf{E} = E_0 e^{i(z/\delta - \omega t)} e^{-z/\delta} \hat{x}\).

Also:

\[
k^2 = i \mu \sigma \omega \quad \rightarrow \quad k = \sqrt{i \mu \sigma \omega} = \frac{1 + i}{\sqrt{2}} \sqrt{\mu \sigma \omega} = \frac{1 + i}{\delta}
\]

Section 9.4: “\(Z = \epsilon/\mu\)” should have been “\(Z = (\mu/\epsilon)^{1/2}\)”.

and similarly “\(Z = Z_0 = \epsilon_0/\mu_0 = 377 \Omega\)” should be “\(Z = Z_0 = (\mu_0/\epsilon_0)^{1/2} \approx 377 \Omega\)”

Lecture 10 Section 10.1: “\(B_\parallel\) changes by \(B'_c\)” would better have been more generally “\(B_\parallel/\mu\) changes by \(j'_c\)”.

Also: “\(\eta = (\sigma/2\epsilon_0\omega)^{-1/2} (1 + i)\)” should be “\(\eta = (\sigma/2\epsilon_0\omega)^{1/2} (1 + i)\)”.

Also: “\(R = \left| \frac{1 - \eta}{1 + \eta} \right|^2 = \left| \frac{1 - \eta_0 + i \eta_0}{1 + \eta_0 + i \eta_0} \right|^2\)” should have been “\(R = \left| \frac{1 - \eta}{1 + \eta} \right|^2 = \left| \frac{1 - \eta_0 - i \eta_0}{1 + \eta_0 + i \eta_0} \right|^2\)”.

Section 10.2: “the reflected fraction = \(\left( \frac{8\epsilon_0\omega}{\sigma} \right)^{1/2} \times 20\%\)” should read “the fraction not reflected = \(\left( \frac{8\epsilon_0\omega}{\sigma} \right)^{1/2} \leq 20\%\)”.
PS 9 Q.1(c) the answer “$R = |r|^2 \approx 1 - \sqrt{8\omega\epsilon/\sigma}$” should have been “$R = |r|^2 \approx 1 - \sqrt{8\omega\epsilon_0/\sigma}$”

Q.2 “where $\epsilon_r\epsilon_0$” should have been “where $\epsilon = \epsilon_r\epsilon_0$”.

Q.3(a) final answer “$z = 6.91 \times \delta/2 = 2.3$ mm” should have been “$z = 6.91 \times \delta/2 = 0.23$ mm”

Q. 3(b) “the noise amplitude by a factor $10^{-3}$?” had people confused as whether I was asking for the amplitude of intensity or electric field, so hopefully it is clearer written as “the intensity of the noise by a factor $10^{-3}$?”

Q.5(e) Added “Condition for wave to stop travelling is $k = 0$, and it will decay if $k < 0$.”

Q.59(f) I wrote “let $\left(1 - \frac{\pi c^2}{a^2 \omega^2}\right)^{1/2} = ib$” but this is true generally for all values of $n$ and so should have been “let $\left(1 - \frac{n^2 \pi c^2}{a^2 \omega^2}\right)^{1/2} = ib$”.