Fourier: Fourier series and Fourier transforms

Lecture 1

1. Introduction

Fourier methods date from 1807 when Fourier asserted that any periodic function \( f(x) \) of repeat length \( 2L \) could be written as an infinite sum of \( \sin \) and \( \cos \) terms:

\[
f(x) = f(x + 2L) = a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{n \pi x}{L}\right) + b_n \sin\left(\frac{n \pi x}{L}\right) \right]
\]

1.1 Example application: The RL low-pass filter

\[
\begin{align*}
\text{differential eqn:} \\
L \frac{dI}{dt} + RI &= V_\text{in}(t) \\
\text{Same maths as driven damped mass} \\
m \frac{dV}{dt} + \lambda V &= F(t)
\end{align*}
\]

For sinusoidal input \( V_\text{in} = V_0 \sin(\omega t) \), \( I \) is sinusoidal:

\[
I = \frac{V_0 \sin(\omega t - \Theta)}{(R^2 + \omega^2 L^2)^{\frac{1}{2}}} \\
\Theta = \tan^{-1}\left(\frac{\omega L}{R}\right)
\]

\[
V_\text{out}(t) = RI = \frac{V_0 \sin(\omega t - \Theta)}{(1 + \frac{\omega^2 L}{R^2})^{\frac{1}{2}}} \\
\text{phase shift} \\
\text{attenuation - high frequencies are suppressed (low pass)}
Eqn (2) is linear so we may superpose solutions

\[ \frac{L}{d} \left( I_1 + I_2 \right) + R \left( I_1 + I_2 \right) = \frac{L}{d} I_1 + R I_1 + \frac{L}{d} I_2 + R I_2 \]

\[ = V_1 + V_2 \]

in other words if \( V_1 \) produces \( I_1 \), and \( V_2 \) produces \( I_2 \), then \( V_1 + V_2 \) produces \( I_1 + I_2 \).

Suppose input signal is sum of two sinusoids

\[ V_{\text{in}} = V_{01} \sin (\omega_1 t) + V_{02} \sin (\omega_2 t) \]

then

\[ V_{\text{out}} = \frac{V_{01} \sin (\omega_1 t - \theta_1)}{\left(1 + \frac{L_1^2}{R^2}\right)^{1/2}} + \frac{V_{02} \sin (\omega_2 t - \theta_2)}{\left(1 + \frac{L_2^2}{R^2}\right)^{1/2}} \]

e.g. if \( \omega_2 \gg \omega_1 \), the filter suppresses the 2nd term.

We can now use Fourier's insight to compute the response to any input periodic signal.
What is response to an square wave, period $2T$, amplitude $A$?

We represent $V_n(t)$ as a Fourier series:

$$V_n = \sum_{n=1,3,5,...}^{\infty} \frac{4A}{n\pi} \sin \left( \frac{n\pi x}{T} \right) = V_1 + V_3 + V_5 + ...$$

Then by superposition:

$$V_{out} = \sum_{n=1,3,5,...}^{\infty} \frac{4A}{n\pi} \sin \left( \frac{n\pi x}{T} - \Theta_n \right) \left( 1 + \frac{n^2\pi^2 L^2}{R^2} \right)^{-\frac{1}{2}}$$

where

$$\Theta_n = \tan^{-1} \left( \frac{n\pi L}{T R} \right)$$

Input showing first two terms of Fourier series.

$V_{in}$

$V_{out}$ = sum of attenuated and phase shifted terms eqn 3
1.2 Fourier methods in physics

- Solving damped harmonic oscillator for periodic driving force

- Solving 2nd order linear partial DE in particular the heat, diffusion, wave, Laplace, Schrödinger eqns (DE course)

- Fundamental in most areas of physics, particularly optics, solid state, QM, cosmology

- Fourier methods are just one of a range of related methods that employ orthogonal basis functions (including Legendre and Hermite polynomials, and Bessel functions)
Lecture 2

2. Fourier Series

2.1 Even and odd functions

Define an even function by
\[ e(x) = e(-x) \]
reflect through \( x = 0 \)

Examples are \( \cos(x), x^2 \)

Define an odd function by
\[ o(x) = -o(-x) \]

Examples are \( \sin(x), x^3 \)

In general a function \( f(x) \) may be written as a sum of an odd and even function
\[ f(x) = e(x) + o(x) \]
\[ f(-x) = e^{-x} + o(-x) = e(x) - o(x) \]

\[ e(x) = \frac{1}{2} \left[ f(x) + f(-x) \right] \]
\[ o(x) = \frac{1}{2} \left[ f(x) - f(-x) \right] \]

For example:
\[ f(x) = e^x \]
\[ e(x) = \frac{1}{2} \left[ e^x + e^{-x} \right] = \cosh(x) \]
\[ o(x) = \frac{1}{2} \left[ e^x - e^{-x} \right] = \sinh(x) \]

Note that:
1. Even times even is even
   \[ f(x) = e(x) \cdot e_2(x) \]
   \[ f(-x) = e(-x) \cdot e_2(-x) = e(x) \cdot e_2(x) = f(x) \]
2. Odd times odd is even
   \[ f(x) = o(x) \cdot o_2(x) \]
   \[ f(-x) = o(-x) \cdot o_2(-x) = +o(x) \cdot o_2(x) = f(x) \]
3. Odd times even is odd
4. \[ -a \int o(x) \, dx = 0 \]
5. \[ a \int e(x) \, dx = 2 \int e(x) \, dx \]
2.2 **Orthogonal Functions**

Two (complex) functions \( f(x) \) and \( g(x) \) are said to be orthogonal on the interval \( a \leq x \leq b \) if

\[
\int_a^b f(x)^* g(x) \, dx = 0 \quad \text{if } \quad * \text{ is complex conjugate.}
\]

For real functions

\[
\int_a^b f(x) g(x) \, dx = 0
\]

The infinite set of functions used in Fourier series

\[
1, \quad \cos \left( \frac{n \pi x}{L} \right), \quad \sin \left( \frac{n \pi x}{L} \right)
\]

form a complete set of orthogonal functions on the interval \(-L \leq x \leq L\).

Completeness: there exists no other function orthogonal to all of the functions.

The completeness is why we can form a Fourier series.

The orthogonality is why the coefficients \( a_n, b_n \) are easy to compute.

[There is an analogy between orthogonal functions and orthogonal vectors]
Clearly 1 is orthogonal to \( \cos \left( \frac{n\pi x}{L} \right) \) and \( \sin \left( \frac{n\pi x}{L} \right) \)

since \( \int_{-L}^{L} \sin \left( \frac{n\pi x}{L} \right) dx = 0 = \int_{-L}^{L} \cos \left( \frac{n\pi x}{L} \right) dx \)

We need to show that

1. \( \int_{-L}^{L} \cos \left( \frac{n\pi x}{L} \right) \sin \left( \frac{m\pi x}{L} \right) dx = 0 \quad \text{all } n, m \)

2. \( \int_{-L}^{L} \cos \left( \frac{n\pi x}{L} \right) \cos \left( \frac{m\pi x}{L} \right) dx = 0 \quad n \neq m \)

3. \( \int_{-L}^{L} \sin \left( \frac{n\pi x}{L} \right) \sin \left( \frac{m\pi x}{L} \right) dx = 0 \quad n \neq m \)

For 1, \( \cos \) is even, \( \sin \) is odd, product is odd, \( \mathcal{O}(1) \), and

\[ \int_{-L}^{L} \cos \left( \frac{n\pi x}{L} \right) dx = 0 \]

For 2 use \( \cos \left( \frac{n\pi x}{L} \right) = \frac{e^{i\frac{n\pi x}{L}} - e^{-i\frac{n\pi x}{L}}}{2} \)

Problem sheet 1
For \( n = m \) we have

\[
I = \int_{-L}^{L} \cos^2 \frac{n \pi x}{L} \, dx
\]

graphically e.g. \( n = 1 \)

Clearly average value of \( \cos^2 \frac{n \pi x}{L} \) is \( \frac{1}{2} \)

\[
\therefore \quad I = \frac{2L}{2} = L
\]

Other important examples of complete sets of orthogonal functions (basis functions) are Legendre and Hermite polynomials, and Bessel functions.

We can construct e.g. Fourier-Legendre series (spherical harmonics in QM)
2.3 The Euler-Fourier formulae

We will assume we can express a periodic function $f(x)$, repeat length $2L$, as a Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

The Euler-Fourier formulae are expressions for the coefficients $a_n, b_n$, and exploit the orthogonality of the basis functions

$$g(x) = 1, \cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)$$

call these $g(x)[a_n], g(x)[b_n]$

To determine the coefficients, multiply $f(x)$ by $g(x)$ and integrate over a period

$$I = \int_{-L}^{L} f(x) g(x) \, dx$$

e.g. to determine $a_n$

$$I = \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx = \int_{-L}^{L} \left[ a_0 + \sum_{m=1}^{\infty} a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right) \right] \cos\left(\frac{n\pi x}{L}\right) \, dx$$

but $\cos\left(\frac{n\pi x}{L}\right)$ is orthogonal to every term in $[ ]$ except $\cos\left(\frac{n\pi x}{L}\right)$, $n = m$

$$I = \int_{-L}^{L} a_n \cos^2\left(\frac{n\pi x}{L}\right) \, dx = a_n \frac{1}{2} \cdot 2L = a_n L$$

\[\text{**Note**: the integral is $\frac{1}{2}$ over the interval.} \]

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx = \frac{1}{L} \int_{-L}^{L} f(x) g(x)[a_n] \, dx$$
Similarly to get $b_n$, write

$$I = \int_{-L}^{L} f(x) \sin \left(\frac{n\pi x}{L}\right) \, dx = b_n \int_{-L}^{L} \sin \left(\frac{n\pi x}{L}\right) \, dx = b_n \frac{1}{2} 2L$$

$\therefore b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \left(\frac{n\pi x}{L}\right) \, dx = \frac{1}{L} \int_{-L}^{L} f(x) g(x) [b_n] \, dx \quad (2)$

And to get $a_0$:

$$I = \int_{-L}^{L} f(x) \, dx = \frac{1}{2} \int_{-L}^{L} \, dx = a_0 \frac{2L}{2} = a_0 L$$

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) \, dx = \frac{1}{L} \int_{-L}^{L} f(x) g(x) [a_0] \, dx \quad (3)$$

Eqs. (1), (2), (3) are the Euler-Fourier formulae.

In general

$$a_0, a_n, b_n = \frac{1}{L} \int_{-L}^{L} f(x) g(x) \, dx$$

This explains why the constant term appears as $\frac{a_0}{2}$.

$\frac{a_0}{2}$ is the average value of $f(x)$ over one period.
Example: square wave

\[
f(x) = \begin{cases} 
-A & -L < x < 0 \\
A & 0 < x < L 
\end{cases}
\]

The Fourier series is e.g.

\[
a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n \pi x}{L}\right) \, dx
\]

\[
a_n = \frac{1}{L} \int_{-L}^{0} (-A) \cos\left(\frac{n \pi x}{L}\right) \, dx + \frac{1}{L} \int_{0}^{L} A \cos\left(\frac{n \pi x}{L}\right) \, dx
\]

\[
a_n = \frac{1}{L} \int_{-L}^{0} (-A) \cos\left(\frac{n \pi x}{L}\right) \, dx + \frac{1}{L} \int_{0}^{L} A \cos\left(\frac{n \pi x}{L}\right) \, dx
\]

\[\text{etc.}\]

Sample:

\[
a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n \pi x}{L}\right) \, dx
\]

\[
a_n = \frac{1}{L} \int_{-L}^{L} \begin{cases} 
-A & \text{odd} \\
A & \text{even} 
\end{cases} \cos\left(\frac{n \pi x}{L}\right) \, dx
\]

\[
\text{odd} \quad \Rightarrow \quad a_n = 0
\]

\[
a_0 = \frac{1}{L} \int_{-L}^{L} f(x) \, dx
\]

\[
\text{odd} \quad \Rightarrow \quad a_0 = 0
\]

\[
b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n \pi x}{L}\right) \, dx
\]

\[
b_n = \frac{1}{L} \int_{-L}^{L} \begin{cases} 
-A & \text{odd} \\
A & \text{odd} 
\end{cases} \sin\left(\frac{n \pi x}{L}\right) \, dx
\]

\[\text{odd} \quad \Rightarrow \quad b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin\left(\frac{n \pi x}{L}\right) \, dx\]
\[ b_n = \frac{2A}{L} \int_0^L \sin \left( \frac{n\pi x}{L} \right) \, dx = \frac{2A}{L} \frac{L}{n\pi} \left[ \cos \left( \frac{n\pi x}{L} \right) \right]_0^L \]

\[ b_n = \frac{2A}{n\pi} \left[ 1 - \cos \left( \frac{n\pi}{L} \right) \right] \]

\[ b_{2,4,6...} = 0 \]

\[ b_{1,3,5...} = \frac{4A}{n\pi} \]

\[ A(x) = \sum_{n=1,3,5...}^{\infty} \frac{4A}{n\pi} \sin \left( \frac{n\pi x}{L} \right) \]

---

**Some tips on calculating Fourier series**

1. If \( f(x) \) is odd, \( f(x) g(x) [a_n] \) is odd
   \[ a_0, a_n = 0 \quad \text{need only sin terms} \]

2. If \( f(x) \) is even, \( f(x) g(x) [b_n] \) is odd
   \[ b_n = 0 \quad \text{need only constant \& cos terms} \]

3. Use
   \[ \int_{-L}^{L} f(x) g(x) \, dx = \int_{-L+q}^{L+q} f(x) g(x) \, dx \quad \text{for any } q \]

because \( f(x) g(x) \) is periodic with period \( 2L \)

i.e. choose interval length \( 2L \) to simplify
3. Suppose you have a function $f(x)$ defined over $0 \leq x \leq a$ for which Fourier series is needed.

3 possibilities:

i.) set $a = 2L$ and repeat

ii.) set $a = L$ and add reflected extension

Function is even, need only $a_0$, $a_n$ terms.
iii) Set \( a = L \) and set \( f(-x) = -f(x) \) to create odd function.

\[ \text{only } n \text{ terms needed} \]

3 different Fourier series created, but each matches the function over \( 0 \leq x \leq a \).
2.4 Dirichlet conditions

If a function satisfies the Dirichlet conditions, its Fourier series converges to the function, or to \( \frac{1}{2} (f^- + f^+) \) at a discontinuity.

The Dirichlet conditions are sufficient but not necessary; i.e., some function may not satisfy the DC, e.g., \( \sin^2(\pi x) \), but may still be represented by a FS.

The Dirichlet conditions:

i) \( f(x) \) must be periodic

ii) \( f(x) \) must be single valued, with a finite number of discontinuities (within one period)

iii) \( f(x) \) must have a finite number of maxima and minima

iv) \( \int_{-L}^{L} |f(x)| \, dx \) must converge.

E.g. \( f(x) = \frac{1}{x} \) \( -L \leq x \leq L \) cannot be expressed as a FS.
2.5 Convergence of Gibbs phenomenon

The square wave is discontinuous in value.

The FS is
\[ f(x) = \frac{4A}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi x}{L}\right) \]

The triangular wave is discontinuous in slope.

The FS is
\[ f(x) = \frac{8A}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi x}{L}\right) \]

Note that
\[ \frac{d}{dx} \left[ \frac{1}{n^2} \sin\left(\frac{n\pi x}{L}\right) \right] = \frac{n\pi}{L} \frac{1}{n} \cos\left(\frac{n\pi x}{L}\right) \]

The derivative of a triangular function is a square wave.

Functions discontinuous in value are characterised by \( a_n, b_n \propto \frac{1}{n} \) and converge slowly.

This is because higher-order terms are needed to fit the discontinuity.

Functions that are continuous converge rapidly. As a rule, in the sequence of derivatives, \( \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots \), convergence becomes more rapid.

We can recognize the problem with \( a_n, b_n \propto \frac{1}{n} \) as follows.

A series \( s = t_1 + t_2 + \ldots + t_n + \ldots \)

converges absolutely if
\[ \lim_{n \to \infty} \frac{p_n}{t_n} = \left| \frac{t_{n+1}}{t_n} \right| < 1 \]

For \( a_n \propto \frac{1}{n} \)
\[ p_n = \left| \frac{n}{n+1} \right| < 1 \] just converges.
At the discontinuity the FS overshoots. For large \( n \) the overshoot remains finite and constant but becomes narrower (see slide). This is the Gibbs phenomenon.

\[
\begin{align*}
\text{\( n = 30 \)} & \quad \text{\( n = 10 \)} \\
\end{align*}
\]

2.6 Parseval's theorem

In a harmonic oscillator the energy \( \propto (\text{amplitude})^2 \).

Parseval's theorem relates the average value of \((f(x))^2\) to the same for the Fourier components.

For a single sin wave

\[
f(x) = \sin(\pi x/L)\]

\[
\frac{1}{2L} \int_{-L}^{L} |f(x)|^2 \, dx = A^2 \frac{1}{2} \frac{2L}{2L} = \frac{A^2}{2}
\]

For a general periodic function \( f(x) \) is

\[
\frac{1}{2L} \int_{-L}^{L} |f(x)|^2 \, dx = \frac{1}{2L} \int_{-L}^{L} \left[ a_0 + \sum \left( a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi x}{L} \right) \right) \right]^2 \, dx
\]
But all the terms are mutually orthogonal so

\[ \frac{1}{2L} \int _{-L}^{L} \left[ f(x) \right] ^2 dx = \frac{1}{2L} \int _{-L}^{L} \left[ \frac{a_0}{2} \right] ^2 + \sum \left( a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi x}{L} \right) \right) dx \]

\[ \text{contact} \quad \text{arc} \quad \frac{1}{2} \]

\[ = \left( \frac{a_0}{2} \right) ^2 + \frac{1}{2} \sum (a_n ^2 + b_n ^2) \]

1.1. average value of square of function is

\[ \text{sum of average value of square of Fourier coefficients} \]
2.7 Complex representation of Fourier series

Consider the set of functions \( e^{\frac{n\pi x}{L}} \) for \( n = -\infty, \infty \).

They form an orthogonal set of basis functions on the interval \(-L \leq x \leq L\).

\[
\int_{-L}^{L} f(x) g(x)^* \, dx = 0 = \int_{-L}^{L} e^{\frac{n\pi x}{L}} e^{-\frac{(m\pi x)/L}{L}} \, dx
\]

\[
= \int_{-L}^{L} \cos \left( \frac{(n-m)\pi x}{L} \right) \, dx = 0 \quad n \neq m
\]

They provide a real way of writing Fourier series that can be easier to manipulate.

\[
\hat{f}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}
\]

\[
= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \left( e^{\frac{n\pi x}{L}} + e^{-\frac{n\pi x}{L}} \right) + \frac{b_n}{2i} \left( e^{\frac{n\pi x}{L}} - e^{-\frac{n\pi x}{L}} \right) \right]
\]

\[
= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ \frac{a_n}{2} \left( e^{\frac{n\pi x}{L}} + e^{-\frac{n\pi x}{L}} \right) + \frac{b_n}{2i} \left( e^{\frac{n\pi x}{L}} - e^{-\frac{n\pi x}{L}} \right) \right]
\]

\[
= \frac{a_0}{2} + \frac{1}{2} \sum_{n=1}^{\infty} \left[ (a_n - ib_n) e^{\frac{n\pi x}{L}} + (a_n + ib_n) e^{-\frac{n\pi x}{L}} \right]
\]
\[ f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i \pi n x / L} \]

\[ c_0 = \frac{a_0}{2} \]

\[ c_n = \frac{1}{2} (a_n - i b_n) \]

\[ c_{-n} = \frac{1}{2} (a_n + i b_n) \]

\[ c_n = \frac{1}{2L} \int_{-L}^{L} f(x) \left[ \cos \frac{n \pi x}{L} - i \sin \frac{n \pi x}{L} \right] dx \]

\[ c_{-n} = \frac{1}{2L} \sqrt{f(x)} \left[ \cos \left( \frac{n \pi x}{L} \right) + i \sin \left( \frac{n \pi x}{L} \right) \right] dx \]

\[ c_{-n} = \frac{1}{2L} \sqrt{f(x)} e^{-i \frac{n \pi x}{L}} dx \]

\[ c_0 = \frac{a_0}{2} = \frac{1}{2L} \int_{-L}^{L} f(x) dx = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i \pi x / L} dx \]

To summarize:

\[ f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i \pi n x / L} \]

\[ c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i \pi n x / L} dx \]
This reflects orthogonality—to pick out \( c_n \)

\[
\int f(x) \overline{g(x)} \overline{[c_n]} \, dx
\]

Note that if \( f(x) \) is real, \( c_n = c_n^* \) ‘Hermite’

2.8 Fourier integral example (introduction to Fourier transforms)

What happens to a Fourier series as \( L \to \infty \)?

Consider the periodic function

![Rectangular function diagram](image)

Function is even \( b_n = 0 \)

\[
a_0 = \frac{2A}{L} \quad a_n = \frac{2A}{L} \frac{\sin(n\pi a/L)}{(n\pi/L)}
\]

Now specialize to the case \( A=1, a=1, L \geq 1 \)

\[
a_0 = \frac{2}{L} \quad a_n = \frac{2}{L} \frac{\sin(n\pi/L)}{(n\pi/L)} = \frac{2}{L} \frac{\sin \omega_n}{\omega_n}
\]

\[
\omega_n = \frac{n\pi}{L}
\]

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi}{L} x \right)
\]
Plot the coefficients as a function of \( C_n = \frac{n \pi}{L} \)

For \( L = 2 \):
- \( \eta = 1, \eta = 2, \eta = 3, \eta = 5 \)

For \( L = 8 \):
- \( \eta = 4, \eta = 12, \eta = 20 \)

As \( L \) increases, the number of terms increases, but the value of each decreases.

Consider the limit \( L \to \infty \)
\[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} \]

\[ = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} \cdot \frac{1}{L} \int_{-L}^{L} f(x') \cos \frac{n\pi x'}{L} \, dx' \]

As \( L \) gets large \( a_0 \to 0 \) ignored.

Now the \( \pi \) spacing of the \( a_n \) is \( \frac{\pi}{L} \), so in an interval \( \Delta \omega \) there are \( \frac{\Delta \omega L}{\pi} \) components.

In that interval the contribution to the sum is

\[ \Delta f(x) = \cos \left( \frac{\omega x}{L} \right) \Delta \omega \frac{1}{\pi} \int_{-L}^{L} f(x') \cos \frac{n\pi x'}{L} \, dx' \]

So in the limit \( L \to \infty \) we can write

\[ f(x) = \int \cos (\omega x) A(\omega) \, d\omega \]

where

\[ A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x') \cos (\omega x') \, dx' \]

the Fourier cosine integral.

An analogous sin integral exists for odd functions.
3. Fourier transforms

3.1. Recap, leading to Fourier transform

Periodic function \( f(x) \), \(-L \leq x \leq L\) represented by

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi x}{L} \right) \right]
\]

where

\[
a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \left( \frac{n\pi x}{L} \right) \, dx, \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \left( \frac{n\pi x}{L} \right) \, dx
\]

\[W_n = \frac{n\pi}{L}\]

\( W \) is angular frequency. The variable \( x \) is unspecified.

If \( x \) is time \( t \), \( W \) is angular frequency

If \( x \) is space, \( W \) is spatial angular frequency

and often symbol \( k \) is used instead, \( k = 2\pi / L \)

Complex representation

\[
f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-\frac{i n\pi x}{L}}
\]

where

\[
c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{i \frac{n\pi x}{L}} \, dx
\]

The \( c_n \) are complex and

\[
c_n = \frac{1}{2} (a_n + b_n), \quad c_{-n} = \frac{1}{2} (a_n - b_n)
\]

and

\[
a_n = c_n + c_{-n}, \quad b_n = i (c_n - c_{-n})
\]

Typically \( c_n \) are easier to compute, and we can then get \( a_n, b_n \) from them.
We can represent the function \( f(x) \) equally well as a curve \( f(x) \), or in frequency space by a plot of \( a_n, b_n \) against \( w \).

![Graph](image.png)

As \( L \) gets larger, spacing gets smaller.

Consider (again) the limit as \( L \to \infty \)

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos(\omega_n x) + b_n \sin(\omega_n x) \right]
\]

\[
= \frac{1}{2L} \int_{-L}^{L} f(x) \, dx + \sum_{n=1}^{\infty} \left[ \frac{1}{L} \int_{-L}^{L} f(x) \cos(\omega_n x') \, dx' \right]
\]

\[
+ \sum_{n=1}^{\infty} \frac{1}{L} \int_{-L}^{L} f(x) \sin(\omega_n x') \, dx'
\]

Provided \( \int_{-L}^{L} f(x) \, dx \) is finite, the first term \( \to 0 \) at \( L \to \infty \)

so ignore

Look at 2nd term + 3rd term

When \( L \) is large, there are many waves contributing to the sum in an interval \( \Delta w \) at angular frequency \( w \)
No of terms in $\Delta w$ is $\frac{\Delta w \cdot L}{\pi}$, each of amplitude $a_l(\omega)$

So contribution to sum is $a_l b_l(\omega) \frac{\Delta w \cdot L}{\pi}$

$$\Delta f(x) = \cos(\omega x) \Delta w \frac{1}{\pi} \int_{-L}^{L} f(x') \cos(\omega x') \, dx' + \sin(\omega x) \Delta w \frac{1}{\pi} \int_{-L}^{L} f(x') \sin(\omega x') \, dx'$$

$$\sum \Delta f(x) = \sum \cos(\omega x) a_l(\omega) \Delta w + \sin(\omega x) b_l(\omega) \Delta w$$

In the limit $L \to \infty$, $w_1 = \frac{\pi}{L} \to 0$.

Define $A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos(\omega x) \, dx$, $B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin(\omega x) \, dx$.

$$f(x) = \lim_{\Delta w \to 0} \sum \Delta f(x) = \int_{0}^{\infty} \cos(\omega x) A(\omega) \, d\omega + \int_{0}^{\infty} \sin(\omega x) B(\omega) \, d\omega$$

Finally, we can use instead the complex exponential form. By analogy,

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} \, d\omega$$

$$\hat{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx$$
\[ f(\omega) = \frac{1}{2} \left[ A(\omega) - \omega B(\omega) \right] \]
\[ f(-\omega) = \frac{1}{2} \left[ A(\omega) + \omega B(\omega) \right] \]

\[ A(\omega) = g(\omega) + g(-\omega) \]
\[ B(\omega) = \omega \left[ g(\omega) - g(-\omega) \right] \]

\( f(\omega) \) is easier to compute, and the real

For \( f(x) \) real, \( A(\omega), B(\omega) \) are real,

and \( f(\omega) \) is complex.

\( f(\omega) \) is easier to compute, and could
then get \( A(\omega), B(\omega) \) for it

If \( f(x) \) is real, \( f(\omega) \) is Hermitian
(real even, imaginary odd)

---

[ people get used to complex representation and
rarely work with \( A(\omega), B(\omega) \) ]

From now on we will use the complex representation.
We are free here to distribute the $2\pi$! We could define

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx$$

in which case

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega) e^{i\omega x} \, d\omega$$

which is a nice symmetrical way of writing things.
There are many different definitions of the FT.

We have used

\[ f(x) = \int_{-\infty}^{\infty} g(w) e^{iwx} \, dw \quad \text{inverse FT} \]

\[ g(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-iwx} \, dx \quad \text{FT} \]

So we can write

\[ f(x) = \int_{-\infty}^{\infty} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x') e^{-iwx'} \, dx' \right] e^{iwx} \, dw \]

We could have written

\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x') e^{-iwx'} \, dx' \right] e^{iwx} \, dw \]

and defined

\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(w) e^{iwx} \, dw \]

\[ g(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-iwx} \, dx \quad \text{nearly symmetric} \]

We could also swap \( e^{iwx} \) and \( e^{-iwx} \)

or use \( t = \frac{w}{2\pi} \) as the variable.

Be careful with other textbooks.
3.2 FT of some important functions: rectangle, Gaussian, delta function

3.2.1 Consider the function \( f(x) \)

![Diagram of rectangle function]

\( f(x) \) is even, expect only \( \cos \) terms

\[
g(w) = \frac{1}{2} \left[ A(w) - B(w) \right]
\]

\[
g(-w) = \frac{1}{2} \left[ A(w) + B(w) \right]
\]

\( g(w) \) will be real and even.

\[
g(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ixw} \, dx
\]

\[
= \frac{A}{2\pi} \int_{-a}^{a} e^{-ixw} \, dx = \frac{A}{\pi w 2\pi} \left[ e^{-iwx} \right]_{-a}^{a}
\]

\[
= \frac{A}{\pi} \left[ e^{-iwa} - e^{-iwa} \right] = \frac{A}{\pi w} \left[ e^{iwa} - e^{-iwa} \right]
\]

\[
g(w) = \frac{A}{\pi} \frac{\sin(wa)}{wa} = \frac{A}{\pi} \frac{\sin(wa)}{wa}
\]

\[
g(w) = \frac{aA \sin(wa)}{\pi \omega a}
\]

\[
\text{Sinc } x = \frac{\sin x}{x}
\]
For the special case is called
\[ a = \frac{1}{2}, \quad A = 1, \quad f(t) \approx \Pi(t) \lor \text{Rect}(t) \]

\[ g(t) = \frac{1}{2\pi} \sin(c)(\frac{t}{2}) \]

\[ A(t) = g(t) + g(-t) \]

\[ A(t) = 2g(t) = 2\pi A \cdot \sin(c)(\frac{t}{2}) = \frac{2\pi A \cdot \sin(c)(\frac{t}{2})}{\pi} \]

\[ g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}} dx \]

\[ A(t) = e^{-\frac{t^2}{2\sigma^2}} - \frac{x^2}{2\sigma^2} - \frac{x^2}{2\sigma^2} \]

Spectrum of cosine

As \( \omega \) increases, width \( f(t) \) decreases.

3.2.2 FT of Gaussian \( f(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}} \)

\[ g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx \]

\[ A(t) = e^{-\frac{t^2}{2\sigma^2}} - \frac{x^2}{2\sigma^2} - \frac{x^2}{2\sigma^2} \]

\[ g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx \]

\[ A(t) = e^{-\frac{t^2}{2\sigma^2}} - \frac{x^2}{2\sigma^2} - \frac{x^2}{2\sigma^2} \]
Substitute

\[ y = x + \frac{\theta}{\sqrt{2\pi}} \]
\[ dy = dx \]

\[ f(\omega) = \frac{1}{2\pi} e^{-\frac{\omega^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{(y-i\omega)^2}{2\sigma^2}} dy \]
\[ = 1 \] (Gaussian integral)

\[ f(\omega) = \frac{1}{2\pi} e^{-\frac{\omega^2}{2}} \]
which is a Gaussian.

The FT of a Gaussian is a Gaussian.

The standard deviation of \( f(x) \) \( \Delta x = \sigma \)

The s. d. of \( f(\omega) \) \( \Delta \omega = \frac{1}{\sigma} \)

Note the fundamental result for a Gaussian \( \Delta x \Delta \omega = 1 \)
### 3.2.3 The FT of the Dirac \( \delta \) function

**How to treat an instantaneous impulse, or the mass per unit length of a point mass.**

*E.g.* a mass is hit over sub-interval \( T \)

\[
\frac{m \, dv}{dt} = F(t)
\]

\[
\frac{T}{2} \int_{-T/2}^{T/2} F(t) \, dt = \int m \, dv = m (v_f - v_0)
\]

Consider the case \( m (v_f - v_0) = \epsilon \) and \( T \) gets smaller.

As \( T \) gets smaller, \( F(t) \) gets higher.

*E.g.* rectangular function

\[
\delta(t) \text{ is the limiting case and satisfies}
\]

\[
\int \delta(t) \, dt = 1 \quad \text{and} \quad \delta(t) = 0 \quad t \neq 0
\]

More proper defined by

\[
\int_{a}^{b} f(x) \delta(x-x_0) \, dx = \begin{cases} f(x_0) & \text{if } a < x_0 < b \\ 0 & \text{otherwise} \end{cases}
\]

\( \delta(x-x_0) \) is a spike at \( x_0 \), i.e. when \( x-x_0 = 0 \)

\( \delta \) function 'shifts' \( f(x) \)
$\delta(x)$ is a "generalised function" or a distribution.

(not strictly a function).

We could equally treat $\delta(x)$ as an infinitely narrow rectangular function or Gaussian function.

FT of Gaussian $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$

$g(\omega) = \frac{1}{2\pi} e^{-\frac{\omega^2}{2\sigma^2}}$

In limit $\sigma \to 0$

$g(\omega) = \frac{1}{2\pi}$ i.e. constant.

All frequencies contribute to $\delta$ function.

$\delta$ function is infinitely wider in $x$ infinitely wide in $\omega$

Or compute normally

$g(\omega) = \left. \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} \, dx \right|_{-\infty}^{\infty} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(x) e^{i\omega x} \, dx$

$=-\frac{1}{2\pi} e^{i\omega 0}$ (sifting)

$g(\omega) = \frac{1}{2\pi}$
Recall
\[ f(x) = \int_{-\infty}^{\infty} f(w) e^{jwx} dw \quad \text{in FT} \]
\[ F(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-jwx} dx \quad \text{FT} \]

If \( g(w) = \delta(w - w_0) \)

\[ f(x) = \int_{-\infty}^{\infty} \delta(w-w_0) e^{jwx} dw = e^{jwx} \]

i.e. \( \delta(w-w_0) \) is FT of \( e^{jwx} \)

\[ \Rightarrow \quad f(x) = \frac{1}{2} \left[ e^{jwx} + e^{-jwx} \right] \]
\[ F(w) = \frac{1}{2} \left[ \delta(w-w_0) + \delta(w+w_0) \right] \]

= FT of \( \sin(w_0 x) \)

\[ A(w) = g(w) + g(-w) = \delta(w-w_0) \]

What is FT of \( \sin(w_0 x) \)?
Blurring, e.g., deforms is a convolution of a function \( f(x) \) by a kernel \( g(x) \).

Usually \( \int g(x) \, dx = 1 \) e.g., flux preserved.

\( g(x) \) may not be symmetric.

\[
\begin{align*}
&\text{we write } h(x) = f(x) \ast g(x) \\
&\text{an element } f(x) \, dx \text{ is blurred by } g(x)
\end{align*}
\]
We can write $h(x)$ two ways

$$h(x) = \int_{-\infty}^{\infty} f(u) g(x-u) \, du \quad \text{i.e. integrate over } f$$

Alternatively, consider the contribution of each part of $f$.

$$h(x) = \int_{-\infty}^{\infty} g(u) f(x-u) \, du \quad \text{i.e. integrate over } g$$

Therefore $f \ast g = g \ast f$ (commutative)
Convolution 2nd try!

Consider a (1D) camera that blurs images by a function $g(x)$ that conserves photons.

$$\int_{-\infty}^{\infty} g(x) \, dx = 1$$

Imagine a scene $f(x)$ with 2 point sources, with $N_1$ and $N_2$ photons detected.

$$N_1 \delta(x-x_1) \quad N_1 g(x-x_1) \quad N_2 \delta(x-x_2) \quad N_2 g(x-x_2)$$

The image recorded is blurred. The blurred scene $h(x)$ is the convolution of $f(x)$ by $g(x)$.

$$h(x) = f(x) * g(x)$$

Note that

$$\int_{-\infty}^{\infty} h(x) \, dx = \int_{-\infty}^{\infty} N_1 \delta(x-x_1) \, dx + \int_{-\infty}^{\infty} N_2 \delta(x-x_2) \, dx = N_1 + N_2$$

But $\int_{-\infty}^{\infty} g(x) \, dx$ could be anything e.g. 0.8 if some photons were lost, or any number. The above representation of convolution is independent of $\int_{-\infty}^{\infty} g(x) \, dx$. 
Now blur a continuous function \( f(x) \) with an arbitrary \( g(x) \). What is \( h(x) \)?

\[
\int f(u) \, dg(x-u)
\]

\((x \text{ variable, } u \text{ fixed})\)

At \( x \), contribution to \( h(x) \) from \( f(u) \, du \) is

\[
\int f(u) \, dg(x-u)
\]

Integrating all contributions at \( x \) we get

\[
h(x) = \int f(u) \, g(x-u) \, du = f * g
\]

\((x \text{ fixed, } u \text{ variable})\)

Conceptually we smear out all columns and at any \( x \), sum up the contributions to get \( h(x) \), the smoothed function at all \( x \).

Eqn 1 shows how to compute \( h(x) \) another way.
Shift $y$ to $x$ and reflect.

Hence, Eqn 1 says multiply $f(u)$ and $g(x-u)$ and integrate to get $h(x)$.

The contributions to $h(x)$ from pt A are the same in these two pictures.

We can show

$$\mathcal{F}(h(x)) = \mathcal{F}(f(u) * g(x-u)) = 2\pi \mathcal{F}(f(u)) \mathcal{F}(g(x-u))$$

in frequency space convolution $\equiv$ multiplication

$$\mathcal{F}(h(x)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(x) e^{-iwx} \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \int f(u) g(x-u) \, du \right] e^{-iwx} \, dx$$

changing the order of integration

$$\mathcal{F}(h(x)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) \left[ \int_{-\infty}^{\infty} g(x-u) e^{-iwx} \, dx \right] \, du$$

and change variables $v = x-u$, $dv = dx$

$$\mathcal{F}(h(x)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) \left[ \int_{-\infty}^{\infty} g(v) e^{-i(wv)} \, dv \right] \, du$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) e^{-iwx} \, du \frac{1}{2\pi} \int_{-\infty}^{\infty} g(v) e^{-i(wv)} \, dv = 2\pi$$

$$\mathcal{F}(h(x)) = \mathcal{F}(f(u)) \mathcal{F}(g(x-u)) 2\pi$$
3.4 Diffraction using FTs

Far-field (Fraunhofer) diffraction at a slit

incident plane wave

at large distance we recover plane waves.

All the waves suffer a phase difference relative to the central wave $\phi(x) = -kx \sin \theta = -kx \theta$ for small angles.

$k = 2\pi/\lambda$ wave number

So to compute the diffraction pattern on a screen at large distance we need to sum all the waves represented by

$$\text{Re} \left[ e^{-i(kr - wt - kx \theta)} \right]$$
Integrating over the aperture

\[ \int_a^b e^{-(kr - wt - kx\theta)} \, dx \]

so the amplitude is proportional to

\[ \int_a^b e^{ikx\theta} \, dx = \int_{-\infty}^{\infty} A(x) e^{ikx\theta} \, dx \]

where \( A(x) \) is rectangular function

Finally define \( w = k\theta \) and we have

\[ f(w) = \int_{-\infty}^{\infty} A(x) e^{iwx} \, dx \]

The angular dependence of the amplitude is the FT of the aperture!

Here \( \omega \) appears as an 'angular frequency' but is proportional to \( \theta \), and therefore \( x \) distance from axis on screen.

From earlier,

\[ f(w) = \frac{\pi}{1 + \frac{\sin(wa)}{ww} \frac{\sin(k\theta a)}{k\theta \pi}} \]
3.4 Diffraction using FTs

Far-field (Fraunhofer) diffraction at a slit

\[ k = \frac{2\pi}{\lambda} \]

incident plane wave

at large distance we recover plane waves.

All the waves suffer a phase difference relative to the central wave at

\[ \phi(x) = -kx \sin \theta = -kx \theta \] for small

k = \frac{2\pi}{\lambda} wave number angles

So to compute the diffraction pattern on a

d Screen at large distance we need to sum

all the waves represented by

\[ \text{Re} \left[ e^{i(kr - \omega t - kx \theta)} \right] \]
integrating over the aperture
\[ \int e^{-i(kr - wt - kx \theta)} \, dx \]

so the amplitude is proportional to
\[ \int e^{-i k \theta x} \, dx = \int A(x) e^{-i k \theta x} \, dx \]

where \( A(x) \) is rectangular function

Finally define \( \omega = k \theta \) and we have
\[ g(\omega) = \int A(x) e^{-i \omega x} \, dx \]

the angular dependence of the amplitude is the FT of the aperture!

Here \( \omega \) appears as an 'angular frequency' but is proportional to \( \theta \), and therefore 2 distance from axis on screen.

From earlier,
\[ g(\omega) = \frac{g}{\pi} \frac{\sin(wa)}{w} \text{ or } \frac{\sin(k \theta a)}{k \theta} \]
More complicated systems, e.g., 2 slit, 3 slit may be treated simply using a set of theorems including

- **Shift theorem**: \( \mathcal{F}[f(x+a)] = e^{-j\omega a} \mathcal{F}[f(x)] \)
- **Exponential multiplication**: \( \mathcal{F}[e^{j\omega x}f(x)] = \mathcal{F}[f(x)] \)
- **Scaling**: \( \mathcal{F}[f(ax)] = \frac{1}{|a|} \mathcal{F}\left(\frac{x}{a}\right) \)
- **Convolution theorem** (above)

See PS2 Q4

3.5 Parseval's theorem for complex notation

**Summary** (see Box A2.7)

**Fourier Series**

- If \( f(x) \) complex and \( |f(x)|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2 \)
- \( c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-jn\omega} dx \)

- If \( f(x) \) real, then \( |f(x)|^2 = f(x)^* = \sum_{n=-\infty}^{\infty} c_n c_{-n} \)

**Fourier Transform**

- \( \int |g(x)|^2 \, dx = \int g(x) g^*(x) \, dx = \frac{1}{2\pi} \int \hat{f}(\omega) \hat{f}(-\omega) \, d\omega \)
- \( \int f(x) \, dx \)
3.6 The uncertainty principle from Fourier theory

In many situations we are interested in \(|T(x)|^2\) and its spread e.g. in QM the spread in \(|\psi(x)|^2\) is \(\Delta x\) the uncertainty in position.

Consider a Gaussian

\[ T(x) \propto e^{-x^2/2\sigma^2} \]

\[ \psi(x) \propto e^{-x^2\sigma^2/2} \]

Then \( T(x)^2 \propto e^{-x^2/\sigma^2} \) \(\propto e^{-x^2/2\sigma^2} \)

\[ \psi^2 (x) \propto e^{-x^2/2\sigma^2} \propto e^{-x^2/\sigma^2} \]

\[ \sigma_{\psi} = \frac{1}{\sqrt{2\sigma}} \]

\[ \sigma_x = \frac{\sigma}{\sqrt{2}} \]

\( \sigma_{\psi} \sigma_x = \frac{1}{2} \) or \( \Delta_{\psi} \Delta_x = \frac{1}{2} \)

The Gaussian case is the limiting case. For other shapes

\( \Delta_{\psi} \Delta_x > \frac{1}{2} \)

Here \( x \) and \( \psi \) are any pair of conjugate variables e.g. position, wave number \( x, k \) time, angular frequency \( t, \omega \)
In QM we have
\[ \rho = \hbar k \quad E = \hbar \omega \]
and \( x, \frac{p}{\hbar}, t, \frac{E}{\hbar} \) are conjugate variables.

Then we recover
\[ \Delta x \Delta p = \hbar \Delta x \Delta k \geq \frac{\hbar}{2} \]
\[ \Delta t \Delta E = \hbar \Delta t \Delta \omega \geq \frac{\hbar}{2} \]