Quantum Mechanics (Prof. Deacon)

TISE
\[
-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V\psi = E\psi \Rightarrow \hat{H}\psi = E\psi
\]

TOSE
\[
-\frac{\hbar^2}{2m} \frac{d^2\gamma}{dx^2} + \frac{i\hbar}{\alpha} \frac{d\gamma}{dt} \Rightarrow \hat{H}\gamma = i\hbar \frac{d\gamma}{dt}
\]

\[E = \hbar \omega, \quad \rho = \hbar \kappa\]

\[\int_{-\infty}^{\infty} |\psi|^2 dx = 1\]

Use separation of variables to solve the TOSE \(\psi(x, t) = u(x) T(t)\)

\[\begin{align*}
&u_L = Ae^{-\gamma x} + Be^{\gamma x} \\
&u_c = Ce^{i\kappa x} + De^{-i\kappa x} \\
&u_R = Fe^{-\gamma x} + Ge^{\gamma x}
\end{align*}\]

Hypocycloids are continuous and have continuous derivatives.

Even vs. odd functions:
- Even
- Odd

Even

\[\gamma^2 + k^2 = \frac{2mV}{\hbar^2}\]

\[\gamma = k \tan (k a)\]

\[\gamma = -k \cot (k a)\]

The highest and lowest energy states have even parity.
For $E < V_0$:

\[ V \]

For $E > V_0$:

\[ V \]

Potential Step ($E > V_0$):

\[ V \]

Reflection coefficient, $R = \frac{\text{reflected flux}}{\text{incident flux}} = \frac{\sqrt{E^2 - 1} A^2}{\sqrt{E^2 + 1} A^2}$

Transmission coefficient, $T = \frac{\text{transmitted flux}}{\text{incident flux}} = \frac{\sqrt{E^2 + 1} A^2}{\sqrt{E^2 + 1} A^2} = \frac{E A}{E A}$

For an SHO, $\psi(n) = \frac{1}{\sqrt{2^n n!}} \psi_0^n x^n$ and $E_n = (n + \frac{1}{2}) \hbar \omega$.

Hamiltonian operator, $\hat{H}$, satisfies...

\[ \int_{-\infty}^{\infty} \langle \psi | \hat{H} | \phi \rangle \, dx = \int_{-\infty}^{\infty} \langle \phi | \hat{H} | \psi \rangle \, dx \]

- **real** eigenvalues, i.e., $g_n^* = g_n$
- **orthonormal** eigenstates, i.e., $\int \phi_n^* \phi_m \, dx = \delta_{mn}$

\[ \hat{p} = -i \hbar \frac{\partial}{\partial x}, \quad \hat{x} = x \]

For $\psi = \sum_n a_n \phi_n$, $a_n = \int \phi_n^* \psi \, dx$ (Overlap integral)

If the wavefunction at $t=0$ was created by making an energy measurement, $E_n$, then it results in the wavefunction collapsing into an energy state $\phi_n$.

\[ \psi(x, t) = \phi_n(x, t) \]
Commutator \[ [\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \] (anti-Hermitian) \[ \times i \] for Hermitian

Anticommutator \[ \{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A} \] (Hermitian)

\[ [\hat{x}, \hat{p}] = i\hbar \quad \therefore \hat{x} \text{ and } \hat{p} \text{ are conjugate variables} \]

Two observables are compatible if their operators have a common set of eigenstates.
- Compatible operators must commute

Expectation Value \[ \langle \hat{A} \rangle = \int_{-\infty}^{\infty} \psi^* \hat{A} \psi \, dx = \sum_n |a_n|^2 \langle \psi_n | \hat{a} | \psi_n \rangle = \langle \psi | \hat{A} | \psi \rangle \]

Ehrenfest Theorem \[ \text{Since } \frac{d}{dt} \langle \hat{A} \rangle = \frac{i}{\hbar} \langle \hat{A} \hat{\phi} \rangle, \]

\[ \frac{d}{dt} \langle \hat{p} \rangle = \frac{d}{dt} \langle \hat{\phi} \rangle \]

"The equations of motion for the expectation values of observables are the same as the equations of motion for their classical counterparts."

Uncertainty \[ \Delta x \Delta p \geq \frac{\hbar}{2} \]

HUP

Schwartz Inequality \[ \left( \int |a(x)|^2 \, dx \right) \left( \int |b(x)|^2 \, dx \right) \geq \left| \int b^*(x) a(x) \, dx \right|^2 \]

Equality if \[ a(x) \propto b(x) \]

Continuous Eigenvalues \[ \psi(x) = \int a(q) \, dq \varphi(q, x) \]

A Gaussian wavepacket is a minimum uncertainty wavepacket.
Wavepackets spread with time.
$\hat{A}^+ = \alpha \hat{a} + i \hat{\rho}$  \hspace{1cm} \text{creation/raising operator}  \\
$\hat{a}^+ = \alpha \hat{a} + i \hat{\rho}$  \hspace{1cm} \text{annihilation/lowering operator}  \\
\{ ladder operators \}

$(\hat{A} + \hat{B})^+ = \hat{A}^+ + \hat{B}^+$  \\
$(\hat{A}\hat{B})^+ = \hat{B}^+\hat{A}^+$

For Hermitian operators $\hat{A}^+ = \hat{A}$ and $\hat{\rho}^+ = \hat{\rho}$

The first-order energy eigenfunction perturbation correction is given by...

$U_n^{(1)} = \sum_{m \neq n} \frac{\int \psi_n^* \hat{V} \psi_m \, dx}{E_n - E_m} \psi_m$  \hspace{1cm} \text{with the corrected energy eigenfunctions $U_n = U_n + U_n^{(1)}$}

For a multi-dimensional SHO, if the potential is the same in all dimensions then it is known as a \textit{cylindrical} potential and has circular symmetry.

\textbf{Degeneracy} is the number of eigenstates that yield the same eigenvalue. (Any superposition of degenerate eigenstates is also a degenerate eigenstate)

In 3-D, an eigenstate $\psi_{n,m}(r)$ has three quantum numbers.

\textbf{Angular Momentum}

$\hat{L} = \hat{\rho} \times \hat{\rho} = \left( \begin{array}{c} \hat{\rho}_x \\ \hat{\rho}_y \\ \hat{\rho}_z \end{array} \right) \times \left( \begin{array}{c} \hat{\rho}_x \\ \hat{\rho}_y \\ \hat{\rho}_z \end{array} \right) = \left( \begin{array}{c} \frac{\hat{\rho}_y \hat{\rho}_z - \hat{\rho}_z \hat{\rho}_y}{} \\ \frac{\hat{\rho}_z \hat{\rho}_x - \hat{\rho}_x \hat{\rho}_z}{} \\ \frac{\hat{\rho}_x \hat{\rho}_y - \hat{\rho}_y \hat{\rho}_x}{} \end{array} \right) = \left( \begin{array}{c} \hat{\rho}_y \hat{\rho}_z - \hat{\rho}_z \hat{\rho}_y \\ \hat{\rho}_z \hat{\rho}_x - \hat{\rho}_x \hat{\rho}_z \\ \hat{\rho}_x \hat{\rho}_y - \hat{\rho}_y \hat{\rho}_x \end{array} \right)$

$[\hat{L}_x, \hat{L}_y] = i \hbar \hat{L}_z$ \\
$[\hat{L}_y, \hat{L}_z] = i \hbar \hat{L}_x$ \\
$[\hat{L}_z, \hat{L}_x] = i \hbar \hat{L}_y$  \hspace{1cm} \text{cyclic ordering}  \\
$\hat{L} \times \hat{L} = i \hbar \hat{L}$

$\hat{L}^2 = \hat{L} \cdot \hat{L} = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$
Angular Momentum Ladder Operator: \[ \hat{L}_z = \hat{L}_x^2 \hat{L}_y \]

Angular Momentum Eigenstates: \[ \gamma_{lm} (\theta, \phi) \]

Eigenvalues of \( \hat{L}_x, \hat{L}_y, \hat{L}_z \): \( m \) \( \hbar \)

Eigenvalues of \( \hat{L}^2 \): \( l(l+1) \hbar^2 \)

The eigenstates of a 3D TISE with a central potential are \( \psi_{lm}(r, \theta, \phi) = R_{lm}(r) Y_{lm}(\theta, \phi) \)

\( \Rightarrow \) \( E_{nl}, R_{nl}(r) \) are independent of angle for a central potential, i.e., do not depend on \( m \).

**Zeeman Effect**: the splitting of atomic energy levels and the associated spectral lines when the atoms are placed/observed through a magnetic field.

**Stern-Gerlach Experiment**: - beam of neutral atoms passed through a non-uniform magnetic field.
- atoms were deflected according to the orientation of their spin moments.
- showed quantization of \( \hat{L} \) (and anomalous Zeeman effect)

* If there were only angular momentum, the beam would be split into an odd number of different components \( (2l+1) \)

\[ J = L + S \]

Spin has the same eigenvalue and commutator relations as \( L \).

\[ \hat{S} \chi = \lambda \chi \]

, where \[ \hat{S} = \frac{\hbar}{2} \hat{\sigma} \]

\[ \chi_+ = \binom{1}{0} \text{ spin up } (1) \]

\[ \chi_- = \binom{0}{1} \text{ spin down } (0) \]
Spin Ladder Operators

\[ \hat{S}^z = \hbar \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \quad \hat{S}^x = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix} \hbar \]

\[ \hat{S}^2 = \hat{I} \quad \therefore \quad \hat{S} = \hat{S}_x + \hat{S}_y + \hat{S}_z = \frac{\hbar^2}{4} \hat{I} + \frac{\hbar^2}{4} \hat{I} + \frac{\hbar^2}{4} \hat{I} = 3\hbar^2 \hat{I} \]

\[ \therefore \text{Any vector is an eigenvector of the spin magnitude.} \quad \lambda = \frac{3\hbar^2}{4} \]

\[ \Rightarrow \quad \hat{S}(s+1) = \frac{s}{s} \quad \Rightarrow \quad s = \frac{1}{2} \]

\[ \langle \hat{S}_z \rangle = \chi^T \hat{S}_z \chi \]