Mathematical Methods

Contents

Part I
Complex Functions

1 Function Definitions

Concerned with functions which $F:\mathbb{C}\to\mathbb{C}$ where $F\left(z\right)=U\left(x,y\right)+iV\left(x,y\right)$

Exponential

$$F\left(z\right)=e^{z}=\sum_{n=0}^{\infty}\frac{z^{n}}{n!}$$

Hyperbolic Function

$$\begin{aligned}\cosh\left(z\right)=&\frac{e^{z}+e^{-z}}{2}\\ \sinh\left(z\right)=&\frac{e^{z}-e^{-z}}{2}\end{aligned}$$

Trignometric Functions

$$\begin{aligned}\cos(z) =& \frac{e^{iz}+e^{-iz}}{2}\\ \sin(z) =& \frac{e^{iz}+e^{-iz}}{2i}\\ \cosh(iz) =& \cos(z)\\ \sinh(iz) =& i\sin(z)\end{aligned}$$

Logarithms

Defined as,

$$\begin{aligned}e^{\ln(z)} =& z\\ z =& re^{i\theta} = re^{i\theta+2\pi ni}\\ \ln(z) =& \ln(r) + i(\theta+2\pi n)\end{aligned}$$

To obtain a unique $\ln(z)$ we need to specify $\theta \in [\theta_0, \theta_0 + 2\pi[$

2 Differentiable Functions

Consider a function $F : \mathbb{C} \to \mathbb{C}$, construct

$$\frac{F(z_0+\Delta) - F(z_0)}{\Delta}$$

The complex function is differentiable if and only if,

$$\lim_{\Delta\to 0}\frac{F(z_0+\Delta) - F(z_0)}{\Delta}$$

exists and is independent of how $\Delta \to 0$

2.1 Riemann - Cauchy relation

A complex function, $F(z) = U(x, y) + iV(x, y)$, is differentiable if and only if the Riemann - Cauchy relations hold

$$\begin{aligned}\frac{\partial U}{\partial x} =& \frac{\partial V}{\partial y}\\ \frac{\partial V}{\partial x} =& -\frac{\partial U}{\partial y}\end{aligned}$$

Proof

1) Assume $f(z)$is differentiable. Let $\Delta = \delta x + i\delta y$ and first consider $\delta y = 0$,

$$\begin{aligned}\frac{\delta f}{\delta z} &= \frac{F(z_0+\Delta)-F(z_0)}{\Delta}\\ &= \frac{F(z_0+\delta x)-F(z_0)}{\delta x}\\ \frac{\delta f}{\delta z} &= \frac{U(x_0+\delta x,y_0)+iV(x_0+\delta x,y_0)-U(x_0,y_0)-iV(x_0,y_0)}{\delta x}\\ &= \frac{U(x_0+\delta x,y_0)-U(x_0,y_0)}{\delta x}+i\frac{V(x_0+\delta x,y_0)-V(x_0,y_0)}{\delta x}\\ \frac{df}{dz} &= \lim_{\delta x\to 0}\frac{\delta f}{\delta z}\\ &= \frac{\partial U}{\partial x}+i\frac{\partial V}{\partial x}\end{aligned}$$

now let $\delta x = 0$, thus

$$\begin{aligned}\frac{\delta f}{\delta z} &= \frac{F(z_0+\Delta)-F(z_0)}{\Delta}\\ &= \frac{F(z_0+i\delta y)-F(z_0)}{i\delta y}\\ \frac{\delta f}{\delta z} &= \frac{U(x_0,y_0+i\delta y)+iV(x_0,y_0+i\delta y)-U(x_0,y_0)-iV(x_0,y_0)}{i\delta y}\\ &= \frac{U(x_0,y_0+i\delta y)-U(x_0,y_0)}{i\delta y}+\frac{V(x_0,y_0+i\delta y)-V(x_0,y_0)}{\delta y}\\ \frac{df}{dz} &= \lim_{\delta y\to 0}\frac{\delta f}{\delta z}\\ &= -i\frac{\partial U}{\partial y}+\frac{\partial V}{\partial y}\end{aligned}$$

$\frac{df}{dz}$ must be unique, independent of how we approach x_0, y_0therefore

$$\frac{\partial U}{\partial x}+i\frac{\partial V}{\partial x} = -i\frac{\partial U}{\partial y}+\frac{\partial V}{\partial y}$$

and thus C-R is true, if our assumption is correct

2) Show that C-R $\Rightarrow$ differentiable

$$
\begin{aligned}
f(z) =& U(x,y) + iV(x,y) \\
df =& \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + i\left[\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy\right] \\
&From\ C-R \\
df =& \frac{\partial U}{\partial x}dx - \frac{\partial V}{\partial x}dy + i\left[\frac{\partial V}{\partial x}dx + \frac{\partial U}{\partial x}dy\right] \\
df =& \frac{\partial U}{\partial x}(dx+idy) + i\frac{\partial V}{\partial x}(dx+idy) \\
df =& \frac{\partial U}{\partial x}dz + i\frac{\partial V}{\partial x}dz \\
\frac{df}{dz} =& \left.\frac{\partial U}{\partial x}\right|_z + i\left.\frac{\partial V}{\partial x}\right|_z
\end{aligned}
$$

This is independent of dz therefore we can conclude $\frac{df}{dz}$ is reached independent of how $dz \to 0$, Thus if C-R holds then f is differentiable, and if f is differentiable then C-R must hold therefore C-R $\iff$ differentiable □

2.1.1 Further C-R Properties

1)

$$
\begin{aligned}
\frac{\partial^2 U}{\partial x^2} =& \frac{\partial}{\partial x}\frac{\partial V}{\partial y} = \frac{\partial}{\partial y}\frac{\partial V}{\partial x} = -\frac{\partial^2 U}{\partial y^2} \\
\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} =& 0 \\
\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} =& 0
\end{aligned}
$$

The real and imaginary parts of a complex function must be solutions to the wave equation to be differentiable

2) Consider

$$
\begin{aligned}
\nabla U =& \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}\right) \\
\nabla V =& \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}\right) \\
\nabla U \cdot \nabla V =& \frac{\partial U}{\partial x}\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\frac{\partial V}{\partial y} \\
=& \frac{\partial U}{\partial x} - \frac{\partial U}{\partial y} + \frac{\partial U}{\partial y}\frac{\partial U}{\partial x} = 0
\end{aligned}
$$

Therefore contours of constant U, V always meet at right angles

2.2 Analytic Functions

Defnition: A Function is analytic at a point x if the function can be represented by a convergent power series at x

Convergence: Ratio Test for convergence

$$S(x) = \sum_{n=0}^{\infty} u_n$$

$$r = \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right|$$

if $r < 1$ then $S(x)$ is convergent as

$$\begin{aligned}\sum_{n=0}^{\infty} u_n =& u_1 + u_2 + ...u_n \left(1 + \frac{u_{n+1}}{u_n} + \frac{u_{n+2}}{u_n} + ...\right) \\ &= \{finite\} + u_n \left(1 + r + r^2 + ...\right)\end{aligned}$$

$\sum_n r^n$ is a convergent series if $r < 1$

For Complex functions Analytic $\Longleftrightarrow$ Differentiable. To construct the power series at z_0 simply Taylor expand

$$\begin{aligned}f(z) =& f(z_0) + f'(z_0)[z - z_0] + \frac{f''(z_0)}{2!}[z - z_0]^2 + ... \\ =& a_0 + a_1 z + a_2 z^2 + ...\end{aligned}$$

2.3 Laurent Expansions

A Laurent Expansion is an expansion about $z_0$of the form

$$f(z) = ... + \frac{b_2}{(z - z_0)^2} + \frac{b_1}{(z - z_0)} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + ...$$

$b_1$is called the residue

Example

$$
\begin{aligned}
f(z) =& \frac{1}{(1-z)(2-z)} = \frac{-1}{(1-z)} - \frac{1}{(2-z)} \\
=& \frac{-1}{(1-z)} - \frac{1}{1+(1-z)} \\
u =& 1-z \\
f(z) =& \frac{-1}{(1-z)} - \left(1+u+u^2+...\right) \\
=& \frac{-1}{(1-z)} - 1 - (1-z) - (1-z)^2 + ...
\end{aligned}
$$

thus $f(x)$ has been laurent expanded about one of its singular points $z_0 = 1$

$$
\begin{aligned}
f(z) =& \frac{1}{(1-z)} - \frac{1}{2}\left[\frac{1}{1-\frac{z}{2}}\right] \\
=& \frac{1}{2} + \frac{2}{4}z + \frac{7}{3}z^2 + ...
\end{aligned}
$$

now $f(x)$ is expanded about a non-singular point $z_0 = 0$

2.4 Classifcation of Singular Points

2.4.1 Removeable Poles:

If $b_1 = b_2 = b_n = 0 \qquad \forall n \in \mathbb{N}$

Example

$$
\begin{aligned}
f(z) =& \frac{\sin(z)}{z} = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} + ...}{z} \\
=& 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + ...
\end{aligned}
$$

Thus expanded about $z_0 = 0 \Rightarrow b_n = 0 \qquad \forall n \in \mathbb{N}$

2.4.2 Essential Singularities

If $b_n \neq 0 \qquad \forall n \in \mathbb{N}$

Example

$$
f(z) = e^{\frac{1}{z-z_0}} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{(z-z_0)^n}
$$

2.4.3 Pole of Order m

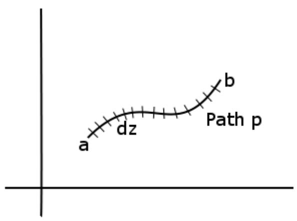
If $b_n = 0 \qquad \forall n > m, n \in \mathbb{N}$

$$f(z) = \frac{b_m}{(z-z_0)^m} + \frac{b_{m-1}}{(z-z_0)^{m-1}} + ... + \frac{b_1}{(z-z_0)} + a_0 + a_1(z-z_0) + ...$$

if $m = 1$ then the pole is called a simple pole

3 Integration of a Complex Function

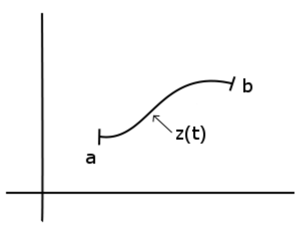
3.1 Principles



For $f : \mathbb{C} \to \mathbb{C}$,

$$\begin{aligned} I &= \int_p f(z)\, dz \\ &= \lim_{N\to\infty} \sum_{n=0}^{N-1} f(z_{n+1})(z_{n+1} - z_n) \\ &= \lim_{N\to\infty} \sum_n f(z_{n+1}) \Delta_{z_{n+1}} \end{aligned}$$

Consider Parametrisation of path



$[0,1] \to \mathbb{C} \qquad z(t) \in P$
Then

$$I = \int_p f(z)\, dz = \int_0^1 f(z(t)) \frac{dz}{dt} dt$$

Assume that $\exists F$ such that $\frac{dF}{dz} = f(z) \quad \forall z \in \mathbb{C}$

$$\begin{aligned}I &= \lim_{N\to\infty}\sum_n f\left(z_{n+1}\right)\Delta_{z_{n+1}}\\ &= \lim_{N\to\infty}\sum_n \left.\frac{dF}{dz}\right|_{z_{n+1}}\Delta_{z_{n+1}}\\ &= \lim_{N\to\infty}\sum_n \left.dF\right|_{z_{n+1}}\\ &= F\left(b\right)-F\left(a\right)\end{aligned}$$

note that if $a=b$, $p=c$, where c is a circular path

$$I=\int_c f\left(z\right)dz=0$$

3.1.1 Properties

1)

$$\int_p \alpha f\left(z\right)+\beta g\left(z\right)dz=\alpha\int_p f\left(z\right)dz+\beta\int_p g\left(z\right)dz$$

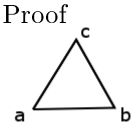
2)

$$\int_{-p} f\left(z\right)dz=-\int_p f\left(z\right)dz$$

3)

$$\left|\int_p f\left(z\right)dz\right|\leq\int_p\left|f\left(z\right)\right|\left|dz\right|$$

Proof



Triangular Inequality $|ab|\leq|ac|+|bc|$

$$\left|\int_p f\left(z\right)dz\right|=\lim_{N\to\infty}\left|\sum_n f\left(z_n\right)\Delta z_n\right|$$

$$\lim_{N\to\infty}\left|\sum_n f\left(z_n\right)\Delta z_n\right|\leq\lim_{N\to\infty}\sum_n\left|f\left(z_n\right)\right|\left|\Delta z_n\right|$$

$$\left|\int_p f\left(z\right)dz\right|\leq\int_p\left|f\left(z\right)\right|\left|dz\right|$$

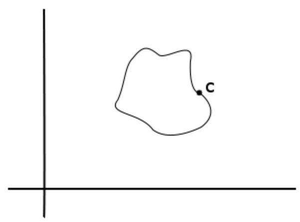
□

4) The modulus of the integral must be $\leq$ the largest value of $f(z)$ on the path $\times$ Length of Path

$$\left|\int_p f(z)\,dz\right| \leq \max_{z\in p}\{|f(z)|\}\,L_p$$

3.2 Cauchy's Integral Theorem

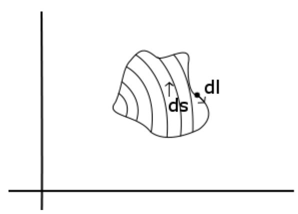
If a function $f(z)$ is analytical on and inside some contour C



$$\int_c f(z)\,dz = 0$$

Proof
Use Stoke's Theorem

$$\oint_c \mathbf{A}\cdot\mathbf{dl} = \iint_s \nabla\times\mathbf{A}\cdot\mathbf{ds}$$



$\mathbf{ds} = dxdy\hat{\mathbf{z}}$

note

$$\int_c f(z)\,dz = \int_c (U+iV)(dx+idy)\,dz$$
$$= \int_c (Udx - Vdy) + i\int_c (Vdx + Udy)$$

Introduce $\mathbf{A} = (H(x,y), G(x,y), 0)$

$$\nabla\times\mathbf{A} = (0, 0, \partial_x G - \partial_y H)$$
$$\oint_c Hdx + Gdy = \iint_s (\partial_x G - \partial_y H)\,dxdy$$

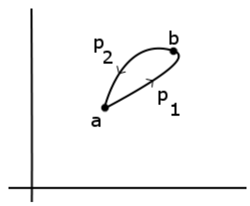
from Stoke's Theorem and C-R

$$
\begin{aligned}
I = &\int_c f(z)\,dz = \int_c (U dx - V dy) + i \int_c (V dx + U dy) \\
&= \iint_s (-\partial_x V - \partial_y U)\,dxdy + i \iint_s (\partial_x U - \partial_y V)\,dxdy \\
&=0
\end{aligned}
$$

□

3.2.1 Conclusion

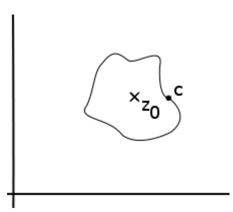
Path Independant



$$C = P_1 + P_2$$

$$
\begin{aligned}
\int_c f(z)\,dz = 0 = \int_{p_1} f(z)\,dz + \int_{p_2} f(z)\,dz \\
\int_{p_1} f(z)\,dz = \int_{-p_2} f(z)\,dz
\end{aligned}
$$

3.3 Cauchy's Integral Formula

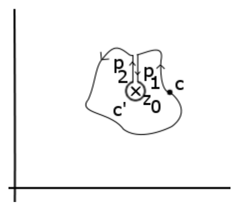


If $f(z)$ is analytic on and inside C except at the point $z = z_0$ where $f(z)$ has a pole

$$I = \int_c f(z)\,dz = 2\pi i\, b_1$$

where b_1 is the coeffcient of the $\frac{1}{(z-z_0)}$ term in the laurent expansion about z_0

Proof



Contour $C+P_1+C'+P_2$does not encircle the pole at z_0

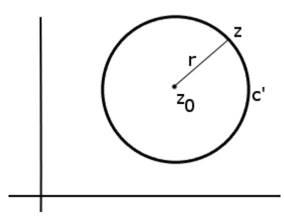
$$\begin{aligned} I &= \int_{C+P_1+C'+P_2} f(z)\,dz = 0 \\ &= \int_c f(z)\,dz + \int_{p_1} f(z)\,dz + \int_{P_2} f(z)\,dz + \int_{c'} f(z)\,dz \\ P_1 &= -P_2 \end{aligned}$$

$$\int_c f(z)\,dz = -\int_{c'} f(z)\,dz$$

express $f(z)$ in the vicinity of the pole $z_0$as a laurent expansion

$$\begin{aligned} f(z) &= ... + \frac{b_2}{(z-z_0)^2} + \frac{b_1}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + ... \\ &= \sum_{n=-\infty}^{\infty} A_n (z-z_0)^n \end{aligned}$$

$$\int_c f(z)\,dz = -\sum_{n=-\infty}^{\infty} \int_{c'} A_n (z-z_0)^n\,dz$$



$$
\begin{aligned}
z - z_0 =& u \\
u =& re^{i\theta} \\
du =& dz \\
=& re^{i\theta} i d\theta \\
=& iu\, d\theta \\
I =& -\sum_{n=-\infty}^{\infty} A_n \int_{2\pi}^{0} iu^{n+1} d\theta \\
=& \sum_{n=-\infty}^{\infty} ir^{n+1} A_n \int_{0}^{2\pi} e^{i(n+1)\theta} d\theta \\
=& \sum_{n=-\infty}^{\infty} ir^{n+1} A_n \begin{cases} \left.\frac{e^{i(n+1)\theta}}{i(n+1)}\right|_0^{2\pi} = 0 & \forall n \neq -1 \\ 2\pi & n = -1 \end{cases} \\
I =& 2\pi i\, A_{n=-1} \\
=& 2\pi i\, b_1 \\
& \square
\end{aligned}
$$

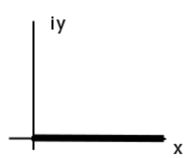
If there are multiple poles inside the contours

$$\int_c f(z)\, dz = \sum_n 2\pi i\, Res\,(z_n)$$

3.4 Contour Integration

Consider

$$I = \int_0^{\infty} \frac{dx}{(1+x)^2}$$



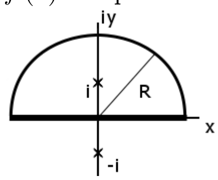
Integrating along the x axis. As the integrand is even

$$I = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$$

Consider

$$f(z) = \frac{1}{(1+z^2)^2}$$

$f(z)$ has poles at $z_0 = \pm i$



Note that

$$\int_{-R}^{R} f(x)\,dx + \int_{semicircle} f(z)\,dz = 2\pi i\ Res\,(i)$$

as $R \to \infty, \int_{-R}^{R} f(x)\,dx \to 2I$

$$\left|\int_{semicircle} f(z)\,dz\right| = \left|\int_{semicircle} \frac{1}{(1+z^2)^2}dz\right|$$
$$z = Re^{i\theta}$$
$$dz = izd\theta$$
$$\left|\int_0^{\pi} \frac{iz}{(1+z^2)^2}d\theta\right| \leq \int_0^{\pi} \frac{|z|}{\left|(1+z^2)^2\right|}d\theta$$
$$\approx \int_0^{\pi} \frac{|z|}{|z^4|}d\theta = \int_0^{\pi} \frac{1}{|z^3|}d\theta$$
$$= \int_0^{\pi} \frac{1}{|R^3|}d\theta$$
$$R \to \infty$$
$$\int_0^{\pi} \frac{1}{|R^3|}d\theta \to 0$$

Therefore the integration around the semicircle in the limit $R \to \infty$must be equal to zero and thus

$$2I = 2\pi i\ Res\,(i)$$

3.5 Finding Residues

3.5.1 Simple Poles

If $f(z)$has a simple pole at $z = z_0$then $Res\,(z_0) = \lim_{z\to z_0}[(z-z_0)f(z)]$. If $\lim_{z\to z_0}[(z-z_0)f(z)] = constant$ then $f(z)$ has a simple pole at $z = z_0$, if it equals 0 then the function is analytic at z_0 if it equals ∞ then $f(z)$ has a higher order pole at $z = z_0$. If $f(z)$can be expressed in the form $f(z) = \frac{g(z)}{h(z)}$ where $g(z_0) \neq 0$ is analytic and $h(z_0) = 0, h'(z_0) \neq 0$, then

$$
\begin{aligned}
Res\left(z_0\right) = & \lim_{z\to z_0}\left[\left(z-z_0\right)\frac{g\left(z\right)}{h\left(z\right)}\right] \\
= & g\left(z_0\right)\lim_{z\to z_0}\left[\frac{\left(z-z_0\right)}{h\left(z\right)}\right] \\
= & \frac{g\left(z_0\right)}{h'\left(z_0\right)}
\end{aligned}
$$

by L'Hopitals Rule

3.5.2 Higher Order Poles

To estimate if $f\left(z\right)$ has a pole of order m then the residue can be found by application of

$$
Res\left(z_0\right) = \frac{1}{\left(m-1\right)!}\frac{d^{m-1}}{dz^{m-1}}\left[\left(z-z_0\right)^m f\left(z\right)\right]\Big|_{z=z_0}
$$

Example, second order pole at $z_0 = i$

$$
\begin{aligned}
f\left(z\right) = & \frac{1}{\left(1+z^2\right)^2} = \frac{1}{\left[\left(z-i\right)\left(z+i\right)\right]^2} \\
b_1 = & \frac{d}{dz}\left[\left(z-i\right)^2\frac{1}{\left(1+z^2\right)^2}\right]\Bigg|_{z_0=i} = \frac{1}{4\pi}
\end{aligned}
$$

3.6 Examples of Contour Integration

1) Integrate

$$
I = -i\frac{2}{b}\int\frac{dz}{z^2+2az+b}
$$

Around the unit circle centered at $z = 0$. First note that there are two poles, namely at the roots of the quadratic on the denominator of the fraction.

$$
\begin{aligned}
I = & -i\frac{2}{b}\int\frac{dz}{\left(z-z_+\right)\left(z-z_-\right)} \\
\left(z-z_+\right)\left(z-z_-\right) = & z^2+2az+b \\
z_+z_- = & 1 \\
z_+ + z_- = & -\frac{2a}{b} \\
z_\pm = & \frac{1}{2}\left[-\frac{2a}{b}\pm\sqrt{\frac{4a^2}{b^2}-4}\right] \\
= & -\frac{a}{b}\pm\sqrt{\frac{a^2}{b^2}-1}
\end{aligned}
$$

Since $z_+ z_- = 1$ only $z_+$is inside the contour,

$$I = 2\pi i \; Res\left[f(z), z_0 = z_+\right]$$
$$Res\left[f(z), z_0 = z_+\right] = -i\frac{2}{b}\left[(z - z_+)\frac{1}{(z - z_+)(z - z_-)}\right]\Bigg|_{z=z_+}$$
$$I = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

2) Find the value of

$$I = \int_0^{2\pi} \frac{\cos(\theta)\, d\theta}{5 - 4\cos(\theta)}$$

Make the substitution $z = e^{i\theta}$ and then integrate around the unit circle. The integral becomes,

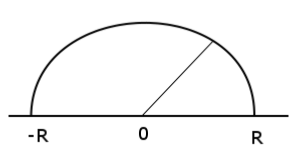
$$I = -\frac{1}{4i}\int_C \frac{z^6 + 1}{z^3\left(z - \frac{1}{2}\right)(z - 2)} dz$$

Notice that within the contour limits there are two poles, a third order at $z_0 = 0$ and a simple at $z_1 = \frac{1}{2}$. Thus

$$I = \sum_n 2\pi i \; Res(z_n)$$
$$Res(z_0) = -\frac{1}{4i}\frac{1}{2!}\frac{d^2}{dz^2}\left[z^3 \frac{z^6 + 1}{z^3\left(z - \frac{1}{2}\right)(z - 2)}\right]\Bigg|_{z=0} = -\frac{1}{4i}\cdot\frac{21}{4}$$
$$Res(z_1) = -\frac{1}{4i}\left[\left(z - \frac{1}{2}\right)\frac{z^6 + 1}{z^3\left(z - \frac{1}{2}\right)(z - 2)}\right]\Bigg|_{z=\frac{1}{2}} = \frac{1}{4i}\cdot\frac{65}{12}$$
$$I = \frac{\pi}{12}$$

4 Fourier Like Integrals

$$I = \int_{-\infty}^{\infty} f(x)\, e^{ikx} dx$$



Close the contour in the upper half plane (UHP) over a range R and then use Cauchys integral theorem and see how the integral behaves as $R \to \infty$

$$\tilde{I} = \int_{-R}^{R} f(x)\, e^{ikx} dx + \int_{UHP} f(z)\, e^{ikz} dz$$

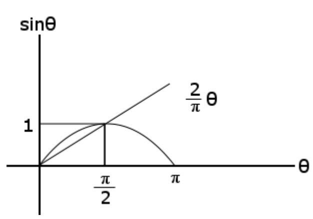
Specifically how does I_{UHP} behave at $R \to \infty$

$$\begin{aligned} \left| \int_{UHP} f(z)\, e^{ikz} dz \right| \leq \int_{UHP} &|f(z)| \left| e^{ikz} \right| |dz| \\ z =& Re^{i\theta} \\ |dz| =& R\, |d\theta| \\ \left| e^{ikz} \right| =& \left| e^{ik(x+iy)} \right| = \left| e^{ikx} \right| \left| e^{-ky} \right| \\ =& e^{-ky} \end{aligned}$$

We are now vindicated in our choice of closing in the UHP as $R \to \infty, y \to \infty, e^{-ky} \to 0$ assuming $k > 0$

$$\begin{aligned} |I_{UHP}| \leq& \int_0^{\pi} Re^{-ky} |f(z)|\, d\theta \\ assume\ |f(z)| \propto& |z|^{\alpha} \\ |I_{UHP}| \leq& R^{\alpha+1} \int_0^{\pi} e^{-kR\sin(\theta)} d\theta \end{aligned}$$

4.1 Jordans Lemma



We can estimate $\sin(\theta)$ as $\frac{2}{\pi}\theta$, since $\sin(\theta) \geq \frac{2}{\pi}\theta \forall \theta \in [0, \pi]$, $e^{-\sin(\theta)} < e^{-\frac{2}{\pi}\theta}$. Since I_{UHP} is symmetric

$$
\begin{aligned}
|I_{UHP}| \leq & 2R^{\alpha+1}\int_0^{\frac{\pi}{2}} e^{-kR\sin(\theta)}d\theta \\
2R^{\alpha+1}\int_0^{\frac{\pi}{2}} e^{-kR\sin(\theta)}d\theta \leq & 2R^{\alpha+1}\int_0^{\frac{\pi}{2}} e^{-kR\frac{2}{\pi}\theta}d\theta \\
= & 2R^{\alpha+1}\left[1-e^{-kR}\right]\frac{\pi}{2Rk} \\
R \to & \infty \\
|I_{UHP}| \leq & R^{\alpha} = 0 \quad \forall \alpha < 1 \\
\therefore \int_{-\infty}^{\infty} f(x)\, e^{ikx} dx = & \sum_n 2\pi i\ Res\,(z_n)
\end{aligned}
$$

4.2 Examples

1) By considering

$$\int \frac{e^{iz}}{(z-i)^2}dz$$

and taking Real Parts show that

$$\int_0^{\infty} \frac{2x\sin(x)-\left(x^2-1\right)\cos(x)}{\left(x^2+1\right)^2}dx = \frac{\pi}{e}$$

Taking the integral about a closing semicircle in the UHP, $f(z)$has a second order pole at $z_0 = i$

$$\int_{UHP} f(z)\,dz + \int_{-R}^{R} f(z)\,dz = 2\pi i\ Res\left[f(z), z_0 = i\right]$$

Show{}

$$
\begin{aligned}
\left|\int_{UHP} f(z)\,dz\right| \leq \int_{UHP} |f(z)|\,|dz| \\
\int_{UHP} |f(z)|\,|dz| = \int_{UHP} \left|\frac{e^{iz}}{(z-i)^2}\right| R\,|d\theta|
\end{aligned}
$$

By Jordan's Lemma

$$\int_{UHP} \left| \frac{e^{iz}}{(z-i)^2} \right| R\,|d\theta| \leq 2\int_0^{\frac{\pi}{2}} \frac{e^{-R\sin\theta}}{R^2} R\,|d\theta| \to 0 \; as \; R \to \infty$$

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x-i)^2} dx = 2\pi i \; Res\left[f(z), z_0 = i\right]$$

$$\frac{e^{ix}}{(x-i)^2} \cdot \frac{(x+i)^2}{(x+i)^2} = \frac{\left[\cos(x) + i\sin(x)\right]\left[x^2 - 1 + 2ix\right]}{(x+1)^2}$$

$$Res\left[f(z), z_0 = i\right] = \frac{d}{dz}\left[(z-i)^2 \frac{e^{iz}}{(z-i)^2}\right]\Bigg|_{z=i}$$

$$= -2\frac{\pi}{e}$$

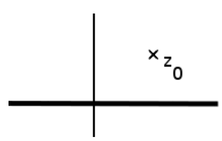
By Taking Real Parts

$$\int_{-\infty}^{\infty} \frac{\left(x^2-1\right)\cos(x) - 2x\sin(x)}{(x+1)^2} = -2\frac{\pi}{e}$$

$$\int_0^{\infty} \frac{2x\sin(x) - \left(x^2-1\right)\cos(x)}{\left(x^2+1\right)^2} dx = \frac{\pi}{e}$$

2)

$$I(k) = \int_{-\infty}^{\infty} \frac{e^{ikx}}{x - z_0} dx$$



Assume $k \in \mathbb{R}$, $\quad Im[z_0] > 0$. If we close the contour in the:

$$UHP \to I(k) = 2\pi i \; Res(z_0)$$

$$LHP \to I(k) = 0$$

note

$$f(x) = \frac{1}{x - z_0}$$

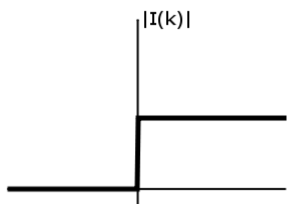
$$|f(z)| \propto |z|^{-1}$$

If $k > 0$, close in the UHP as $\left|e^{ikz}\right| = e^{-ky}$, else close in the LHP

$$k > 0 \to I(k) = 2\pi i \; Res(z_0)$$
$$k < 0 \to I(k) = 0$$

$f(z)$is already laurent expanded around z_0

$$f(z) = \frac{e^{ikz}}{z - z_0}$$
$$b_1 = e^{ikz_0}$$
$$k > 0 \to I(k) = 2\pi i e^{ikz_0}$$



$I(k)$ is an integral form of the heavyside function

3) Evaluate

$$\int_{-\infty}^{\infty} \frac{e^{ikx}}{1 + x^2} dx$$

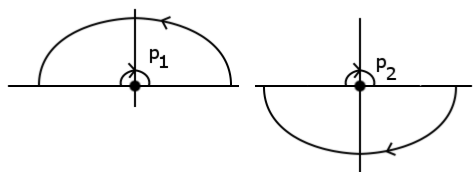
By Jordans Lemma the integral around the semicircle is zero in the limit of $R \to \infty$ but for $k > 1$ we must close in the UHP and for $k < 1$ we must close in the LHP. $f(z)$ has two poles at $z_0 = i$ and a simple at $z_1 = -i$.

$$\begin{aligned}
I(k > 1) &= 2\pi i \; Res\left(\frac{e^{ikz}}{(z - i)(z + i)}, z_0 = i\right) \\
&= 2\pi i \left[(z - i)\frac{e^{ikz}}{(z - i)(z + i)}\right]\Bigg|_{z_0 = i} \\
&= \pi e^{-k} \\
I(k < 1) &= -2\pi i \; Res\left(\frac{e^{ikz}}{(z - i)(z + i)}, z_0 = -i\right) \\
&= -2\pi i \left[(z + i)\frac{e^{ikz}}{(z - i)(z + i)}\right]\Bigg|_{z_0 = -i} \\
&= \pi e^{k} \\
\therefore I &= \pi e^{-|k|}
\end{aligned}$$

4) Evaluate

$$I = \int_{-\infty}^{\infty} e^{ikx} \frac{\sin(x)}{x} dx$$

First express $\sin(x) = \frac{1}{2i}\left[e^{ix} - e^{-ix}\right]$ then notice that each new integral has a pole at $z_0 = 0$, which is on the integration contour. We must intriduce a path segment to avoid this pole, we will construct two paths with will be used ifthe integral needs closing in the UHP or the LHP. The large semicircular segment $\to 0$ as $R \to \infty$ by Jordan's Lemma, and the small semicircular segment can be made arbitrarily small so its contribution is negligable.



$$2i \cdot I = \underbrace{\int_{-\infty}^{\infty} e^{i(k+1)x} dx}_{I_1} - \underbrace{\int_{-\infty}^{\infty} e^{i(k-1)x} dx}_{I_2}$$

$$I_1\{k+1>0\} = \int_{p_1} e^{i(k+1)x} dx = 0$$

$$I_1\{k+1<0\} = \int_{p_2} e^{i(k+1)x} dx = -2\pi i \; Res\left[f_1(z), z_0 = 0\right]$$

$$= -\pi$$

$$I_2\{k-1>0\} = \int_{p_1} e^{i(k-1)x} dx = 0$$

$$I_2\{k-1<0\} = \int_{p_2} e^{i(k-1)x} dx = 2\pi i \; Res\left[f_{12}(z), z_0 = 0\right]$$

$$= \pi$$

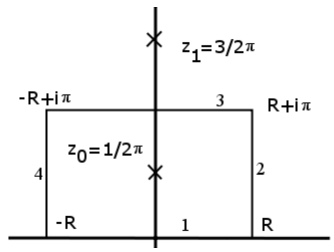
$$I = \begin{cases} -\pi + \pi = 0 & k < -1 \\ \pi + 0 = \pi & 1 > k > -1 \\ 0 + 0 = 0 & k > 1 \end{cases}$$

$$I(k) = \begin{cases} \pi & |k| < 1 \\ 0 & |k| > 1 \end{cases}$$

5) Evaluate

$$I = \int_{-\infty}^{\infty} Sech(x)\, e^{ikx} dx = \int_{-\infty}^{\infty} \frac{e^{ikx}}{Cosh(x)} dx$$

$f(z)$has poles at $z_0 = 0 + (2n+1)\frac{\pi}{2}$, close a rectangular contour encapturing one pole



$$\int_c = \int_1 + \int_2 + \int_3 + \int_4$$

$$\int_1 = I$$

$$\int_2 = \lim_{R\to\infty} \int_R^{R+i\pi} \frac{e^{ikR}e^{-ky}}{Cosh\,(R+iy)} dy = 0$$

$$\int_3 = \int_{R+i\pi}^{-R+i\pi} \frac{e^{ikz}}{Cosh\,(z)} dx = -e^{-k\pi} \int_R^{-R} \frac{e^{ikx}}{Cosh\,(x)} dx = e^{-k\pi} I$$

$$\int_4 = \lim_{R\to\infty} \int_{-R+i\pi}^{-R} \frac{e^{ikR}e^{-ky}}{Cosh\,(-R+iy)} dy = 0$$

$$\left(1+e^{-k\pi}\right) I = 2\pi i\ Res\left[f\left(z\right), z_0 = i\frac{\pi}{2}\right]$$

$$Res = \lim_{z\to i\frac{\pi}{2}} \left[\left(z - i\frac{\pi}{2}\right) \frac{e^{ikz}}{Cosh\,(z)}\right]$$

$$By\ L'Hopital \Longrightarrow$$

$$= e^{-k\frac{\pi}{2}} \lim_{z\to i\frac{\pi}{2}} \left[\frac{1}{Sinh\,(z)}\right] = -ie^{-k\frac{\pi}{2}}$$

$$I = \frac{2\pi e^{-k\frac{\pi}{2}}}{\left(1+e^{-k\pi}\right)} = \frac{\pi}{Cosh\left(\frac{k\pi}{2}\right)}$$

Part II
Fourier Transforms

5 Fourier Transforms

5.1 Definitions

Consider a function $f\left(x\right)$ which vanishes at $|x| \to \infty$, $f\left(x\right) = \frac{A}{|x|^{\alpha}}$ $\alpha, A > 0$, The FT $\hat{f}\left(k\right)$ of $f\left(x\right)$and its inverse are given by

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx = \mathcal{F}[f(x)]$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk = \mathcal{F}^{-1}\left[\hat{f}(k)\right]$$

5.1.1 Example

$$\hat{f}(k) = \int_{-\infty}^{\infty} \frac{e^{-ikx}}{1+x^2} dx = \int_{-\infty}^{\infty} \frac{e^{-ikz}}{(z-i)(z+i)} dz$$

As the contribtion from the semi-circular contour closure needs $\to 0$ as $R \to \infty$, There are two cases to consider, $k > 0,\ k < 0$. For $k > 0 \Rightarrow \left|e^{-ikz}\right| = \left|e^{ky}\right|$, must close in the LHP, semicircular segment is integrated in a Left Hand fashion $\therefore I = \int_{-c} f(z)\, dz = -2\pi i\, b_1$

$$\hat{f}(k) = -\ 2\pi i\ Res(z_0)$$

Simple Pole in the LHP located at $z_0 = -i$

$$Res[f(z), -i] = (z+i) \left. \frac{e^{-ikz}}{(z-i)(z+i)} \right|_{z=-i}$$
$$= i \frac{e^{-k}}{2}$$
$$\hat{f}(k) = \pi e^{-k}$$

$k < 0, \Rightarrow \left|e^{-ikz}\right| = \left|e^{ky}\right|$, must close in the UHP

$$\hat{f}(k) = 2\pi i\ Res(z_0)$$

Simple Pole in the LHP located at $z_0 = i$

$$Res[f(z), i] = (z-i) \left. \frac{e^{-ikz}}{(z-i)(z+i)} \right|_{z=i}$$
$$\hat{f}(k) = \pi e^{k}$$
$$\therefore \mathcal{F}\left[\int_{-\infty}^{\infty} \frac{e^{-ikx}}{1+x^2} dx\right] = \pi e^{-|k|}$$

And the Inverse

$$
\begin{aligned}
f(x) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} \pi e^{-|k|} e^{ikx} dk \\
=& \frac{1}{2\pi}\int_{-\infty}^{0} \pi e^{k} e^{ikx} dk + \frac{1}{2\pi}\int_{0}^{\infty} \pi e^{-k} e^{ikx} dk \\
=& \frac{1}{2}\left[\left.\frac{e^{(1+ix)k}}{1+ix}\right|_{-\infty}^{0} + \left.\frac{-e^{-(1-ix)k}}{1-ix}\right|_{0}^{\infty} \right] \\
=& \frac{1}{2}\left[\frac{1}{1+ix} + \frac{1}{1-ix}\right] \\
=& \frac{1}{1+x^2}
\end{aligned}
$$

5.2 Properties of the FT

1) $\mathcal{F}[f(x)]$ exists and is bounded

2) $\mathcal{F}[f(x)]$ is a continuus function of k even if $f(x)$ is discontinuous

3) The Transform is linear

4) Riemann-Le Besque Lemma

$$\hat{f}(x) \to 0 \quad as \ |k| \to \infty$$

5) Shift Property

$$
\begin{aligned}
\mathcal{F}\left[f(x) e^{-ik\alpha}\right] =& \hat{f}(k+\alpha) \\
\mathcal{F}[f(x-\beta)] =& \hat{f}(k) e^{-i\beta x}
\end{aligned}
$$

6) If $f(x)$ is odd/even then $\hat{f}(k)$is odd/even

7) Transform of Derivatives

$$\mathcal{F}\left[\frac{df}{dx}\right] = ik\hat{f}(k)$$

5.3 Dirac Delta Function

Defined as

$$\int_a^b \delta(x-x_0) f(x) dx = f(x_0) \quad \forall x_0 \in [a,b]$$

5.3.1 FT of the δ

$$\hat{\delta}(k) = \int_{-\infty}^{\infty} \delta(x) e^{-ikx} dx = e^{-ik\cdot 0} = 1$$
$$\delta(k) = \int_{-\infty}^{\infty} \hat{\delta}(k) e^{ikx} \frac{dk}{2\pi} = \int_{-\infty}^{\infty} e^{ikx} \frac{dk}{2\pi}$$

8) FT of Convolutions

$$f(x) = \int_{-\infty}^{\infty} g(x-y) h(y) dy = g \star h$$
$$\hat{f}(k) = \hat{g}(k) \hat{h}(k)$$

Proof

$$\begin{aligned}\hat{f}(k) &= \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \\ &= \int_{-\infty}^{\infty} e^{-ikx} dx \int_{-\infty}^{\infty} g(x-y) h(y) dy \\ &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\ g(x-y) e^{-ik(x-y)} h(y) e^{-iky}\end{aligned}$$

Sub in $u = x - y$ for fixed $y, \Rightarrow du = dx$

$$\begin{aligned}\hat{f}(k) &= \int_{-\infty}^{\infty} dy \left[\int_{-\infty}^{\infty} du\ g(u) e^{-ik(u)}\right] h(y) e^{-iky} \\ &= \int_{-\infty}^{\infty} du\ g(u) e^{-ik(u)} \cdot \int_{-\infty}^{\infty} dy\ h(y) e^{-iky} \\ &= \hat{g}(k) \hat{h}(k)\end{aligned}$$

□

5.4 Diffusion

Diffusion equation

$$\frac{\partial n}{\partial t} = \gamma \frac{\partial^2 n}{\partial x^2} + g(x,t)$$

Where $g(x,t)$ is called the source term. The Source term is usually of the form $g(x,t) = \delta(x) \chi(t)$. Use FT to express the solution $n(x,t)$

$$
\begin{aligned}
n\left(x,t\right) &= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\hat{n}\left(k,\omega\right)e^{ikx+i\omega t} \\
g\left(x,t\right) &= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\hat{g}\left(k,\omega\right)e^{ikx+i\omega t} \\
\frac{\partial n}{\partial t} &= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\hat{n}\left(k,\omega\right)\left[i\omega\right]e^{ikx+i\omega t} \\
\frac{\partial^2 n}{\partial x^2} &= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\hat{n}\left(k,\omega\right)\left[ik\right]^2 e^{ikx+i\omega t} \\
\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\hat{n}\left(k,\omega\right)\left[i\omega\right]e^{ikx+i\omega t} &= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\left[\hat{n}\left(k,\omega\right)\gamma\left[ik\right]^2+\hat{g}\left(k,\omega\right)\right]e^{ikx+i\omega t} \\
i\omega\hat{n}\left(k,\omega\right) &= \hat{n}\left(k,\omega\right)\gamma\left(ik\right)^2+\hat{g}\left(k,\omega\right) \\
\hat{n} &= \frac{\hat{g}}{i\omega+\gamma k^2}
\end{aligned}
$$

From this we can conclude

$$
n\left(x,t\right) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{\hat{g}}{i\omega+\gamma k^2}e^{ikx+i\omega t}
$$

5.5 Correlations

Auto Correlation function

$$
\begin{aligned}
N\left(t\right) &= n\left(x_0,t\right)-\left\langle n\left(x_0,t\right)\right\rangle \\
C\left(t\right) &= \left.\left\langle N\left(t_0\right)N\left(t_0+t\right)\right\rangle\right|_{t_0}
\end{aligned}
$$

Correlation Coefficients

$$
C_{AB} = \left\langle AB\right\rangle - \left\langle A\right\rangle\left\langle B\right\rangle
$$

5.5.1 Relations between $C\left(t\right)$and $\left|\hat{N}\left(\omega\right)\right|^2$

$$
\mathcal{F}\left[C\left(t\right)\right] = \int_{-\infty}^{\infty} dt\, C\left(t\right)e^{-i\omega t}
$$

Represent $\left\langle A\right\rangle$ by some sum

$$
\left.\left\langle N\left(t_0\right)N\left(t_0+t\right)\right\rangle\right|_{t_0} = \sum_{t_0}\frac{N\left(t_0\right)N\left(t_0+t\right)}{m}
$$

m is the number of terms in the sum

$$
\begin{aligned}
\hat{C}(\omega) &= \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt_0 N(t_0) N(t_0+t) e^{-i\omega t} \\
&= \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt_0 N(t_0) e^{i\omega t_0} N(t_0+t) e^{-i\omega(t+t_0)}
\end{aligned}
$$

For Fixed $t_0 \Rightarrow u = t_0 + t \Rightarrow du = dt$

$$
\begin{aligned}
\hat{C}(\omega) &= \underbrace{\int_{-\infty}^{\infty} dt_0 N(t_0) e^{i\omega t_0}}_{\hat{N}(\omega)^\star} + \underbrace{\int_{-\infty}^{\infty} du\ N(u) e^{-i\omega u}}_{\hat{N}(\omega)} \\
\hat{C}(\omega) &= \left|\hat{N}(\omega)\right|^2
\end{aligned}
$$

We need to find $\hat{N}(\omega)$

$$
\begin{aligned}
\hat{N}(\omega) &= \int_{-\infty}^{\infty} dt\ N(t) e^{-i\omega t} \\
&= \int_{-\infty}^{\infty} dt\ \left[n(x_0,t) - \langle n(x_0,t)\rangle\right] e^{-i\omega t} \\
&= \int_{-\infty}^{\infty} dt\ n(x_0,t) e^{-i\omega t} - \langle n(x_0,t)\rangle \int_{-\infty}^{\infty} dt\ e^{-i\omega t} \\
&= \int_{-\infty}^{\infty} dt\ n(x_0,t) e^{-i\omega t} - \langle n(x_0,t)\rangle 2\pi\delta(\omega)
\end{aligned}
$$

Ignore the $-\langle n(x_0,t)\rangle 2\pi\delta(\omega)$ term

$$
\begin{aligned}
n(x_0,t) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\hat{g}}{i\omega + \gamma k^2} e^{ikx + i\omega t} \\
\hat{N}(\omega) &= \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\hat{g}}{i\omega + \gamma k^2} e^{ikx_0 + i\omega' t} e^{-i\omega t}
\end{aligned}
$$

If $g(x,t) = \delta(x)\chi(t)$, $\hat{g}(k,\omega) = \hat{\chi}(\omega)$

$$
\begin{aligned}
\hat{N}(\omega) &= \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\hat{\chi}(\omega)}{i\omega' + \gamma k^2} e^{ikx_0} e^{i(\omega'-\omega)t} \\
&= \underbrace{\int_{-\infty}^{\infty} dt e^{i(\omega'-\omega)t}}_{\to 2\pi\delta(\omega'-\omega)} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{\hat{\chi}(\omega)}{i\omega' + \gamma k^2} e^{ikx_0} \\
&= \hat{\chi}(\omega) \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{1}{i\omega + \gamma k^2} e^{ikx_0}
\end{aligned}
$$

Assume $x_0 > 0$ $\rightarrow$ Close in UHP. Poles at $k^2 = -i\frac{\omega}{\gamma}$, $k_\pm = \pm\sqrt{\frac{\omega}{\gamma}}\left(-\frac{1}{2} + \frac{i}{\sqrt{2}}\right)$. Need to find the residue from k_+, Note:

$$\begin{aligned}
\frac{e^{ikx_0}}{i\omega + \gamma k^2} =& \frac{1}{\gamma}\frac{e^{ikx_0}}{k^2 + i\frac{\omega}{\gamma}} \\
=& \frac{1}{\gamma}\frac{e^{ikx_0}}{(k - k_+)(k - k_-)} \\
Res\,(k_+) = (k - k_+)\,& \frac{1}{\gamma}\frac{e^{ikx_0}}{(k - k_+)(k - k_-)}\bigg|_{k=k_+} \\
=& \frac{1}{\gamma}\frac{e^{ik_+x_0}}{(k_+ - k_-)}
\end{aligned}$$

Subbing back into $\hat{N}(\omega)$

$$\begin{aligned}
\hat{N}(\omega) =& \hat{\chi}(\omega)\,\frac{2\pi i}{\gamma}\,\frac{e^{-\sqrt{\frac{\omega}{2\gamma}}x_0}e^{-i\sqrt{\frac{\omega}{2\gamma}}x_0}}{\sqrt{\frac{2\omega}{\gamma}}\,(i-1)} \\
\left|\hat{N}(\omega)\right|^2 =& \frac{\left|\hat{\chi}(\omega)\right|^2}{4\gamma\omega}e^{-\sqrt{\frac{2\omega}{\gamma}}x_0}
\end{aligned}$$

5.6 Example: Heat Equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Valid for $-\infty < x < \infty, t > 0$. Initial Conditions, $u(x,0) = f(x)$. Introduce an FT in x

$$
\begin{aligned}
u(x,t) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \hat{u}(k,t)\, e^{ikx} \\
\frac{\partial^2 u}{\partial x^2} &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left[-k^2 \hat{u}(k,t)\right] e^{ikx} \\
\frac{\partial u}{\partial t} &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\partial}{\partial t} \hat{u}(k,t)\, e^{ikx}
\end{aligned}
$$

$$
\begin{aligned}
\therefore -k^2 \hat{u}(k,t) =& \frac{\partial}{\partial t} \hat{u}(k,t) \\
\hat{u}(k,t) =& \hat{u}(k,0)\, e^{-k^2 t}
\end{aligned}
$$

$$
\begin{aligned}
\hat{u}(k,0) &= \int_{-\infty}^{\infty} dx\ \hat{u}(x,0)\, e^{-ikx} = \int_{-\infty}^{\infty} dx\ f(x)\, e^{-ikx} \\
u(x,t) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left[\hat{u}(k,0)\, e^{-k^2 t}\right] e^{ikx} \\
&= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left[\int_{-\infty}^{\infty} dx'\ f(x')\, e^{-ikx'}\right] e^{-k^2 t} e^{ikx} \\
&= \int_{-\infty}^{\infty} dx' \left[\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x')-k^2 t}\right] f(x')
\end{aligned}
$$

Define

$$
\begin{aligned}
K(x,t) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx-k^2 t} \\
u(x,t) &= \int_{-\infty}^{\infty} dx' K(x-x',t)\, f(x')
\end{aligned}
$$

Assuming $f(x) = \delta(x)$

$$
\begin{aligned}
u(x,t) =& \int_{-\infty}^{\infty} dx' K(x-x',t)\, \delta(x') \\
=& K(x,t)
\end{aligned}
$$

Want to complete the square so we can do $\int_{-\infty}^{\infty} dx\ e^{Ax-Bx^2}$

$$
K(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx-k^2 t}
$$

Sub in $k' = k\sqrt{t}$

$$K(x,t) = \int_{-\infty}^{\infty} \frac{dk'}{2\pi\sqrt{t}} e^{ik'\frac{x}{\sqrt{t}} - k'^2}$$

$$-\left[k'^2 - ik'\frac{x}{\sqrt{t}}\right] = -\left[\left(k' - i\frac{x}{2\sqrt{t}}\right)^2 - \left(i\frac{x}{2\sqrt{t}}\right)^2\right]$$

$$K(x,t) = \frac{1}{2\pi\sqrt{t}} e^{-\frac{x^2}{4t}} \int_{-\infty}^{\infty} dk' e^{-\left(k' - i\frac{x}{2\sqrt{t}}\right)^2}$$

Standard integral

$$\int_{-\infty}^{\infty} dx\ e^{-(x-A)^2} = \sqrt{\pi}$$

Therefore

$$K(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

6 Half Transforms

Consider problems involving intial conditions of the form ,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Valid for $0 \le x < \infty,\ t > 0$. Boundary Conditions, $u(0,t) = g(t)$or $u'(0,t) = J(t)$. Initial Conditions, $u(x,0) = f(x)$. Need to use Half Range transforms by extending the function to $-\infty < x < \infty$, either by an even or odd extension.

6.0.1 Derrivation of inversion relation for Even Extension transform

Even Extension

$$\hat{f}(k) = \int_{-\infty}^{\infty} dx\ f(x) e^{-ikx}$$
$$= \int_{-\infty}^{\infty} dx\ f(x) [\cos(kx) + i\sin(kx)]$$

$\int_{-\infty}^{\infty} dx f(x) i \sin(kx) = 0$ as $f(x)$ is even and $\sin(kx)$is odd, therefore this is an integral of an odd function over all space.

$$\begin{aligned}\hat{f}(k) &= \int_{-\infty}^{\infty} dx\ f(x)\cos(kx) \\ &= 2\int_0^{\infty} dx\ f(x)\cos(kx) \\ &= \hat{f}(-k)\end{aligned}$$

Inverse

$$\begin{aligned}f(x) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\ \hat{f}(k)\, e^{ikx} \\ &= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\ \hat{f}(k)\cos(kx) \\ &= \int_0^{\infty} \frac{dk}{\pi}\ \hat{f}(k)\cos(kx)\end{aligned}$$

By convention$u'(0,t) = J(t)$

$$\hat{f}_c(k) = \frac{1}{2}\hat{f}(k)$$

6.1 Sin and Cos Transforms

6.1.1 Cos

$$\begin{aligned}\hat{f}_c(k) &= \int_0^{\infty} dx\ f(x)\cos(kx) \\ f(x) &= \frac{2}{\pi}\int_0^{\infty} dk\ \hat{f}_c(k)\cos(kx)\end{aligned}$$

6.1.2 Sin

$$\begin{aligned}\hat{f}_s(k) &= \int_0^{\infty} dx\ f(x)\sin(kx) \\ f(x) &= \frac{2}{\pi}\int_0^{\infty} dk\ \hat{f}_s(k)\sin(kx)\end{aligned}$$

6.2 Transforms of Derivatives

6.2.1 Cos

$$\mathcal{F}_c[f'(x)] = \int_0^{\infty} dx\ f'(x)\cos(kx)$$

By Parts

$$\mathcal{F}_c\left[f'(x)\right] = \left[f(x)\cos(kx)\right]_0^\infty + k\int_0^\infty dx\ f(x)\sin(kx)$$
$$\mathcal{F}_c\left[f'(x)\right] = -\ f(0) + k\ \hat{f}_s(k)$$
$$\mathcal{F}_c\left[f''(x)\right] = -\ f'(0) - k^2\ \hat{f}_c(k)$$

6.2.2 Sin

$$\mathcal{F}_s\left[f'(x)\right] = -\ k\ \hat{f}_c(k)$$
$$\mathcal{F}_s\left[f''(x)\right] = kf(0) - k^2\ \hat{f}_s(k)$$

We can see that we should use an even extension and thus a cos transform if our boundary conditions are of the form $u'(0,t) = J(t)$and use an odd extension and thus a sin transform if the boundary condition is of the form $u(0,t) = g(t)$

6.3 Example: Heat Flow in a Semi-Infinite Bar

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Valid for $0 \leq x < \infty,\ t > 0$. Boundary Condition, $u(0,t) = u_0$. Initial Condition, $u(x,0) = 0$. As this problem is only valid from $0 \leq x < \infty$ cannot use standard FT must use a half range. Boundary condtion refers to function not function derrivative therefore use a sin transform

$$\hat{u}_s(k,t) = \int_0^\infty dx\ u(x,t)\sin(kx)$$
$$\mathcal{F}_s\left[\frac{\partial u}{\partial t}\right] = \int_0^\infty dx\ \frac{\partial}{\partial t}u(x,t)\sin(kx) = \frac{\partial}{\partial t}\int_0^\infty dx\ u(x,t)\sin(kx)$$
$$= \frac{\partial}{\partial t}\hat{u}_s(k,t)$$
$$\mathcal{F}_s\left[\frac{\partial^2 u}{\partial x^2}\right] = ku(0,t) - k^2\ \hat{u}_s(k,t)$$

Transform the equation

$$\frac{\partial}{\partial t}\hat{u}_s(k,t) = ku_0 - k^2\ \hat{u}_s(k,t)$$

Solution

Homogeneous: Assume $u_0 = 0$

$$\begin{aligned}\frac{\partial}{\partial t}\hat{u}_s\left(k,t\right) =& -k^2\ \hat{u}_s\left(k,t\right)\\ \hat{u}_s\left(k,t\right) =& Ae^{-k^2 t}\end{aligned}$$

Inhomogeneous: Assume $\hat{u}_s\left(k,t\right) = constant = c$

$$\begin{aligned}0 =& ku_0 - k^2 c\\ c =& \frac{u_0}{k}\\ \hat{u}_s\left(k,t\right) =& Ae^{-k^2 t} + \frac{u_0}{k}\end{aligned}$$

From the initial conditions

$$u\left(x,0\right) = 0 \Longrightarrow \hat{u}_s\left(k,0\right) = 0$$

$$\hat{u}_s\left(k,t\right) = \frac{u_0}{k}\left(1 - e^{-k^2 t}\right)$$

$$\begin{aligned}u\left(x,t\right) =& \frac{2}{\pi}\int_0^\infty dk\ \hat{u}_s\left(k\right)\sin\left(kx\right)\\ =& \frac{2u_0}{\pi}\int_0^\infty dk\ \frac{\sin\left(kx\right)}{k}\left(1 - e^{-k^2 t}\right)\end{aligned}$$

Sub in $s = kx \Rightarrow ds = xdk$

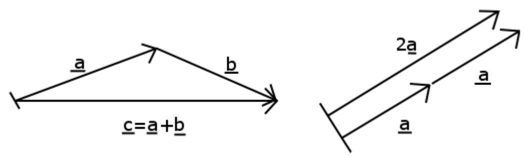
$$u\left(x,t\right) = \frac{2u_0}{\pi}\int_0^\infty ds\ \frac{\sin\left(s\right)}{s}\left(1 - e^{-s^2\frac{t}{x^2}}\right)$$

Standard Integrals

$$\begin{aligned}\int_0^\infty ds\frac{\sin\left(s\right)}{s} =& \frac{\pi}{2}\\ \int_0^\infty ds\frac{\sin\left(\gamma s\right)}{s}e^{-\beta s^2} =& \frac{\gamma e^{-\frac{\gamma^2}{4\beta}}}{2\sqrt{\beta}}\Gamma\left(\frac{1}{2}\right)_1 F_1\left(1,\frac{3}{2},\frac{\gamma^2}{4\beta}\right)\end{aligned}$$

Part III
Linear Vector Spaces

7 Properties of a Linear Vector Space



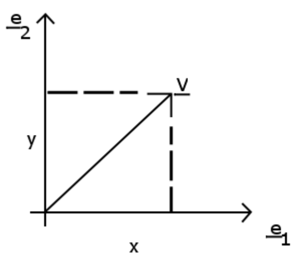
Zero Vector

$$\mathbf{a} + \mathbf{0} = \mathbf{a}$$

Unit For Multiplication

$$1 \times \mathbf{a} = \mathbf{a}$$

7.0.1 Introduce a Basis



$\mathbf{V} = x\mathbf{e_1} + y\mathbf{e_2}$

7.1 Properties

Let $x, y, z \in \mathcal{S}$ denote three elements in the space $\mathcal{S}$. Let $\alpha, \beta \in \mathbb{C}$

1. $x + y = y + x$
2. $x + (y + z) = (x + y) + z$
3. $0 \in \mathcal{S}: \ 0 + x = x$
4. $-x \in \mathcal{S}: \ -x + x = 0$
5. $\alpha (\beta x) = (\alpha\beta) x$
6. $(\alpha + \beta) x = \alpha x + \beta x$
7. $\alpha (x + y) = \alpha x + \alpha y$

8. $1 \times x = x,\ 0 \times x = 0$

Assume these 8 rules are fulfilled an that

1. $x, y \in \mathcal{S} \Rightarrow x + y \in \mathcal{S}$
2. $\alpha \in \mathbb{C}, x \in \mathcal{S} \Rightarrow \alpha x \in \mathcal{S}$\mathbf{e_{2}}

If all of these are fulfilled then $\mathcal{S}$ is said to be a linear vector space

7.2 Examples

1. $\mathbb{R}^2 = \{(x, y) \,|x, y \in \mathbb{R}\}$
   (a) addition $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$
   (b) scalar multiplication $\alpha (x, y) = (\alpha x, \alpha y)$
2. Similarly
   (a) $\mathbb{R}^n = \{(x_1, x_2 ... x_n) \,|x_i \in \mathbb{R}\}$
   (b) $\mathbb{C}^n = \{(z_1, z_2 ... z_n) \,|z_i \in \mathbb{C}\}$
3. The set of all Polynomials of degree $\leq n$, Denote it by **P**
   (a) $p_1 = a_0 + a_1 x + ... a_k x^k \quad k \leq n$
   (b) $p_2 = b_0 + b_1 x + ... b_m x^m \quad m \leq n$
   (c) $p_1 + p_2$ will be a polynomial of degree $\leq n$
4. The set of all continuous functions defined on $[a, b] :\to \mathbb{R}$. Denoted by $C_n ([a, b], \mathbb{R})$ where n denotes the number of times a function in the set must be differentiable

7.3 Linear Dependence

If $x_i \in \mathcal{S} \quad \forall\, i \leq n,\ i \in \mathbb{N}$, then

$$x = \alpha_1 x_1 + \alpha_2 x_2 + ... \alpha_n x_n \in \mathcal{S}$$

and if

$$\sum_{i=1}^{n} \alpha_i x_i = 0$$

where not all $\alpha_i = 0$ then $x_1, x_2 ... x_n$ is said to be linearly dependent. In contrast if

$$\sum_{i=1}^{n} \alpha_i x_i = 0, \quad \alpha_i = 0\, \forall\, i \leq n,\ i \in \mathbb{N}$$

7.4 A Basis For $\mathcal{S}$

7.4.1 Spanning Set

A Spanning Set $\mathcal{T} \subseteq \mathcal{S}$ fulfills that any $x \in \mathcal{S}$ can be written as below where, $t_i \in \mathcal{T} \quad \forall\, i \leq n,\ i \in \mathbb{N}$

$$x = \alpha_1 t_1 + \alpha_2 t_2 ... \alpha_n t_n$$

A Basis for $\mathcal{S}$ is a Minimal Spanning Set

7.4.2 Example for $\mathbb{R}^2$

$$\mathcal{T} = \{(1,0), (0,1), (0,2)\}$$
$$\mathcal{B} = \{(1,0), (0,1)\}$$

$\mathcal{T}$ is a spanning set as any $x \in \mathbb{R}^2$ can be expressed as

$$x = \alpha (1,0) + \beta (0,1) + \gamma (0,2)$$

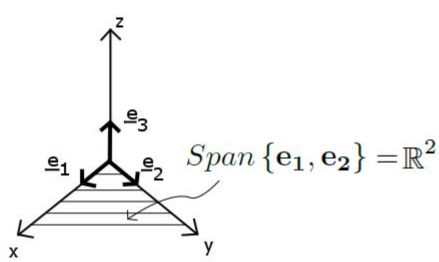
$\mathcal{B}$ is a basis as it is a minimal spanning set, $(0,2)$ is unessecary

7.4.3 Notation

$$Span \{a_1, a_2 ... a_m\}$$
$$= \{x \in \mathcal{S} | x = \alpha_1 a_1 + \alpha_2 a_2 + ... \alpha_m a_m, \alpha_i \in \mathbb{R}\}$$

Example for $\mathbb{R}^3$

$$Span \{\mathbf{e_1}, \mathbf{e_2}\}$$
$$\mathbf{e_1} = (1,0)$$
$$\mathbf{e_2} = (0,1)$$



8 Normed Vector Spaces

8.1 Norms

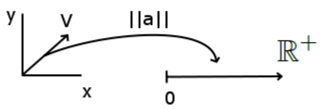
$x \in \mathcal{S} \quad \alpha \in \mathbb{C} \quad \alpha x \in \mathcal{S}$

8.1.1 Define Norm

$$||x||: \; x \in S \to \mathbb{R}^{+} \cup \{0\}$$

8.1.2 Properties

1. $||x_1 + x_2|| \leq ||x1|| + ||x_2||$, Triangular Inequality
2. $||\alpha x|| = |\alpha| \, ||x||$
3. $||x|| \geq 0$
4. $||x|| = 0 \iff x = 0$



1) Consider $\mathbb{C}^n = \{(z_1, z_2 ... z_n) \, |z_i \in \mathbb{C}\}$

$$\mathbf{a} \in \mathbb{C}^n, \quad \mathbf{a} = (a_1, a_2 ... a_n)$$
$$||\mathbf{a}||_2 = \left[|a_1|^2 + |a_2|^2 + ... \, |a_n|^2 \right]^{\frac{1}{2}}$$

This norm is denoted by $||\mathbf{a}||_2$

2) Consider $n \to \infty$, $\mathbb{C}^{\infty}$

$$||\mathbf{a}||_2 = \left[\sum_{i=1}^{\infty} |a_i|^2 \right]^{\frac{1}{2}}$$

The Subset of $\mathbb{C}^{\infty}$ for which $||\mathbf{a}||_2$ is finite is called l_2. I.E. $l_2 \subset \mathbb{C}^{\infty}$or,

$$l_2 = \{x \in \mathbb{C}^{\infty} | \; ||x||_2 < \infty\}$$

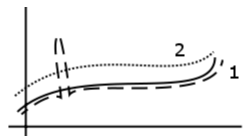
The p - norm on $\mathbb{C}^n$

$$||\mathbf{a}||_p = \left[\sum_{i=1}^{\infty} |a_i|^p \right]^{\frac{1}{p}}$$

The Maximum norm on $\mathbb{C}^n$

$$||\mathbf{a}||_{max} = max \{|a_1|, |a_2| ... |a_n|\}$$

Note



Path 1 is mostly along the original path but has a large deviation for a short distance whereas Path 2 has a constant deviation. The Max norm would say Path 1 has the largest deviation but the 2 norm would say that path 2 has the largest average deviation.

Observe

$$||\mathbf{a}||_{max} = \lim_{p\to\infty} ||\mathbf{a}||_p$$

Proof

$$\begin{aligned} ||\mathbf{a}||_p &= \left[\sum_{i=1}^{\infty} |a_i|^p\right]^{\frac{1}{p}} \\ &= \left[|a_{max}|^p \sum_{i=1}^{\infty} \left(\frac{|a_i|}{|a_{max}|}\right)^p\right]^{\frac{1}{p}} \\ \left(\frac{|a_i|}{|a_{max}|}\right) &< 1 \\ \lim_{p\to\infty} \left(\frac{|a_i|}{|a_{max}|}\right)^p &= 0 \quad \forall\, a_i \neq a_{max} \\ \lim_{p\to\infty} \left[||\mathbf{a}||_p\right] &= \left[|a_{max}|^p\right]^{\frac{1}{p}} = |a_{max}| \\ &= ||\mathbf{a}||_{max} \\ & \square \end{aligned}$$

Consider

$C_0([a,b], \mathbb{C})$

Introduce the norm

$$||f||_2 = \left(\int_a^b |f(x)|^2 \, dx\right)^{\frac{1}{2}}$$

The space l_2 is defined by

$$l_2 = \{f \in C_0([a,b], \mathbb{C}) \mid ||f(x)||_2 < \infty\}$$

Note that

$$||f||_p = \left(\int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

$$||f||_p \to_{p\to\infty} ||f||_{max} = max \{ |f(x)| \, | x \in [a,b] \}$$

9 Banach or Complete Spaces

Reminder

Closed Interval $[0,1] = \{x \in \mathbb{R} | 0 \leq x \leq 1\}$

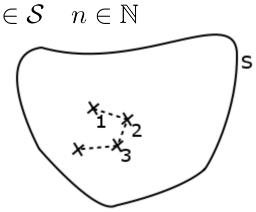
Open Interval $]0,1[= \{x \in \mathbb{R} | 0 < x < 1\}$

Convergence and open verses closed

Open Set: $a_n = \frac{1}{n}, \ n \geq 2, \ a_n \in \,]0,1[$. Note the limit $\lim_{n\to\infty} a_n = 0$ is not included in the interval, but $\lim_{n\to\infty} a_n \in [0,1]$

9.1 Sequence

$x_n \in \mathcal{S} \quad n \in \mathbb{N}$



9.1.1 Strong Convergence

$\forall \, \varepsilon > 0 \, \exists \, n_0 | \, \forall \, n > n_0 \Longrightarrow ||x_n - x|| < \varepsilon$

Where $x = \lim_{n\to\infty} x_n$

9.1.2 Cauchy Sequence

$\forall \, \varepsilon > 0 \, \exists \, n_0 | \, \forall \, n, m > n_0 \Longrightarrow ||x_n - x_m|| < \varepsilon$

Theorem: If a sequence $x_n \in \mathcal{S}$ is strongly convergent then it is also Cauchy

Proof: Estimate the distance between $x_n$and x_m

$$\begin{aligned} ||x_n - x_n|| &= ||x_n - x + x - x_m|| \\ &\leq ||x_n - x|| + ||x - x_m|| \end{aligned}$$

Since $x_n$is a strongly convergent sequency we know that $\forall \, \varepsilon > 0$ we can find n_0 such that,

$$||x_n - x|| < \frac{\varepsilon}{2}$$
$$||x - x_m|| < \frac{\varepsilon}{2}$$

when $n, m > n_0$ we can conclude

$$||x_n - x_n|| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

□

Hence the Cauchy condition is satisfied for x_n

9.1.3 Example

Is it clear that $x_n = \frac{1}{n}$ is Cauchy? Strongly convergent ? $x_n = \frac{1}{n}$is strongly convergent on $\mathcal{S} = [0, 1]$, but it is not strongly convergent on $\mathcal{S} =]0, 1[$

Consider Fucntion Space $\mathcal{S} = C_0\left([-\pi, \pi], \mathbb{R}\right)$. Any $f \in \mathcal{S}$ can be written as

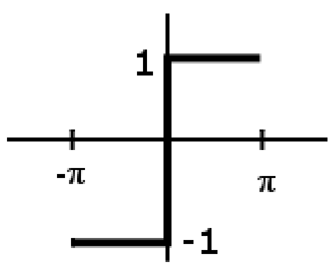
$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty}\left[a_k \cos(kx) + b_k \sin(kx)\right]$$

Also note that for any $\{a_k, b_k\}$ we have

$$f_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}\left[a_k \cos(kx) + b_k \sin(kx)\right],\ f_n(x) \in \mathcal{S}$$

$$f(x) = \lim_{n \to \infty} f_n(x)$$

9.1.4 Example



$$f(x) = \begin{cases} 1 & 0 < x < \pi \\ 0 & x = \pi \\ -1 & -\pi < x < 0 \end{cases}$$

$f(x) \notin \mathcal{S} = C_0([-\pi,\pi],\mathbb{R})$ as it is not a continuous function. From Fourier we know, $a_k = 0\ \forall k \in \mathbb{N}$ as it is an odd function, $b_k = \frac{1}{k}\left(1-(-1)^k\right)$

$$\begin{aligned} f(x) &= \lim_{n\to\infty} f_n(x) \\ &= \lim_{n\to\infty}\sum_{k=1}^{n}\left[b_k\sin(kx)\right] \end{aligned}$$

We want to show that $f_n(x)$ is Cauchy. First define a useful norm on $\mathcal{S}$, let $g \in \mathcal{S}$ s.t.

$$g(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty}\left[a_k\cos(kx) + b_k\sin(kx)\right]$$

So we can use $\left\{\frac{1}{2},\cos(x),\cos(2x)\ldots\sin(x),\sin(2x)\ldots\right\}$as a basis for $\mathcal{S}$ and consider the following 2-norm

$$||g||_2 = \left[\left(\sum_k\left[|a_k|^2 + |b_k|^2\right] + |a_0|^2\right)\right]^{\frac{1}{2}}$$

Hence

$$||f_n - f_m||_2 = \left|\left|\sum_{k=1}^{n}\left[b_k\sin(kx)\right] - \sum_{k=1}^{m}\left[b_k\sin(kx)\right]\right|\right|$$

Assume $n > m$

$$\begin{aligned} ||f_n - f_m||_2 &= \left|\left|\sum_{k=n+1}^{m}\left[b_k\sin(kx)\right]\right|\right| \\ &= \left[\sum_{k=n+1}^{m}\left[b_k\right]^2\right]^{\frac{1}{2}} \\ b_k &= \frac{1}{k}\left(1-(-1)^k\right) \\ &= \left[\sum_{k=n+1}^{m}\frac{1}{k^2}\left(1-(-1)^k\right)^2\right]^{\frac{1}{2}} \end{aligned}$$

Now including the extra, even terms in the sum

$$||f_n - f_m||_2 \leq \left[\sum_{k=n+1}^{m}\frac{4}{k^2}\right]^{\frac{1}{2}}$$

Notating that this sum is convergent

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

for a convergent series

$$\sum_{m=n}^{\infty} |a_n| \rightarrow_{n\rightarrow\infty} 0$$

must be valid and therefore

$$||f_n - f_m||_2 \leq 2 \left[\sum_{k=n+1}^{m} \frac{1}{k^2} \right]^{\frac{1}{2}}$$
$$\lim_{n\rightarrow\infty} \sum_{k=n+1}^{m} \frac{1}{k^2} = 0$$

$\therefore f_n$ is Cauchy

9.1.5 Definition: Banach Space

A Normed vector space is said to be Banach (or complete) if every cauchy sequence is strongly convergent within the space.

9.2 Hilbert Space

9.2.1 Definition: Hilbert Space

A Banach space in which the norm is obtained from a scalar product

9.2.2 Definition: Scalar Product

A scalar product $\langle x, y\rangle : \mathcal{S} \rightarrow \mathbb{C}$. For $\mu_i \in \mathbb{C}, \quad x_i, y_i \in \mathcal{S}$, must satisfy

1. $\langle x_1, (\mu_1 y_1 + \mu_2 y_2)\rangle = \mu_1 \langle x_1, y_1\rangle + \mu_2 \langle x_2, y_2\rangle$
2. $\langle x_1, y_1\rangle^\star = \langle y_1, x_1\rangle$
3. $\langle x_1, x_1\rangle \geq 0$
4. $\langle x_1, x_1\rangle = 0 \Longleftrightarrow x_1 = 0$

9.2.3 Example

1. $\mathcal{S} = \mathbb{C}^n, \quad x, y \in \mathcal{S}$
   - $\langle x, y\rangle = \sum_{i=1}^{n} x_i^\star y_i, \quad \begin{cases} x = & \{x_1, x_2 ... x_n\} \\ y = & \{y_1, y_2 ... y_n\} \end{cases}$

2. Real Space, $\mathbb{R}^3$: $\quad \mathbf{a} \in \mathbb{R}^3,\ \mathbf{b} \in \mathbb{R}^3$

   - $\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=1}^{3} a_i b_i = \mathbf{a} \cdot \mathbf{b}$

3. Real Space, $\mathbb{R}^\infty$

   - $\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=1}^{\infty} a_i b_i \qquad \text{If} \begin{cases} \langle \mathbf{a}, \mathbf{a} \rangle & < \infty \\ \langle \mathbf{b}, \mathbf{b} \rangle & < \infty \\ \langle \mathbf{a}, \mathbf{b} \rangle & < \infty \end{cases}$

4. $C_0\left([a,b], \mathbb{C}\right) = \mathcal{S}$

   - $f, g \in \mathcal{S}$
   - $\langle f, g \rangle = \int_a^b f^\star(x)\, g(x)\, dx$

9.3 Schwartz Inequality

For any $x, y \in \mathcal{S}$

$$|\langle x, y \rangle| \leq \langle x, x \rangle^{\frac{1}{2}} \langle y, y \rangle^{\frac{1}{2}}$$

9.3.1 Proof:

Consider $\omega = x + \mu y,\ \omega \in \mathcal{S}\ \mu \in \mathbb{C}$

$$\begin{aligned} 0 &\leq \langle x + \mu y, x + \mu y \rangle \\ &= \langle x, x \rangle + \langle \mu y, x \rangle + \langle x, \mu y \rangle + \langle \mu y, \mu y \rangle \\ &= \langle x, x \rangle + \mu^\star \langle y, x \rangle + \mu \langle x, y \rangle + |\mu|^2 \langle y, y \rangle \end{aligned}$$

Choose $\mu = -\frac{\langle y, x \rangle}{\langle y, y \rangle}$

$$\begin{aligned} \langle x + \mu y, x + \mu y \rangle &= \langle x, x \rangle + \left(-\frac{\langle y, x \rangle}{\langle y, y \rangle} \right)^\star \langle y, x \rangle - \frac{\langle y, x \rangle}{\langle y, y \rangle} \langle x, y \rangle + \left| \frac{\langle y, x \rangle}{\langle y, y \rangle} \right|^2 \langle y, y \rangle \\ &= \langle x, x \rangle + \frac{|\langle y, x \rangle|^2}{\langle y, y \rangle} - 2 \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle} \\ |\langle x, y \rangle|^2 &\leq \langle x, x \rangle \langle y, y \rangle \\ & \square \end{aligned}$$

Define the norm

$$||x|| = \langle x, x \rangle^{\frac{1}{2}}$$

9.3.2 Check the Triangular inequality

$$
\begin{aligned}
||x+y||^2 &= \langle x+y, x+y\rangle \\
&= |\langle x,x\rangle + \langle y,x\rangle + \langle x,y\rangle + \langle y,y\rangle| \\
&\leq |\langle x,x\rangle| + |\langle y,y\rangle| + 2\,|\langle x,y\rangle| \\
&= ||x||^2 + ||y||^2 + 2\,|\langle x,y\rangle|
\end{aligned}
$$

By the Schwartz Inequality

$$
\begin{aligned}
|\langle x,y\rangle| &\leq \langle x,x\rangle^{\frac{1}{2}} \langle y,y\rangle^{\frac{1}{2}} \\
||x+y||^2 &\leq ||x||^2 + ||y||^2 + 2\,||x||\,||y|| \\
||x+y||^2 &\leq (||x|| + ||y||)^2
\end{aligned}
$$

9.4 HUP

Let H and B be observables, I.E. Hermitian Operators. Consider and arbitrary quantum state ψ

$$
\begin{aligned}
\Delta A =& A - \langle \psi, A\psi\rangle \\
\Delta B =& B - \langle \psi, B\psi\rangle
\end{aligned}
$$

Schwartz inequality gives

$$
\begin{aligned}
\langle \Delta A\psi, \Delta A\psi\rangle \langle \Delta B\psi, \Delta B\psi\rangle &\geq |\langle \Delta A\psi, \Delta B\psi\rangle|^2 \\
\left\langle \psi, (\Delta A)^2 \psi\right\rangle \left\langle \psi, (\Delta B)^2 \psi\right\rangle &\geq |\langle \psi, \Delta A\Delta B\psi\rangle|^2
\end{aligned}
$$

Note that,

$$
\Delta A\Delta B = \frac{1}{2}\left([\Delta A, \Delta B] + \{\Delta A, \Delta B\}\right)
$$

Where $[k,m] = km - mk, \{k,m\} = km + mk$

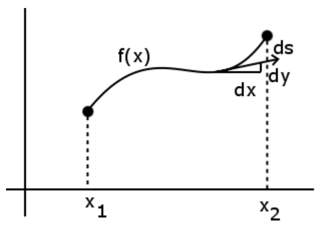
$$
\begin{aligned}
\langle \psi, \Delta A\Delta B\psi\rangle =& \frac{1}{2}\underbrace{\langle \psi, [\Delta A, \Delta B]\,\psi\rangle}_{\text{Imaginary}} + \frac{1}{2}\underbrace{\langle \psi, \{\Delta A, \Delta B\}\,\psi\rangle}_{\text{Real}} \\
\Longrightarrow& |z|^2 = (Re\,[z])^2 + (Im\,[z])^2 \\
|\langle \psi, \Delta A\Delta B\psi\rangle|^2 =& \frac{1}{4}\,|\langle \psi, [\Delta A, \Delta B]\,\psi\rangle|^2 + \frac{1}{4}\,|\langle \psi, \{\Delta A, \Delta B\}\,\psi\rangle|^2 \\
\geq& \frac{1}{4}\,|\langle \psi, [\Delta A, \Delta B]\,\psi\rangle|^2
\end{aligned}
$$

By the Schwartz Inequality

$$\left\langle \psi, (\Delta A)^2 \psi \right\rangle \left\langle \psi, (\Delta B)^2 \psi \right\rangle \geq \frac{1}{4} \left| \langle \psi, [\Delta A, \Delta B] \psi \rangle \right|^2$$

Part IV
Calculus of Variation

10 Calculus of Variation



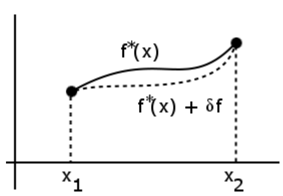
Total length of the curve is $L = \int_{x_1}^{x_2} ds$. Where

$$\begin{aligned} ds =& \sqrt{dx^2 + dy^2} \\ =& dx \sqrt{1 + \left(\frac{dy}{dx} \right)^2} \end{aligned}$$

$$L = \int_{x_1}^{x_2} dx \sqrt{1 + f'(x)^2}$$

If $f^\star(x)$ represents the shortest path between the two points then

$$L[f^\star(x)] \leq L[f^\star(x) + \delta f]$$



10.0.1 Extrema:

Recalling how to find the extrema of a function of a single variable

$$\left. \frac{df}{dx} \right|_{x=x_0} = 0$$

By analogy, to minimise a function

$$\delta L\left[f^{\star}(x)\right]=0$$

11 Functionals

Assume $y:\mathbb{R}\rightarrow\mathbb{R},\ y\in C_0$, and let $J:C_0\rightarrow\mathbb{R}$

$$J[y]=\int_{x_1}^{x_2} dx\ f\left(y,y',x\right)$$

11.1 Examples of functionals

1. $f\left(y,y',x\right)=\sqrt{1+y'^2}$ - Path Length
2. $f\left(y,y',x\right)=m^2\frac{y'^2}{2m}-u(y)$- Lagrangian

Note that $y^{\star}$ is an extremal function of $J[y]$ if

$$\delta J\left[y^{\star}\right]=J[y]-J\left[y^{\star}\right]=0$$

to the first order in dy

11.2 Derive the Euler-Lagrange equation for J

$$\begin{aligned}\delta J\left[y^{\star}\right]&=\int_{x_1}^{x_2} dx\ f\left(y,y',x\right)-\int_{x_1}^{x_2} dx\ f\left(y^{\star},y^{\star\prime},x\right)\\&=\int_{x_1}^{x_2} dx\ \left\{f\left(y,y',x\right)-f\left(y^{\star},y^{\star\prime},x\right)\right\}\end{aligned}$$

Observe

$$\begin{aligned}f\left(y,y',x\right)=&f\left(y^{\star}+\delta y,\frac{\partial}{\partial x}\left(y^{\star}+\delta y\right),x\right)\\=&f\left(y^{\star}+\delta y,y^{\star\prime}+\frac{\partial}{\partial x}\delta y,x\right)\end{aligned}$$

Assume δy is small and smooth.

11.2.1 Two variable Taylor Expansions

$$g\left(x_1+\delta x_1,x_2+\delta x_2\right)=g\left(x_1,x_2\right)+\frac{\partial g}{\partial x_1}\delta x_1+\frac{\partial g}{\partial x_2}\delta x_2+...$$

$$
\begin{aligned}
f\left(y,y',x\right) =& f\left(y^{\star}+\delta y, y^{\star\prime}+\frac{\partial}{\partial x}\delta y, x\right)\\
=& f\left(y^{\star},y^{\star\prime},x\right)+\left.\frac{\partial f}{\partial y}\right|_{y^{\star},y^{\star\prime}}\delta y+\left.\frac{\partial f}{\partial y'}\right|_{y^{\star},y^{\star\prime}}\frac{\partial}{\partial x}\delta y+...
\end{aligned}
$$

$\implies \delta J$

$$
\begin{aligned}
\delta J\left[y^{\star}\right] &= \int_{x_1}^{x_2} dx\ \left\{f\left(y,y',x\right)-f\left(y^{\star},y^{\star\prime},x\right)\right\}\\
&= \int_{x_1}^{x_2} dx\ \left\{\left[f\left(y^{\star},y^{\star\prime},x\right)+\frac{\partial f}{\partial y}\delta y+\frac{\partial f}{\partial y'}\frac{\partial}{\partial x}\delta y\right]-f\left(y^{\star},y^{\star\prime},x\right)\right\}\\
&= \int_{x_1}^{x_2} dx\ \left\{\frac{\partial f}{\partial y}\delta y+\frac{\partial f}{\partial y'}\frac{\partial}{\partial x}\delta y\right\}\\
&= \int_{x_1}^{x_2} dx\frac{\partial f}{\partial y}\delta y+\int_{x_1}^{x_2} dx\frac{\partial f}{\partial y'}\frac{\partial}{\partial x}\delta y\\
\text{By Parts, } & \begin{cases} u=\frac{\partial f}{\partial y'} & dv=\frac{d}{dx}\delta y dx\\ du=\frac{\partial f}{\partial y'}dx & v=\delta y\end{cases}\\
&= \int_{x_1}^{x_2} dx\frac{\partial f}{\partial y}\delta y+\left[\frac{\partial f}{\partial y'}\delta y\right]_{x_1}^{x_2}-\int_{x_1}^{x_2} dx\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\delta y
\end{aligned}
$$

Conclude that, because δy is the deviation from the minimal path, all paths must have the same end points and thus the deviation must be zero so $\left[\frac{\partial f}{\partial y'}\delta y\right]_{x_1}^{x_2}=0$

$$
\delta J\left[y^{\star}\right]=\int_{x_1}^{x_2} dx\left\{\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right\}\delta y
$$

Hence if $\delta J\left[y^{\star}\right]=0 \quad \forall \delta y^{\star}$, Then we arrive at the E-L Equation

$$
\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0
$$

11.3 Alternate Form of E-L

$$
y'\frac{\partial f}{\partial y}=y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)
$$

Also note that the total derivative of $f\left(y,y',x\right)$

$$
\begin{aligned}
\frac{d}{dx}f =& \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \\
=& \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} \\
=& \frac{\partial f}{\partial x} + y'\frac{d}{dx}\frac{\partial f}{\partial y'} + y''\frac{\partial f}{\partial y'} \\
=& \frac{\partial f}{\partial x} + \frac{d}{dx}\left[y'\frac{\partial f}{\partial y'}\right]
\end{aligned}
$$

$$
\frac{\partial f}{\partial x} - \frac{d}{dx}\left[f - y'\frac{\partial f}{\partial y'}\right] = 0
$$

May be more simple when $f = f\left(y, y'\right)$ to use

$$
f - y'\frac{\partial f}{\partial y'} = constant
$$

11.4 E-L Equation for Functionals containing a second derivative

$$
\delta J\left[y^\star\right] = \int_{x_1}^{x_2} dx\ f\left(y, y', y''x\right) - \int_{x_1}^{x_2} dx\ f\left(y^\star, y^{\star\prime}, y^{\star\prime\prime}, x\right)
$$

Observe

$$
\begin{aligned}
f\left(y, y', y'', x\right) =& f\left(y^\star + \delta y, y^{\star\prime} + \frac{\partial}{\partial x}\delta y, y^{\star\prime\prime} + \frac{\partial^2}{\partial x^2}\delta y, x\right) \\
=& f\left(y^\star, y^{\star\prime}, x\right) + \left.\frac{\partial f}{\partial y}\right|_{y^\star, y^{\star\prime}}\delta y + \left.\frac{\partial f}{\partial y'}\right|_{y^\star, y^{\star\prime}}\frac{\partial}{\partial x}\delta y + \left.\frac{\partial f}{\partial y''}\right|_{y^\star, y^{\star\prime}}\frac{\partial^2}{\partial x^2}\delta y + \ldots
\end{aligned}
$$

$\Longrightarrow \delta J$

$$
\delta J\left[y^\star\right] = \int_{x_1}^{x_2} dx\left\{\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\frac{\partial}{\partial x}\delta y + \frac{\partial f}{\partial y''}\frac{\partial^2}{\partial x^2}\delta y\right\}
$$

By Parts, $\begin{cases} u = \frac{\partial f}{\partial y'} & dv = \frac{d}{dx}\delta y dx \\ du = \frac{\partial f}{\partial y'}dx & v = \delta y \end{cases}$

$$
= \int_{x_1}^{x_2} dx\left\{\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y''}\frac{\partial^2}{\partial x^2}\delta y\right\} + \left[\frac{\partial f}{\partial y'}\delta y\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} dx\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\delta y
$$

Conclude that, because δy is the deviation from the minimal path, all paths must have the same end points and thus the deviation must be zero so $\left[\frac{\partial f}{\partial y'}\delta y\right]_{x_1}^{x_2} = 0$. Now just concentrating on the term containing the second differential in x.

$$\int_{x_1}^{x_2} dx \frac{\partial f}{\partial y''}\frac{\partial^2}{\partial x^2}\delta y \begin{cases} u = \frac{\partial f}{\partial y''} & dv = \frac{\partial^2}{\partial x^2}\delta y dx \\ du = \frac{d}{dx}\frac{\partial f}{\partial y''}dx & v = \frac{\partial}{\partial x}\delta y \end{cases}$$

$$= \left[\frac{\partial f}{\partial y''}\frac{\partial}{\partial x}\delta y\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} dx \frac{\partial}{\partial x}\delta y \frac{d}{dx}\frac{\partial f}{\partial y''} \begin{cases} u = \frac{d}{dx}\frac{\partial f}{\partial y''} & dv = \frac{\partial}{\partial x}\delta y dx \\ du = \frac{d^2}{dx^2}\frac{\partial f}{\partial y''}dx & v = \delta y \end{cases}$$

$$= \left[\frac{\partial f}{\partial y''}\frac{\partial}{\partial x}\delta y\right]_{x_1}^{x_2} - \left[\frac{\partial f}{\partial y''}\delta y\right]_{x_1}^{x_2} + \int_{x_1}^{x_2} dx \delta y \frac{d^2}{dx^2}\frac{\partial f}{\partial y''}$$

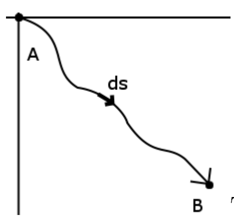
For the same reasons as above we can conclude that both the evaluated terms but equate to zero and we are just left with the integral term, returning to the whole expression

$$\delta J\left[y^\star\right] = \int_{x_1}^{x_2} dx \left\{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)\right\}\delta y$$

Hence if $\delta J\left[y^\star\right] = 0 \quad \forall \delta y$, Then we arrive at the E-L Equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 0$$

11.5 The Brachistochrone Problem



Time taken to travel $ds \rightarrow \frac{ds}{v}$

Hence we want to minimise the functional

$$t\left[y\right] = \int_A^B \frac{ds}{v}$$

We now need to express $v = f\left(x\right)$, make use of $E_{tot} = constant$

$$E_{tot} = \frac{1}{2}mv^2 - mgy$$

Assume: $E_{pot}\left(y = 0\right) = 0$ and $v\left(t = 0\right) = 0$

$$\begin{aligned} \therefore E_{tot} =& 0 \\ \frac{1}{2}mv^2 =& mgy \\ v =& \sqrt{2gy} \end{aligned}$$

Sub into the Functional

$$\begin{aligned} t[y] &= \int_0^{x_0} dx \frac{\sqrt{1 + y'(x)^2}}{\sqrt{2gy}} \\ &= \int_0^{x_1} dx\ f(y, y', x) \end{aligned}$$

Make use of Euler Lagrange formula, since $f = f(y, y')$ we can use the second form

$$\begin{aligned} f - y'\frac{\partial f}{\partial y'} =& constant \\ \frac{\partial f}{\partial y'} = \frac{\partial}{\partial y'}\frac{\sqrt{1 + y'(x)^2}}{\sqrt{2gy}} &= \frac{1}{\sqrt{2g}}\frac{y'}{\sqrt{y(1 + y'^2)}} \end{aligned}$$

E-L $\Longrightarrow$

$$\begin{aligned} \frac{1}{\sqrt{2g}}\sqrt{\frac{1 + y'^2}{y}} - \frac{1}{\sqrt{2g}}\frac{y'^2}{\sqrt{y(1 + y'^2)}} &= constant \\ y\left(1 + y'^2\right) =& constant \end{aligned}$$

Can solved by substitution $y' = \cot(\theta)$, find solutions

$$\begin{aligned} x =& A(\phi - \sin\phi) \\ y =& A(1 - \cos\phi) \\ \phi =& 2\theta \end{aligned}$$

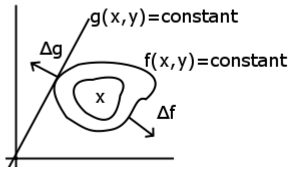
11.5.1 Situation with Several Variables

$$J[y_1, y_2 ... y_n] = \int_{x_1}^{x_2} (y_1, y_2 ... y_n, y_1', y_2' ... y_n', x)$$

Make variations in each variable, $y_i = y_i^\star + \delta y_i$. With some boundary conditions, $\delta y_i(x_0) = \delta y_i(x_1) = 0$

12 Lagrange Multipliers

Consider $f : \mathbb{R}^2 \to \mathbb{R}$ want to find the minimum of f



Minimise $f(x, y)$under the constraint that $g(x, y) = constant$, by Solving the problem

$$\nabla f \parallel \nabla g \Leftrightarrow \nabla f \propto \nabla g$$

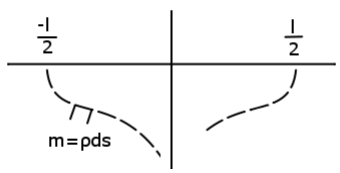
Introduce,

$$J = f + \lambda g$$

find the minimum for $\nabla J = 0$ of the unconstrained problem

12.1 Example: Hanging Bicycle Chain

The functional form of the curve of length L, that minimises the potential energies. Chain fixed at two points, how does it hang ?



$dE_{pot} = \rho y ds$
The functional to find the extremal of

$$E_{pot} = \int_{-\frac{l}{2}}^{\frac{l}{2}} ds \rho y$$

The functional of the contraint

$$L = \int_{-\frac{l}{2}}^{\frac{l}{2}} dx \sqrt{1 + y'^2}$$

Consider

$$\begin{aligned} \tilde{J} =& E_{pot} + \lambda L \\ =& \int_{-\frac{l}{2}}^{\frac{l}{2}} dx \begin{cases} (\rho y - \lambda) \sqrt{1 + y'^2} \\ f(y, y', x) \end{cases} \end{aligned}$$

Choose $\rho = 1$, Sub in $u = y + \lambda \Rightarrow u' = \lambda'$

$$\tilde{J} = \int_{-\frac{l}{2}}^{\frac{l}{2}} dx u\sqrt{1+u'^2}$$

Observe no x dependence, use the alternate form of the E-L equation

$$\begin{aligned} f - u'\frac{\partial f}{\partial u'} &= constant \\ u\sqrt{1+u'^2} - u'u\frac{u'}{\sqrt{1+u'^2}} &= C \\ \frac{u}{\sqrt{1+u'^2}} &= C \end{aligned}$$

$$\begin{aligned} \frac{du}{dx} &= \sqrt{\frac{u^2}{c^2} - 1} \\ x + b &= \int \frac{du}{\sqrt{u^2-1}} = C\ arccosh\left(\frac{u}{z}\right) \\ y &= C\ \cosh\left(\frac{x+b}{C}\right) - \lambda \end{aligned}$$

Determine constants from boundary conditions, $y\left(\frac{l}{2}\right) = 0, \quad y'(0) = 0$

$$\lambda = C\ \cosh\left(\frac{L}{2C}\right)$$

Part V

Tensor Analysis

13 Vectors in $\mathbb{R}^3$

13.1 Index/Suffix notation

$$\mathbf{v} \in \mathbb{R}^3, \mathbf{v} = (v_1, v_2, v_3)$$

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = \sum_{i=1}^{3} a_ib_i$$

This is often written in suffix notation without the summation symbol, $\mathbf{a} \cdot \mathbf{b} = a_ib_i$. If an index is on both sides of an equation it is not included in the summation.

Examples

1. $c_i = a_{ij} b_j$
   - $c_1 = a_{11} b_1 + a_{12} b_2 + a_{13} b_3$
   - $c_2 = a_{21} b_1 + a_{22} b_2 + a_{23} b_3$
2. $\frac{\partial V_i}{\partial x_i} = \frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2} + \frac{\partial V_3}{\partial x_3} = \partial_i V_i = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = \nabla \cdot \mathbf{V}$

13.2 Kronecher Delta Symbol

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Note similarity to the Dirac δ function

$$\int_{-\infty}^{\infty} dx \delta \left(x - x_0 \right) f \left(x \right) = f \left(x_0 \right)$$

Kronecher

$$\sum_i \delta_{ij} f \left(i \right) = f \left(j \right)$$

A matrix formed of Kronecher Delta's corresponds to the identity matrix

$$\{\delta_{ij}\}_{matrix} = \left\{ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right\} = \underline{\underline{\mathbf{I}}}$$

13.2.1 Properties

1. $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$
2. $\delta_{ij} a_j = a_i$
3. $\delta_{ij} \delta_{ik} = \delta_{jk}$
4. Let x_i denote the cartesian coordinates of$\mathbb{R}^3$
   - $\frac{\partial x_i}{\partial x_j} = \delta_{ij}$

13.3 Permutation Symbol

$$\varepsilon_{ijk} = \begin{cases} 0 & \text{if any two } i, j, k \text{ are equal} \\ 1 & \text{if } i, j, k \text{ is a cyclic permutation} \\ -1 & \text{if } i, j, k \text{ is an anti-cyclic permutation} \end{cases}$$

Cyclic, $i, j, k = 1, 2, 3$

$$\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = 1$$

Anti-cyclic, $i, j, k = 1, 3, 2$

$$\varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1$$

Everything else $\varepsilon_{ijk} = 0$

13.4 Uses of ε_{ijk}

Consider

$$\begin{aligned} \mathbf{a} \times \mathbf{b} =& \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ =& \mathbf{i} \left(a_2 b_3 - a_3 b_2 \right) - \mathbf{j} \left(a_1 b_3 - a_3 b_1 \right) + \mathbf{k} \left(a_1 b_2 - a_2 b_1 \right) \end{aligned}$$

note $(\mathbf{a} \times \mathbf{b})_i = \varepsilon_{ijk} a_j b_k$, test for $(\mathbf{a} \times \mathbf{b})_1$

$$\begin{aligned} (\mathbf{a} \times \mathbf{b})_1 =& (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{i} \\ =& \varepsilon_{1jk} a_j b_k \end{aligned}$$

The only permutation which do not have a duplicate suffixes are

$$\begin{aligned} \varepsilon_{1jk} a_j b_k =& \varepsilon_{123} a_2 b_3 + \varepsilon_{132} a_3 b_2 \\ (\mathbf{a} \times \mathbf{b})_1 =& a_2 b_3 - a_3 b_2 \\ (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{i} =& (a_2 b_3 - a_3 b_2) \end{aligned}$$

13.4.1 Permutation Identities

Given without proof,

$$\varepsilon_{pqr} \det \underline{\underline{\mathbf{A}}} = \varepsilon_{ijk} a_{pi} a_{qj} a_{rk}$$

$$\varepsilon_{ijk} \varepsilon_{pqk} = \delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp}$$

13.4.2 Vector Identity Proofs

The permutation symbol can be used to prove many vector identites,

Triple Scalar product

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

Proof

$$\begin{aligned}
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= a_i \varepsilon_{ijk} b_j c_k \\
&= b_j \varepsilon_{jki} c_k a_i \\
&= \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) \\
&= c_k \varepsilon_{kij} a_i b_j \\
&= \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \\
& \quad \square
\end{aligned}$$

Determinants

$$\underline{\underline{\mathbf{A}}} = \{a_{ij}\}, \det \underline{\underline{\mathbf{A}}} = \varepsilon_{ijk} a_{1i} a_{2j} a_{3k}$$

Proof
Cycling through all non-zero permutations for ε_{ijk}

$$\begin{aligned}
\varepsilon_{ijk} a_{1i} a_{2j} a_{3k} &= \varepsilon_{123} a_{11} a_{22} a_{33} + \varepsilon_{312} a_{13} a_{21} a_{32} + \varepsilon_{231} a_{12} a_{23} a_{31} \\
&\quad + \varepsilon_{321} a_{13} a_{22} a_{31} + \varepsilon_{213} a_{12} a_{21} a_{33} + \varepsilon_{132} a_{11} a_{23} a_{3} \\
&= a_{11} a_{22} a_{33} + a_{13} a_{21} a_{32} + a_{12} a_{23} a_{31} \\
&\quad - a_{13} a_{22} a_{31} - a_{12} a_{21} a_{33} - a_{11} a_{23} a_{3} \\
&= a_{11} (a_{22} a_{33} - a_{23} a_{32}) - a_{12} (a_{21} a_{33} - a_{23} a_{31}) \\
&\quad + a_{13} (a_{21} a_{32} - a_{22} a_{31}) \\
&= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{2} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
& \square
\end{aligned}$$

Determinants of multiplied matricies

$$\underline{\underline{\mathbf{C}}} = \underline{\underline{\mathbf{AB}}}$$

Let $\underline{\underline{\mathbf{A}}}, \underline{\underline{\mathbf{B}}}, \underline{\underline{\mathbf{C}}}$ be 3×3 matricies, then

$$\det \underline{\underline{\mathbf{C}}} = \det \underline{\underline{\mathbf{A}}} \det \underline{\underline{\mathbf{B}}}$$

Proof

$$\begin{aligned}\det \underline{\underline{\mathbf{C}}} =& \varepsilon_{ijk} c_{1i} c_{2j} c_{3k} \\ =& \varepsilon_{ijk} a_{1\alpha} b_{\alpha i} a_{2\beta} b_{\beta j} a_{3\gamma} b_{\gamma k} \\ =& a_{1\alpha} a_{2\beta} a_{3\gamma} \varepsilon_{ijk} b_{\alpha i} b_{\beta j} b_{\gamma k}\end{aligned}$$

we know that,

$$\varepsilon_{pqr} \det \underline{\underline{\mathbf{A}}} = \varepsilon_{ijk} a_{pi} a_{qj} a_{rk}$$

therefore

$$\begin{aligned}a_{1\alpha} a_{2\beta} a_{3\gamma} \overbrace{\varepsilon_{ijk} b_{\alpha i} b_{\beta j} b_{\gamma k}}^{\varepsilon_{\alpha\beta\gamma} \det \underline{\underline{\mathbf{B}}}} &= \det \underline{\underline{\mathbf{B}}} \overbrace{\varepsilon_{\alpha\beta\gamma} a_{1\alpha} a_{2\beta} a_{3\gamma}}^{\det \underline{\underline{\mathbf{A}}}} \\ \det \underline{\underline{\mathbf{C}}} &= \det \underline{\underline{\mathbf{A}}} \det \underline{\underline{\mathbf{B}}}\end{aligned}$$

$\square$

Triple Cross Product

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\, \mathbf{c}$$

Proof

$$\begin{aligned}(\mathbf{a} \times (\mathbf{b} \times \mathbf{c}))_i =& \varepsilon_{ijk} a_j (\mathbf{b} \times \mathbf{c})_k \\ =& \varepsilon_{ijk} a_j \varepsilon_{kpq} b_p c_q \\ =& \varepsilon_{ijk} \varepsilon_{pqk} a_j b_p c_q\end{aligned}$$

we know that,

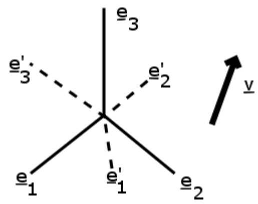
$$\varepsilon_{ijk} \varepsilon_{pqk} = \delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp}$$

therefore

$$\begin{aligned}(\delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp})\, a_j b_p c_q =& a_j b_i c_j - a_j b_j c_i \\ (\mathbf{a} \times (\mathbf{b} \times \mathbf{c}))_i =& (\mathbf{a} \cdot \mathbf{c})\, b_i - (\mathbf{a} \cdot \mathbf{b})\, c_i\end{aligned}$$

14 Vector Coordinate Transformations

14.1 Rotational coordinate transform



Right Hand coordinate system $\mathcal{S}$ wil orthogonal **unit** vectors as a basis

$$\mathbf{e_i} \cdot \mathbf{e_j} = \delta_{\mathbf{ij}}$$

Any vector **v** can be written as

$$\mathbf{v} = v_i \mathbf{e_i}$$

In the rotated system $\mathcal{S}'$

$$\begin{aligned} e'_i \cdot e'_j =& \delta_{ij} \\ \mathbf{v} =& v'_i \mathbf{e'_i} \end{aligned}$$

\det\underline{\underline{\mathbf{A}}}

The transformation from $\mathcal{S} \to \mathcal{S}'$

$$\mathbf{e'_i} = a_{ij} \mathbf{e_j}$$

The transformation matrix $\underline{\underline{\mathbf{A}}}$

$$\begin{aligned} \mathbf{e'_i} \cdot \mathbf{e_j} =& a_{ik} \mathbf{e_k} \cdot \mathbf{e_j} = a_{ik} \delta_{kj} = a_{ij} \\ \underline{\underline{\mathbf{A}}} =& \{a_{ij}\} = \{\mathbf{e'_i} \cdot \mathbf{e_j}\} \end{aligned}$$

Backwards transformation

$$\mathbf{e_i} = b_{ij} \mathbf{e'_j}$$

note,

$$\begin{aligned} \mathbf{e_i} \cdot \mathbf{e'_j} =& b_{ij} \\ \mathbf{e'_j} \cdot \mathbf{e_i} =& a_{ji} \\ a_{ji} =& b_{ij} \end{aligned}$$

Conclusion

$$\underline{\underline{\mathbf{B}}} = \underline{\underline{\mathbf{A^T}}}$$

14.2 Transformation matrix properties

The transformation matrix transforms a vector expressed in one coordinate basis to another

$$\begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix} = \underline{\underline{\mathbf{A}}} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

As $\underline{\underline{\mathbf{A}}}$ sends $\mathcal{S} \to \mathcal{S}'$ and $\underline{\underline{\mathbf{B}}}$ sends $\mathcal{S}' \to \mathcal{S}$ then

$$\underline{\underline{\mathbf{AB}}} = \underline{\underline{\mathbf{AA^T}}} = \underline{\underline{\mathbf{I}}}$$

Therefore the transpose of $\underline{\underline{\mathbf{A}}}$ must also be it's inverse

$$a_{ik}a_{jk} = \delta_{jk}$$

The determinant of $\underline{\underline{\mathbf{A}}}$ must be ± 1

$$\begin{aligned} 1 = \det \underline{\underline{\mathbf{I}}} &= \det \left(\underline{\underline{\mathbf{AA^T}}}\right) \\ &= \det \left(\underline{\underline{\mathbf{A^T}}}\right) \det \left(\underline{\underline{\mathbf{A}}}\right) \\ &= \left[\det \left(\underline{\underline{\mathbf{A}}}\right)\right]^2 \\ \det \left(\underline{\underline{\mathbf{A}}}\right) &= \pm\, 1 \end{aligned}$$

With Respect to the basis, $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$

$$\mathbf{e_i'} = \begin{pmatrix} a_{i1} \\ a_{i2} \\ a_{i3} \end{pmatrix}$$

We can show that,

$$\begin{aligned} \mathbf{e_1'} \cdot (\mathbf{e_2'} \times \mathbf{e_3'}) =& a_{i1}\varepsilon_{ijk}a_{j2}a_{k3} \\ =& \det \left(\underline{\underline{\mathbf{A}}}\right) \\ =& \begin{cases} 1 & RHS \\ -1 & LHS \end{cases} \end{aligned}$$

15 Cartesian Tensor

Definition: Any Set of 3^n quantities, $T'_{i_1,i_2\ldots i_n}, i_k = 1,2,3$ which transforms according to $T'_{i_1,i_2\ldots i_n} = a_{i_1j_1}a_{i_2j_2}\ldots a_{i_nj_n}T_{j_1,j_2\ldots j_n}$ when refered to two coordinate systems $\mathcal{S},\mathcal{S}'$ given terms of basis $\mathbf{e_i}$ and $\mathbf{e_i'}$ which are related through $\mathbf{e_j'} = a_{ij}\mathbf{e_j}$ is called a Tensor of rank n

Example: Conductivity

$$\mathbf{J} = \underline{\underline{\sigma}}\mathbf{E}$$

Let $\phi(x)$ be a scalar field, a scalar is a tensor of rank $n = 0$. Consider the gradient of this field, $\nabla\phi = \partial_i\phi$ the gradient transforms as a tensor of rank 1.

Proof

$$\begin{aligned} v_i =& \partial_i \phi \in \mathcal{S} \\ v_i' =& \frac{\partial \phi}{\partial x_i'} = \partial_i^{'} \phi \in \mathcal{S}' \\ =& \frac{\partial x_j}{\partial x_i'} \frac{\partial \phi}{\partial x_j} = \frac{\partial x_j}{\partial x_i'} v_j \end{aligned}$$

note

$$\begin{aligned} x_j =& a_{kj} x_k' \\ \frac{\partial x_j}{\partial x_i'} =& a_{kj} \frac{\partial x_k'}{\partial x_i'} = a_{kj} \delta_{ki} \\ =& a_{ij} \end{aligned}$$

Hence

$$v_i' = \frac{\partial x_j}{\partial x_i'} \frac{\partial \phi}{\partial x_j} = a_{ij} \frac{\partial \phi}{\partial x_j} = a_{ij} v_j$$

15.0.1 Transformation properties of the Kronecher Delta:

We want $\delta_{ij} = \delta_{ij}'$ and we also want δ_{ij} to be a tensor of rank 2, check if these are consistent.

Assuming δ to be a tensor of rank 2 we have

$$\underline{\underline{\delta_{ij}'}} = a_{ik} a_{jl} \underline{\underline{\delta_{kl}}}$$

Directly from this

$$\underline{\underline{\delta_{ij}'}} = a_{ik} a_{jk} = \underline{\underline{\delta_{ij}}}$$

which validates our first requirement.

15.0.2 Transformation properties of the permutation symbol

The permutation symbol is defined to transform as a psuedo-tensor, this is defined as, $T_{i_1,i_2...i_n}' = \det \underline{\underline{\mathbf{A}}} a_{i_1 j_1} a_{i_2 j_2} ... a_{i_n j_n} T_{j_1,j_2...j_n}$ so that,

$$\varepsilon_{ijk}' = \det \underline{\underline{\mathbf{A}}} a_{ip} a_{jq} a_{kr} \varepsilon_{pqr} = \left(\det \underline{\underline{\mathbf{A}}}\right)^2 \varepsilon_{ijk} = \varepsilon_{ijk}$$

15.0.3 Psuedo Vectors

Consider $\mathbf{c} = (\mathbf{d} \times \mathbf{b})$

$$\begin{aligned} c'_i =& \varepsilon'_{ijk} d'_j b'_k = \varepsilon_{ijk} d'_j b'_k \\ =& \varepsilon_{ijk} a_{jp} d_p a_{kq} b_q \end{aligned}$$

introduce a δ

$$c'_i = \delta_{im} \varepsilon_{mjk} a_{jp} d_p a_{kq} b_q$$

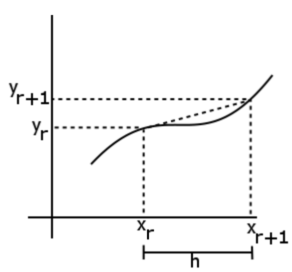
split this new δ into two matrix components

$$\begin{aligned} c'_i =& a_{is} a_{ms} \varepsilon_{mjk} a_{jp} d_p a_{kq} b_q \\ =& a_{is} d_p b_q \varepsilon_{mjk} a_{ms} a_{jp} a_{kq} \\ \varepsilon_{mjk} a_{ms} a_{jp} a_{kq} =& \varepsilon_{spq} \det \underline{\underline{\mathbf{A}}} \\ c'_i =& \det \underline{\underline{\mathbf{A}}} a_{is} \varepsilon_{spq} d_p b_q \\ =& \det \underline{\underline{\mathbf{A}}} a_{is} (\mathbf{d} \times \mathbf{b})_s \\ c'_i =& \det \underline{\underline{\mathbf{A}}} a_{is} c_s \end{aligned}$$

Part VI
Numerical Analysis

16 Integration Methods

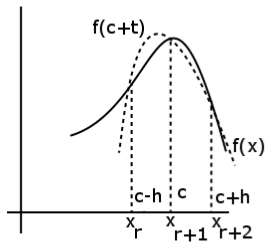
16.1 Trapezium Rule



By Approximating the area under a section of curve by the area of the trapezium formed from the boundaries of the curve section we arrive at, $\int_{x_r}^{x_{r+1}} dx f(x) \approx \frac{h}{2}(y_r + y_{r+1})$. This can be extended over N trapeziums of equal width,

$$I_n = \sum_{r=0}^{N-1} \frac{h}{2}(y_r + y_{r+1}) = \frac{h}{2}(y_0 + 2(y_1 + y_2 + ...y_{N-1}) + y_N)$$

16.2 Simpson's Rule



Approximate $f(x)$ on the interval $[x_r, x_{r+2}]$ by a parabola $f(c+t)$

$$f(c+t) \approx f(c) + f'(c)\,t + \frac{1}{2} f''(c)\,t^2$$

The area under $f(c+t)$

$$\begin{aligned}
A_r \approx & \int_{-h}^{h} dt \left[f(c) + f'(c)\,t + \frac{1}{2} f''(c)\,t^2 \right] \\
= & 2h f(c) + f'(c) \left[\frac{t^2}{2} \right]_{-h}^{h} + \frac{1}{2} f''(c) \left[\frac{t^3}{3} \right]_{-h}^{h} \\
A_r \approx & 2h \left[f(c) + \frac{1}{3!} f''(c)\,h^2 \right]
\end{aligned}$$

Next eliminate $f''(c)$

$$\begin{aligned}
f(c+h) = & f(c) + f'(c)\,h + \frac{1}{2} f''(c)\,h^2 \\
f(c-h) = & f(c) - f'(c)\,h + \frac{1}{2} f''(c)\,h^2 \\
f''(c) = & \frac{f(c-h) - 2f(c) + f(c+h)}{h^2} \\
A_r = & \frac{h}{3} \left[f(c-h) + 4f(c) + f(c+h) \right]
\end{aligned}$$

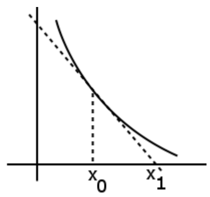
where $f(c-h) = f(x_r) = y_r$, therefore the total area under the curve,

$$A \approx \sum_r A_r = \frac{h}{3} \left[y_0 + 4\,(y_1 + y_3 + ...y_{N-1}) + 2\,(y_2 + y_4 + ...y_{N-1}) + y_N \right]$$

17 Iterative Methods for Finding Solutions to Equations

17.1 Newton-Raphson

Used to find zeros to equations.



Examining the tangent we find

$$f'(x) \approx \frac{\Delta y}{\Delta x} = \frac{0 - f(x)}{x_{n+1} - x_n}$$
$$x_{n+1} = x_n - \frac{f(x)}{f'(x)}$$

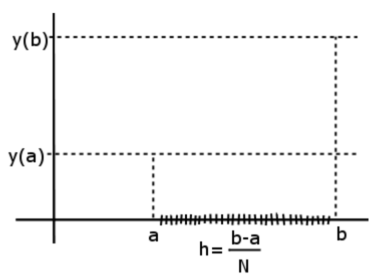
17.2 Runge-Kutta

Integrate a function in two steps

Step 1 Integration using Trapezium rule

Step 2 Taylor Expand
Example

$$\frac{dy}{dx} = x^2 y^3$$



Step 1

$$y_1 - y_0 = \int_a^b y'(x)\,dx = \int_a^b f(x, y(x))\,dx$$
$$\int_a^b f(x, y(x))\,dx \approx \frac{h}{2}\{f(x_0, y_0) + f(x_1, y_1)\}$$
$$y_1 \approx y_0 + \frac{h}{2}\{f(x_0, y_0) + f(x_1, y_1)\}$$

Step 2

$$y_1 = y(x_1) \approx y(x_0) + y'(x_0)(x_1 - x_0)$$

Combine these two

$$y_1 = y_0 + \frac{h}{2}\left\{f(x_0, y_0) + f[x_1, y_0 + f(x_0, y_0)h]\right\}$$

Note iterative formula expressed as,

$$\begin{aligned} y_{n+1} =& y_n + \frac{h}{2}(k_1 + k_2) \\ k_1 =& f(x_n, y_n) \\ k_2 =& f(x_n, y_n + hk_1) \end{aligned}$$

18 Matrix Analysis

18.1 Gaussian Elimination

We want to solve $\underline{\underline{\mathbf{A}}}\mathbf{x} = \mathbf{b}$, take linear combinations of the equations to eliminate some of the variables.

18.1.1 Example

Consider a system of linear equations where $x_i \in \mathbb{R}$

$$\begin{aligned} 2x_1 - x_2 + 3x_3 =& 3 \\ x_1 + 2x_3 =& 3 \\ x_1 + x_2 + 2x_3 =& 4 \end{aligned}$$

this can be expressed as

$$\underbrace{\begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \\ 1 & 1 & 2 \end{bmatrix}}_{\underline{\underline{\mathbf{A}}}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}}_{b}$$

Usually more tractable if written in extended matrix notation

$$\begin{array}{lccccc} (1) & 2 & -1 & 3 & | & 3 \\ (2) & 1 & 0 & 2 & | & 3 \\ (3) & 1 & 1 & 2 & | & 4 \end{array}$$

First try to make zeros in the first column below the diagonal of $\underline{\underline{\mathbf{A}}}$

$$\begin{array}{rccc|c}
(1)' = (1) & 2 & -1 & 3 & 3 \\
(2)' = (2) - 1/2\,(1) & 0 & 1/2 & 1/2 & 3/2 \\
(3)' = (3) - 1/2\,(1) & 0 & 3/2 & 1/2 & 5/2
\end{array}$$

Now make zeros in the second column below the diagonal

$$\begin{array}{rccc|c}
(1)'' = (1)' & 2 & -1 & 3 & 3 \\
(2)'' = (2)' & 0 & 1/2 & 1/2 & 3/2 \\
(3)'' = (3)' - 3\,(2)' & 0 & 0 & -1 & -2
\end{array}$$

now we have an upper triangular matrix, this is easily solvable by working from bottom to top.

$$\begin{aligned}
(3)'' &\Rightarrow -x_3 = -2 \\
(2)'' &\Rightarrow \frac{1}{2}x_2 + x_3 = \frac{3}{2} \\
(1)'' &\Rightarrow 2x_1 - x_2 + 3x_3 = 3
\end{aligned}$$

therefore $x_3 = 2, x_2 = 1, x_1 = -1$

18.2 Gauss-Jordan Elimination

Utilising Gaussian Elimination to find the inverse of a matrix, suppose we wish to find $\underline{\underline{\mathbf{A}}}^{-1}$ where,

$$\underline{\underline{\mathbf{A}}} = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

write in extended matrix form

$$\begin{array}{cccc|ccc}
(1) & 2 & -1 & 3 & 1 & 0 & 0 \\
(2) & 1 & 0 & 2 & 0 & 1 & 0 \\
(3) & 1 & 1 & 2 & 0 & 0 & 1
\end{array}$$

Method: make all the elements below the diagonal zero, then all the elements above zero, the elements on the diagonal equal to unity,

$$\begin{array}{rccc|ccc}
(1)' = (1) & 2 & -1 & 3 & 1 & 0 & 0 \\
(2)' = (2) - 1/2\,(1) & 0 & 1/2 & 1/2 & -1/2 & 1 & 0 \\
(3)' = (3) - 1/2\,(1) & 0 & 3/2 & 1/2 & -1/2 & 0 & 1
\end{array}$$

$$\begin{array}{rccc|ccc}
(1)'' = (1)' & 2 & -1 & 3 & 1 & 0 & 0 \\
(2)'' = (2)' & 0 & 1/2 & 1/2 & -1/2 & 1 & 0 \\
(3)'' = (3)' - 3\,(2)' & 0 & 0 & -1 & 1 & -3 & 1
\end{array}$$

$$
\begin{array}{c|ccc|ccc}
(1)''' = (1)'' + 3\,(3)'' & 2 & -1 & 0 & 4 & -9 & 3 \\
(2)''' = (2)'' + 1/2\,(3)'' & 0 & 1/2 & 0 & 0 & -1/2 & 1/2 \\
(3)''' = (3)'' & 0 & 0 & -1 & 1 & -3 & 1
\end{array}
$$

$$
\begin{array}{c|ccc|ccc}
(1)^{IV} = (1)''' + 2\,(2)''' & 2 & 0 & 0 & 4 & -10 & 4 \\
(2)^{IV} = (2)''' & 0 & 1/2 & 0 & 0 & -1/2 & 1/2 \\
(3)^{IV} = (3)''' & 0 & 0 & -1 & 1 & -3 & 1
\end{array}
$$

$$
\begin{array}{c|ccc|ccc}
(1)^{V} = \frac{1}{2}\,(1)^{IV} & 1 & 0 & 0 & 2 & -5 & 2 \\
(2)^{V} = 2\,(2)^{IV} & 0 & 1 & 0 & 0 & -1 & 1 \\
(3)^{V} = -1\,(3)^{IV} & 0 & 0 & 1 & -1 & 3 & -1
\end{array}
$$

Then

$$
\underline{\underline{\mathbf{A}}}^{-1} = \begin{bmatrix} 2 & -5 & 2 \\ 0 & -1 & 1 \\ -1 & 3 & -1 \end{bmatrix}
$$