Fourier

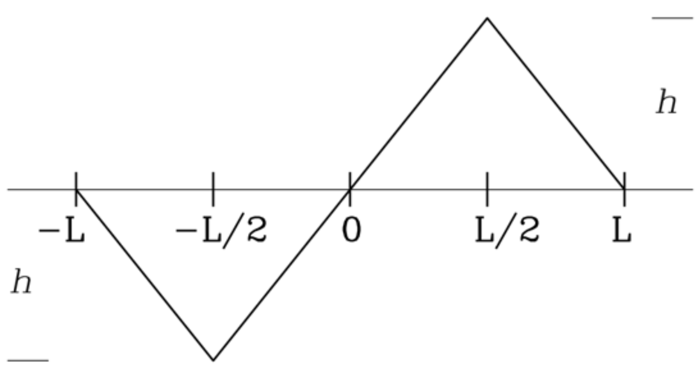
Classwork 1 October 14, 2011

Evaluation of Fourier series

The evaluation of the Fourier coefficients of a periodic function may often be greatly simplified by exploiting the symmetry of the problem. A few minutes sketching the functions and recognising the symmetries can save a great deal of unnecessary integration.

The function $f(x)$ plotted below, of repeat length $2L$, is used in determining the vibration of a string plucked at its mid-point (see Differential Equations course). In this Classwork we will determine the Fourier series representation of the function

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} \left[a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L)\right] \ .$$



1. Express the function $f(x)$ as three separate functions, for the intervals $-L \leq x \leq -L/2$, $-L/2 \leq x \leq L/2$, $L/2 \leq x \leq L$.

2. Write down the full expressions for the terms a_0, a_n, b_n i.e. insert the functions into the Euler-Fourier formulae:
$$\begin{aligned} a_0 &= \tfrac{1}{L}\textstyle\int_{-L}^{L} f(x)dx \ , \\ a_n &= \tfrac{1}{L}\textstyle\int_{-L}^{L} f(x)\cos(n\pi x/L)dx \ , \\ b_n &= \tfrac{1}{L}\textstyle\int_{-L}^{L} f(x)\sin(n\pi x/L)dx \ . \end{aligned}$$

3. A brute-force approach is to evaluate the three integrals for each term. More sophisticated is to note that $f(x)$ is an odd function, so that $a_0 = 0$ and $a_n = 0$. Recognise this visually, by sketching the relevant functions in the integrals (i.e. $f(x)$ and $\cos(n\pi x/L)$), and observing (trivially) how parts of the integral cancel with each other, so that the overall integral is zero, and so $a_0 = 0$ and $a_n = 0$.

4. It remains to determine the terms b_n. Again by sketching the functions, recognise the following:

   - That regardless of the value of n

   $$\int_{-L}^{L} f(x)\sin(n\pi x/L)dx = 2\int_{0}^{L} f(x)\sin(n\pi x/L)dx \ .$$

   - That for $n = 2, 4, 6...$

   $$\int_{0}^{L} f(x)\sin(n\pi x/L)dx = 0 \ ,$$

     and so $b_n = 0$, $n = 2, 4, 6...$

   - That for $n = 1, 3, 5...$

   $$\int_{-L}^{L} f(x)\sin(n\pi x/L)dx = 4\int_{0}^{L/2} f(x)\sin(n\pi x/L)dx \ .$$

5. By these means show that the Fourier representation of $f(x)$ is given by:

$$f(x) = \frac{8h}{\pi^2}\sum_{n=1,3,5,...}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{L}\right) \ .$$

Fourier

Classwork 2 October 21, 2011

Differentiation and integration of Fourier series

In some cases it may be simpler to compute a Fourier series by integrating or differentiating a known Fourier series.

1. Determine the Fourier series

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} [a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L)]$$

   for the periodic function defined by $f(x) = x^2$ on the interval $-L \leq x \leq L$.

2. Differentiate the series and compare to the Fourier series for $f(x) = x$ over the same interval

$$f(x) = \sum_{n=1}^{\infty} -\frac{2L}{n\pi} \cos(n\pi) \sin(n\pi x/L) \ .$$

3. If instead we integrate the Fourier series for $f(x) = x$, in comparing to the Fourier series for $f(x) = x^2$, we recognise that the constant of integration is $a_0/2$ i.e. the average value of the function over the interval. With this in mind, integrate the Fourier series for $f(x) = x^2$, to show that the Fourier series for $f(x) = x^3$ is given by

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{12L^3}{n^3\pi^3} - \frac{2L^3}{n\pi} \right) \cos(n\pi) \sin(n\pi x/L) \ .$$

Fourier

Classwork 3 November 1, 2011

Convolution

This classwork uses the notation used in lectures that the inverse FT and the FT are, respectively

$$f(x) = \int_{-\infty}^{\infty} g(\omega) e^{i\omega x} d\omega \ ,$$

$$g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \ .$$

1. Two rectangular functions $f(x)$, $g(x)$ are defined by

$$f(x) = \begin{cases} 0 & -\infty < x < -a \\ A & -a < x < a \\ 0 & a < x < \infty \end{cases}$$

$$g(x) = \begin{cases} 0 & -\infty < x < -b \\ B & -b < x < b \\ 0 & b < x < \infty \end{cases}$$

where $a > b$. [Note that the g of $g(x)$ has nothing to do with the g of $g(\omega)$, above. It is just convenient notation for a second function.]

(a) Without performing any integrations, by considering the nature of convolution (i.e. smearing each element by the convolution function, or kernel) determine and sketch the function $h(x) = f(x) * g(x)$ which is the convolution of the two functions, labeling all relevant quantities on the diagram. [Consider a small column of $f(x)$, height A, width dx, and smear it out by $g(x)$, i.e. spread it over width $2b$. Now at each x sum up the contribution from all dx.]

(b) By appropriate integration, using the expression for convolution

$$h(x) = \int_{-\infty}^{\infty} f(u) g(x-u) du \ ,$$

compute expressions for the different parts of $h(x)$ and compare to your previous result.

2. Derive an expression for the convolution $h(x) = f(x) * g(x)$ of the two normalised Gaussian functions $f(x) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_1^2}}$, $g(x) = \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_2^2}}$, by applying the convolution theorem *i.e.* by taking their Fourier transforms, multiplying together, and then transforming back. The inverse FT of a normalised Gaussian function is

$$f(x) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\omega^2}{2\sigma^2}} e^{i\omega x} dx = e^{-\frac{x^2\sigma^2}{2}} \ .$$

Fourier

Classwork 1 answers October 14, 2011

Evaluation of Fourier series

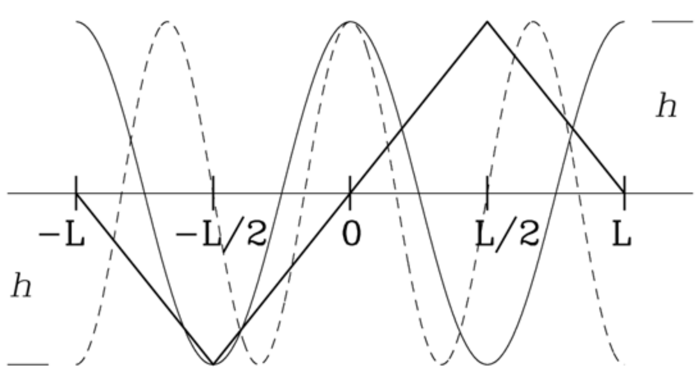
1. There are three linear segments:

$$f(x) = \begin{cases} -2h(L+x)/L & (-L \le x \le -L/2) \\ 2hx/L & (-L/2 \le x \le L/2) \\ 2h(L-x)/L & (L/2 \le x \le L) \end{cases}$$

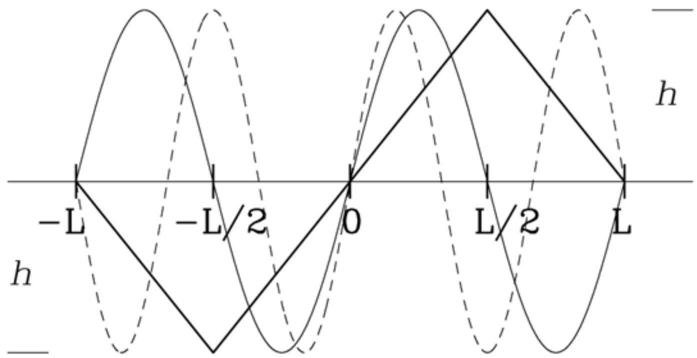
2. Inserting the three functions into the Euler-Fourier formulae gives

$$\begin{aligned} a_0 &= -\tfrac{1}{L}\int_{-L}^{-L/2} \tfrac{2h(L+x)}{L}dx + \tfrac{1}{L}\int_{-L/2}^{L/2} \tfrac{2hx}{L}dx + \tfrac{1}{L}\int_{L/2}^{L} \tfrac{2h(L-x)}{L}dx\ , \\ a_n &= -\tfrac{1}{L}\int_{-L}^{-L/2} \tfrac{2h(L+x)}{L}\cos(n\pi x/L)dx + \tfrac{1}{L}\int_{-L/2}^{L/2} \tfrac{2hx}{L}\cos(n\pi x/L)dx \\ &\quad +\tfrac{1}{L}\int_{L/2}^{L} \tfrac{2h(L-x)}{L}\cos(n\pi x/L)dx\ , \\ b_n &= -\tfrac{1}{L}\int_{-L}^{-L/2} \tfrac{2h(L+x)}{L}\sin(n\pi x/L)dx + \tfrac{1}{L}\int_{-L/2}^{L/2} \tfrac{2hx}{L}\sin(n\pi x/L)dx \\ &\quad +\tfrac{1}{L}\int_{L/2}^{L} \tfrac{2h(L-x)}{L}\sin(n\pi x/L)dx\ . \end{aligned}$$

3. There is no need to evaluate all these integrals. The function $f(x)$ is plotted below, together with $\cos(n\pi x/L)$, $n = 2$ (solid) and $n = 3$ (dashed). The expression for a_0, above, involves simply integrating the function $f(x)$ over $-L \le x \le L$, and is zero, by inspection. The expression for a_n involves integrating the product of $f(x)$ and each $\cos$ term. By reference to the figure, because $f(-x) = -f(x)$, i.e. $f(x)$ is odd, while $\cos(n\pi x/L)$ is even for every contribution to the integrals for $x > 0$ there is an identical negative contribution at $-x$, which cancels, with the result that $a_n = 0$. So all a terms are zero if $f(x)$ is odd.



4. Again, with a little thought, we can avoid doing most of the integrals. The sketch below plots $\sin(n\pi x/L)$, $n = 2$ (solid) and $n = 3$ (dashed). We can see:



- that both $f(x)$ and $\sin(n\pi x/L)$ are odd, which means that at every point x the contribution to the integral is matched by an identical term at the point $-x$. Therefore, regardless of the value of n

$$\int_{-L}^{0} f(x)\sin(n\pi x/L)dx = \int_{0}^{L} f(x)\sin(n\pi x/L)dx \ ,$$

so that

$$\int_{-L}^{L} f(x)\sin(n\pi x/L)dx = 2\int_{0}^{L} f(x)\sin(n\pi x/L)dx \ .$$

This statement is always true if $f(x)$ is odd.

- that for $n = 2, 4, 6...$

$$\int_{0}^{L/2} f(x)\sin(n\pi x/L)dx = -\int_{L/2}^{L} f(x)\sin(n\pi x/L)dx \ ,$$

so that

$$\int_{0}^{L} f(x)\sin(n\pi x/L)dx = 0 \, .$$

and so $b_n = 0$, $n = 2, 4, 6...$ This statement is always true for an odd function of period $2L$, which is also even over the interval $0 < x < L$, about $x = L/2$.

- that for $n = 1, 3, 5...$

$$\int_{0}^{L/2} f(x)\sin(n\pi x/L)dx = \int_{L/2}^{L} f(x)\sin(n\pi x/L)dx \ ,$$

so that

$$\int_{0}^{L} f(x)\sin(n\pi x/L)dx = 2\int_{0}^{L/2} f(x)\sin(n\pi x/L)dx \ ,$$

and

$$\int_{-L}^{L} f(x)\sin(n\pi x/L)dx = 4\int_{0}^{L/2} f(x)\sin(n\pi x/L)dx \ .$$

This statement is always true for an odd function of period $2L$, which is also even over the interval $0 < x < L$, about $x = L/2$.

5. We need to evaluate the integral

$$b_n = \frac{4}{L}\int_0^{L/2} f(x)\sin(n\pi x/L)dx \; n = 1, 3, 5... \,,$$

$$b_n = \frac{8h}{L^2}\int_0^{L/2} x\sin\left(\frac{n\pi x}{L}\right)dx \,.$$

Integrating by parts we obtain

$$b_n = \frac{8h}{L^2}\left[\frac{-xL}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right)\right]_0^{L/2} = \frac{8h}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) \,.$$

So the required Fourier sin series is

$$f(x) = \frac{8h}{\pi^2}\sum_{n=1,3,5,...}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{L}\right) \,.$$

Fourier

Classwork 2 answers October 21, 2011

Differentiation and integration of Fourier series

1. The function $f(x) = x^2$ is even, so all $b_n = 0$. Then we determine the a terms.

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx = \frac{1}{L}\int_{-L}^{L} x^2 dx = \frac{1}{L}\left[\frac{x^3}{3}\right]_{-L}^{L} = 2L^2/3 \; .$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\pi x/L)dx = \frac{1}{L}\int_{-L}^{L} x^2\cos(n\pi x/L)dx = \frac{2}{L}\int_{0}^{L} x^2\cos(n\pi x/L)dx \; .$$

Integrating successively by parts leads to:

$$a_n = \frac{2}{L}\left(\left[\frac{x^2 L}{n\pi}\sin(n\pi x/L)\right]_0^L - \frac{2L}{n\pi}\int_0^L x\sin(n\pi x/L)dx\right) = 0 - \frac{4}{n\pi}\int_0^L x\sin(n\pi x/L)dx \; ,$$

$$= -\frac{4}{n\pi}\left(\left[-\frac{xL}{n\pi}\cos(n\pi x/L)\right]_0^L + \frac{L}{n\pi}\int_0^L \cos(n\pi/L)dx\right) \; ,$$

$$= \frac{4L^2}{n^2\pi^2}\cos(n\pi) + 0 \; .$$

So the Fourier series is

$$f(x) = x^2 = \frac{L^2}{3} + \sum_{n=1}^{\infty}\frac{4L^2}{n^2\pi^2}\cos(n\pi)\cos(n\pi x/L) \; .$$

2. Differentiating the series and dividing by 2 we obtain

$$f(x) = x = \sum_{n=1}^{\infty} -\frac{2L}{n\pi}\cos(n\pi)\sin(n\pi x/l) \; .$$

As expected, this matches the Fourier series for $f(x) = x$, given in the question.

3. We first integrate the expression for the Fourier series for $f(x) = x^2$, and multiply by 3, which gives

$$f(x) = x^3 = L^2 x + \sum_{n=1}^{\infty}\frac{12L^3}{n^3\pi^3}\cos(n\pi)\sin(n\pi x/L) + C \; ,$$

where C is a constant of integration. The above is not a Fourier series since it contains x. We can substitute the Fourier series for x to give

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{12L^3}{n^3\pi^3} - \frac{2L^3}{n\pi}\right)\cos(n\pi)\sin(n\pi x/L) + C \; .$$

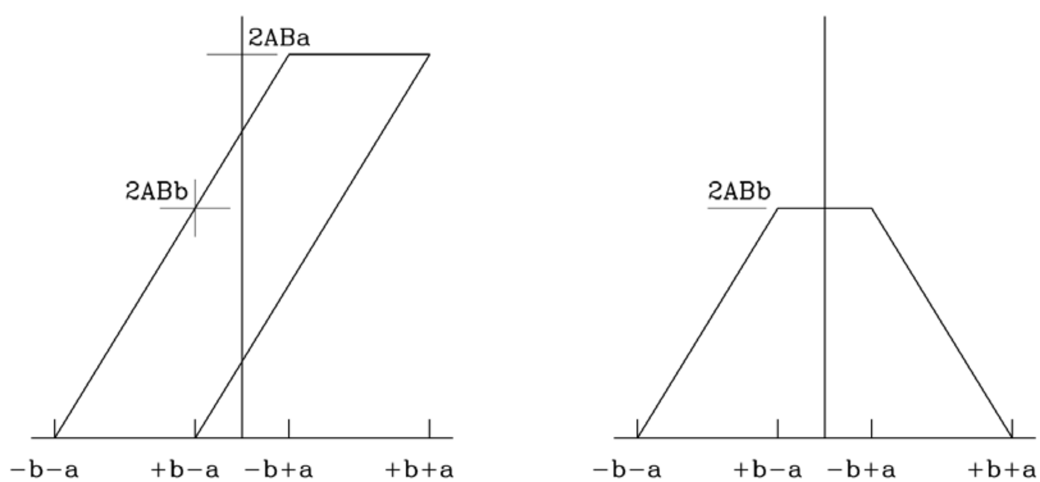
The constant term C must match $a_0/2$, the average value of the function $f(x) = x^3$. But this is zero, since x^3 is odd, leaving the expression in the question.

Fourier

Classwork 3 answers November 1, 2011

Convolution

1. (a) Within the range $-a < x < a$, each thin column Adx at x is smeared out over the interval $x - b$ to $x + b$, to create a thin flat rectangle of thickness (height) $ABdx$, centred on x. [The area of the rectangle is $2ABb\,dx$ which is Adx times the area of $g(x)$.] These rectangles, centred at the relevant value of x, may be imagined as stacked on top of each other, resembling a pack of cards that has been sheared over. Then $h(x)$ is the sum of all the thicknesses of the cards at any x. This imaginary stack is pictured below, *LHS*. [If this is not clear, slice $f(x)$ into a specific number of columns, say 10, and stack the smeared rectangles.] The thickness of the stack increases over the interval $-b - a < x < +b - a$. The thickness of the stack at the point $x = +b - a$ is $h(x) = 2ABb$, and $h(x)$ remains at this value as far as $x = 0$. For $x > 0$, $h(x)$ is the even-function reflection of $h(x)$ for $x < 0$. Viewed in this way, we expect $h(x)$ to be the function plotted and labeled below, *RHS*.



(b) While the above reasoning in terms of smearing provides the right answer, a more straightforward way is to flip the convolution function $g(x)$ and then slide it along the x axis. At each point we form the product of f and g (flipped) and integrate. This is what the formula for $h(x)$ is saying. By drawing $f(x)$ and $g(x)$ we can see that over the range $-b - a < x < +b - a$ the rectangle g partially overlaps f by the amount $x - (-b - a) = x + b + a$, and since the height of the two functions are A and B, the integral $h(x) = \int_{-\infty}^{\infty} f(u)g(x - u)du = AB(x + b + a)$. This function is a straight line passing through $x, y = -b - a, 0$ and reaching a height $2ABb$ when $x = +b - a$. This is exactly as expected, and plotted above (*RHS*). At this point the smaller (g) rectangle lies entirely within the broader (f) rectangle and then $h(x)$ becomes constant, until $x = 0$. From symmetry considerations, the remainder of $h(x)$ is the mirror image over the region

$x < 0$. This solution therefore agrees with the analysis in the first part. Note that we have convolved f with g. You might want to try convolving g with f to convince yourself that convolution is commutative i.e. $f * g = g * f$.

2. Writing the Fourier transform as $\mathcal{F}f(x)$, then the convolution theorem states that if $h(x) = f(x)*g(x)$, then $\mathcal{F}h(x) = 2\pi\mathcal{F}f(x)\mathcal{F}g(x)$. Now the FT of a normalised Gaussian of dispersion σ is $\frac{1}{2\pi}e^{-\frac{\sigma^2\omega^2}{2}}$. Therefore we find

$$\mathcal{F}h(x) = 2\pi\frac{1}{2\pi}e^{-\frac{\sigma_1^2\omega^2}{2}}\frac{1}{2\pi}e^{-\frac{\sigma_2^2\omega^2}{2}} = \frac{1}{2\pi}e^{-\frac{(\sigma_1^2+\sigma_2^2)\omega^2}{2}} \ .$$

This is a Gaussian of dispersion $\sigma^2 = 1/(\sigma_1^2 + \sigma_2^2)$. It would need to be multiplied by a factor $\sqrt{2\pi}/\sigma = \sqrt{2\pi(\sigma_1^2 + \sigma_2^2)}$ to be properly normalised. Therefore when we use the provided formula for the inverse FT, which applies when the Gaussian is normalised, we need to divide by this factor. This leads to

$$h(x) = \frac{1}{\sqrt{2\pi(\sigma_1^2 + \sigma_2^2)}}e^{-\frac{x^2}{2(\sigma_1^2+\sigma_2^2)}} \ .$$

This says that when convolving two normalised Gaussians (centred on 0) the result is a normalised Gaussian where the variances have been added.