Notes on Atmospheric Physics

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These notes contain only the basic information discussed in the lectures (the latter are where emphasis is on the physical interpretation and schematics). My aim in writing them is to provide you with a support and a clear knowledge of what is examinable (=what is in the notes). I recommend the excellent textbook by Wallace and Hobbs (“Atmospheric Sciences: an introductory survey”) as a companion for the course (the library has many copies).

Each chapter contains a set of problems, whose solutions will be provided as we go along. Some sections in the notes are highlighted with a ⋆ which indicates that they are a little more challenging.

Please do not hesitate to come to Office Hours (Thursdays, 11.30-12.30; Fridays, 1-2pm) for further help or to give me feedback on the course. You are also welcome to drop any suggestions by email at a.czaja@imperial.ac.uk.
Chapter 1

An overview of the atmosphere

key concepts: well mixed gases, “dry” air, measures of water vapour in air, top-of-the-atmosphere (TOA), global budgets of mass, heat and angular momentum.

1.1 Main features of the atmosphere

Based on the ppt slides discussed in the first lecture (to which you are referred to for illustrations), the main features of the atmosphere are:

- a well mixed structure up to $\approx 100\,km$ in terms of constituents, with a near exponential decay of pressure and number densities with height. The associated scale is on the order of $8\,km$ for the well mixed layer$^1$.

- a rich temperature structure, with, in some regions, temperature decreasing with height and poleward, but in some regions temperature increasing upward and poleward. The simplest view (global average as a function of height) is schematized in Fig. 1.1, introducing the troposphere, the stratosphere and the mesosphere. The course will focus on the first two of these where 99.9 % of the mass of the atmosphere resides.

- the presence of strong zonal (along a latitude circle) time mean jets (windspeeds in excess of $30\,m/s$). These are mostly found going from

$^1$From Boltzmann’s principle we would expect the ratio of distribution of a molecule of mass $m$ at height $z_1$ and $z_2$ to obey $n_1/n_2 = e^{mg(z_1-z_2)/k_BT}$ where $g$ is gravity and $T$ temperature. This provides a different scale height for each molecule according their mass $(k_BT/mg)$, which is not observed below $100km$ (the “turbopause”). It is observed above $100km$. 

CHAPTER 1. AN OVERVIEW OF THE ATMOSPHERE

west to east (e.g., the tropospheric Jet Stream) but also exists seasonally from east to west (mesospheric jets). At the Earth’s surface westerlies are found poleward of 30° of latitude and easterlies (“Trade winds”) are found equatorward of that latitude. The atmosphere is in a state of “superrotation”, an air parcel in the tropospheric Jet Stream coming back to its initial position in about 23h, not 24h!

- the presence of smaller time mean velocities in the North-South direction (a few \( ms^{-1} \)). These are predominantly seen in the Hadley cell at low latitudes, with rising motions near the equator and descending motion along \( \approx 30° \). Such “meridional cells” (in the latitude-height plane) also exist in the stratosphere and mesosphere but the associated mass transport is much weaker than that of the Hadley cell.

- its convective nature (Fig. 1.2) on scales ranging from a few \( km \) to thousands of \( km \) (planetary scale). Updraft motions are associated with phase change and the formation of rain, snow and other hydrometeors. The convection involves mostly upward/downward motions in the Tropics, but sloping (i.e., upward and poleward, downward and equatorward) motions at higher latitudes.

- The fundamental role of water vapour. Not only does it affect atmospheric motions through its effect on buoyancy (condensational heating, evaporative cooling add or remove buoyancy to air parcels, as we’ll see in Chapter 3), but water vapour is also the main greenhouse gas (as we’ll see in Chapter 2). Because the oceans occupy 70% of the Earth’s surface and because surface evaporation depends on surface temperature, water vapour couples the state of the oceans to that of the atmosphere.

- It is only one component among many (oceans, cryosphere, biosphere, the deep Earth, etc) setting the Earth’s climate.

The state of the atmosphere (temperature, moisture, winds, etc), averaged over a long enough time period defines the climate. The weather is the state of the atmosphere at a given time.

1.2 Atmospheric composition

The most abundant substance in the atmosphere is diatomic nitrogen (\( N_2 \)), which accounts for 78% of the air molecules we breath. Most of the nitrogen
Figure 1.1: Global, annual mean atmospheric temperature as a function of height/pressure.
CHAPTER 1. AN OVERVIEW OF THE ATMOSPHERE

Figure 1.2: Global composite infrared map on 9 March 2004. White is cold on this map and, in most regions, indicates the presence of upper level clouds. Notice the “spotty” nature of the convection in the Tropics and the “waveness” in middle and high latitudes. You can find many of those maps (as well as animations) on the MetOffice website.

on Earth is actually stored in the atmosphere \((3.9 \times 10^{18} \text{ kg})\), with the Lithosphere (Earth’s crust) coming second \((\approx 2 \times 10^{18} \text{ kg})\). The large atmospheric reservoir of nitrogen reflects the outgassing from the Earth’s interior in the earliest stage of its history and the great stability of the \(N_2\) molecule.

Next in abundance comes “free” oxygen \((O_2)\), which represents 21% of atmospheric molecules. The Earth is unique in having so much of its atmosphere made up of diatomic oxygen, and there is little doubt that this reflects the presence of life early in its history (it is believed that oxygen started to accumulate in the atmosphere about 2Gyr ago, when the production of \(O_2\) by bacteria exceeded the consumption of \(O_2\) by iron ions dissolved in the oceans).

The percentages given above assume that any given sample of air has the same composition. In practice, this is only true for gases whose residence time in the atmosphere is long compared to the time it takes for atmospheric motions to mix (from a few days to a few months). This is for example the case for argon \((Ar, \approx 0.9\% \text{ of air molecules})\) and carbon dioxide \((CO_2, \approx 0.04\% \text{ of air molecules})\), the next two most abundant species after \(N_2\) and \(O_2\). Water vapour has a highly variable composition (which can be on the order of \(Ar\) concentration locally) depending on time and location because it
is quickly removed from the atmosphere through rainfall. In addition, above approximately 100 km, the mean free path becomes large enough (about 1 m) that the concept of a sample made of different gases does not make any sense.

For the troposphere (the lowest layer of atmosphere where temperature decreases with height, roughly from the Earth’s surface to a height $z = 10$ km) and stratosphere (the layer above the troposphere where temperature increases with height, from about $z = 10$ km to $z = 50$ km), which will be the focus of the course, it is convenient to simplify atmospheric composition by considering “dry air”, a mixture of $N_2$, $O_2$, $Ar$, $CO_2$ and other trace gases, and “moist air” (water vapour). The concentrations are large enough that the ideal gas law model is accurate and so at a given temperature $T$ and volume $V$, the pressure of “dry air” $P_d$ obeys,

$$P_d V = N_d k_B T$$ (1.1)

while for water vapour at pressure $e$,

$$e V = N_v k_B T$$ (1.2)

In these two equations, $k_B$ is Boltzmann’s constant while $N$ denotes the number of molecules (the subscripts $d$ and $v$ will be used throughout the course for dry air and water vapour, respectively). Note that the total pressure $P$ of a given sample of air is simply the sum of $P_d$ (the partial pressure of dry air) and $e$ (the partial pressure of water vapour), as result known as Dalton’s law,

$$P = P_d + e$$ (Dalton’s law) (1.3)

Atmospheric pressures are usually expressed in hPa where 1 hPa = 100 Pa (you might also find pressures expressed as millibar (1 mb = $10^{-3}$ bar), in which 1 bar = $10^5$ Pa).

Because of the very large number of molecules in the atmosphere, it is convenient to rewrite the ideal gas law as,

$$\frac{P_d V}{N_d \mu_d} = \frac{k_B T}{\mu_d}$$ (1.4)

in which $\mu_d$ is the mass of a “dry air molecule” ($\mu_d = \sum N_i \mu_i / \sum N_i$ where $\mu_i$ is the mass of molecule $i$ of which there are $N_i$ in the sample considered—the sum is carried over $i = N_2, O_2, Ar, CO_2, etc$). Introducing the specific volume of dry air $\alpha_d$, and the gas constant for dry air $R_d = k_B / \mu_d = 287 J kg^{-1} K^{-1}$, this becomes,

$$P_d \alpha_d = R_d T$$ (1.5)

Likewise, for water vapour,

$$e \alpha_v = R_v T$$ (1.6)

with $R_v = k_B / \mu_{H_2O} = 461 J kg^{-1} K^{-1}$. 

1.3 Mass in the atmosphere

1.3.1 Pressure as a measure of mass

If the atmosphere were at rest and its properties uniform in the horizontal, the state of mechanical equilibrium would be that in which gravity is solely opposed by the vertical pressure gradient force,

\[ \rho g = -\frac{\partial P}{\partial z} \]  \hspace{1cm} (hydrostatic equation)  \hspace{1cm} (1.7)

Note the minus sign, which expresses that pressure must decrease with height to be able to oppose the downward acceleration due to gravity. This equation is called the hydrostatic equation.

One interesting use of this equation is to integrate it in the vertical as,

\[ P(z) = \int_{z}^{+\infty} \rho g dz \]  \hspace{1cm} (1.8)

in which we have used the fact that pressure vanishes at sufficiently large heights. The discussion in section 1 showed that this is a good approximation for \( z \gg 8\text{km} \) and this loosely defines the “top-of-the-atmosphere” (TOA throughout the course). The corresponding layer of air is still very thin compared to the Earth radius so that one can approximate \( g \) in the integral by its surface value \( g = 9.81\text{ms}^{-2} \),

\[ \frac{P(z)}{g} = \int_{z}^{+\infty} \rho dz \]  \hspace{1cm} (1.9)

This shows that atmospheric pressure can be thought of as a mass measurement since \( \rho dz \) is simply the mass per unit area sandwiched between heights \( z \) and \( z + dz \). A couple of straightforward applications of this equation are worth mentioning. On average the surface pressure is \( P_s = 1013\text{hPa} \) while the tropopause pressure is 250\text{hPa}. This shows that the troposphere contains about \((1013 - 250)/1013 = 75\% \) of the mass of the atmosphere. Conversely, since the pressure \( P \) in (1.9) is the total pressure \( (P = P_d + e) \), and that \( e \) at the Earth’s surface is at most 10\text{hPa} in the global and annual mean, water vapour contributes to less than \( 10/1013 \leq 1\% \) of atmospheric mass.

*Technical sidenote: units of pressure. Pressure is usually expressed in hPa = 100Pa in atmospheric sciences. You might also find the use of millibars (mb, \( 1\text{mb} = 10^{-3}\text{bar} \) where \( 1\text{bar} = 10^{5}\text{Pa} \)).*
1.3. MASS IN THE ATMOSPHERE

1.3.2 Measures of water in air

A given sample of air is described, besides its temperature and pressure, by its mass of dry air \( m_d \) (see previous section), water vapour \( m_v \), liquid water \( m_l \) and ice water \( m_i \). It is common practice to introduce ratios of these quantities:

\[
q_v \equiv \frac{m_v}{m_d + m_v + m_l + m_i} \quad \text{(specific humidity)} \quad (1.10)
\]

\[
q_l \equiv \frac{m_l}{m_d + m_v + m_l + m_i} \quad \text{(specific liquid water content)} \quad (1.11)
\]

\[
q_i \equiv \frac{m_i}{m_d + m_v + m_l + m_i} \quad \text{(specific ice water content)} \quad (1.12)
\]

\[
q_d \equiv \frac{m_d}{m_d + m_v + m_l + m_i} \quad \text{(specific mass of dry air)} \quad (1.13)
\]

In terms of size, because there is so little amount of water in the air, \( q_d \gg q_v \), and typically \( q_v \gg (q_l, q_i) \). Note that \( q_d = 1 - (q_v + q_l + q_i) \). The total specific amount of water is denoted by \( q_t \equiv q_v + q_l + q_i \).

Sometimes, mass mixing ratio, rather than specific humidity is used. The difference is that, for example for water vapour (mixing ratio \( r_v \)), mixing ratio involves taking the ratio of \( m_v \) to \( m_d \) rather than \( m_v \) to \( m_d + m_v + m_l + m_i \), i.e.,

\[
r_v \equiv \frac{m_v}{m_d} \quad \text{(mass mixing ratio)} \quad (1.14)
\]

The air density \( \rho \) is defined according to,

\[
\rho = \frac{m_d + m_v + m_l + m_i}{V} \quad (1.15)
\]

in which \( V \) is the total volume occupied by the sample (the sum of the volume occupied by the gas, liquid and solid phases). As a result, the density of dry air \( \rho_d = m_d/V = (m/V)(m_d/m) = \rho q_d \), and, likewise, the density of water is \( \rho_l = (m_v + m_l + m_i)/V = (m/V)(m_v + m_l + m_i)/m = \rho q_l \).

1.3.3 The atmosphere as an open system

There is continual exchange of gases at the Earth’s surface between either the ocean and atmosphere, or the land and the atmosphere. However, to zero order, the main exchange of material is that associated with water. The latter is added to the atmosphere by surface evaporation \( \mathcal{E} \) and removed from
the atmosphere by precipitation $P$ (both $P$ and $E$ having units of $kgm^{-2}s^{-1}$).

These statements can be expressed mathematically as,

$$\frac{\partial}{\partial t} \iiint \rho q dV = \iint (E - P) dx dy$$  \hspace{1cm} (1.16)

for water ($dx dy$ referring to an infinitesimal area element at the Earth’s surface), and,

$$\frac{\partial}{\partial t} \iiint \rho q dV = 0$$  \hspace{1cm} (1.17)

for dry air. In steady state, we do not learn much from (1.17), but (1.16) shows that there must be an approximate balance between the net evaporation and the net precipitation,

$$\bar{E} \equiv \iint E dx dy \approx \iint P dx dy \equiv \bar{P}$$  \hspace{1cm} (1.18)

The value of either side of this equation gives a measure of the intensity of the hydrological cycle. This is typically on the order of $1m/yr$, i.e., fluxes on the order of $\approx 3 \times 10^{-5}kgm^{-2}s^{-1}$ (NB: when precipitation is given in $m/yr$ or $mm/day$, this refers to the height of water collected over a $1m^2$ unit surface. For example, to convert $1m/yr$ in $kgs^{-1}m^{-2}$, multiply by $1m^2$ and by density of water ($1000kgm^{-3}$), and divide by one year ($\approx 3.10^7s$), yielding $3 \times 10^{-5}kgm^{-2}s^{-1}$ as quoted.)

A few comments are in order. First, an approximate equality sign was used because the balance between evaporation and precipitation is only true in a statistical sense. At a given location, moisture is taken off the ocean or land by a storm but the latter may well precipitate this moisture several hundreds of $km$ (or more) away from the evaporation region. It is only by averaging spatially and temporally over many storms that the approximate balance (1.18) begins to hold. Indeed, maps of long term mean $E - P$ indicate that there are vast regions of net precipitation and net evaporation, i.e., regions where the two never balance: the approximation (1.18) is only true in a global sense.

Second, the distribution of precipitation or evaporation has a “long tail” and is very sensitive to extreme events. Illustrations of this are, for example, the intense and sporadic rainfall in a Summer storm (the number of which set the summertime “mean rainfall”).

A simple calculation highlights how efficient are storms of various kind (extra-tropical cyclones, tropical cyclones, hurricanes) at collecting moisture from vast areas and precipitating it. The total amount of water vapour in
1.4 Angular momentum

Viewed by an observer above the North pole, the Earth rotates anticlockwise at the angular velocity $\Omega$ (magnitude $\Omega = 2\pi/1\text{day}$). If the atmosphere was

$$\Omega = 2\pi/1\text{day}$$

an atmospheric column, or total precipitable water (TPW), is

$$TPW = \int_0^\infty \rho q_v dz$$

Using the simple model $\rho = \rho_s e^{-z/H_s}$ and $q_v = q_s e^{-z/H_q}$ in which $H_s$ is the scale height, $H_q$ a scale height for moisture (Fig. 1.3) and $\rho_s, q_s$ refer to surface density and specific humidity, respectively, one can estimate that,

$$TPW \approx \rho_s q_s H \quad \text{with} \quad H = \frac{H_s H_q}{H_s + H_q}$$

Expressed in $mm$ of water per unit area by dividing this quantity by the density of water, we find typically that the atmosphere holds something like 20$mm$ of precipitable water in vapour form (for $\rho_s = 1.2kgm^{-3}, q_s = 10g/kg, H_s = 7km, H_q = 3km$). Observed values (Fig. 1.4) are within that range. A quick look at the MetOffice website\(^2\) shows that storms typically produce more rain than that, up to an order of magnitude or more in particularly severe storms.

\[^2\text{http://www.metoffice.gov.uk/public/weather/climate-extremes/}\]
Figure 1.4: Mean total precipitable water (in mm) averaged over the oceans for November 2013. This map was produced using passive microwave measurement from satellites.

exactly in solid body rotation around this axis, the velocity of a sample of air at a distance \( r \) to the Earth’s center (see Fig. 1.4) would be \( \mathbf{U}_\Omega \),

\[
\mathbf{U}_\Omega = \Omega \times \mathbf{r}
\]  

(1.21)

In this frame in solid body rotation, an air parcel has a relative velocity \( \mathbf{U} \) whose coordinates are denoted by \( (u, v, w) \) in a local set of orthonormal axes \( i \) (oriented from west to east), \( j \) (oriented from south to North) and \( k \) oriented parallel to \( r \) (see Fig. 1.4),

\[
\mathbf{U} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}
\]  

(1.22)

For example, the observations discussed in the lecture show that the atmosphere is actually in a state of super-rotation at high altitudes and poleward of about 30° (i.e., \( u > 0 \), from west to east). We define an angular momentum per unit mass of air along \( \Omega \) and passing through the Earth’s center from,

\[
L = [\mathbf{r} \times (\mathbf{U} + \mathbf{U}_\Omega)] \cdot \Omega / \Omega
\]  

(1.23)

In the local set of coordinates, one has \( \mathbf{r} = r\mathbf{k} \),

\[
\mathbf{U}_\Omega = \Omega r \cos \phi \mathbf{i}
\]  

(1.24)
1.4. ANGULAR MOMENTUM

Figure 1.5: Geometry. The Earth’s angular velocity vector $\Omega$ is shown in blue, the vector position $r$ in black and the local coordinate axes in gray.

and

$$\Omega = \Omega(\sin \phi \hat{k} + \cos \phi \hat{j})$$

in which $\phi$ is latitude (Fig. 1.4). As a result,

$$L = \left[ r \times (ui + vj + wk + \Omega r \cos \phi i) \right] \cdot \Omega/\Omega$$

$$= \left[ (\Omega r^2 \cos \phi + u)j - rv i \right] \cdot (\sin \phi \hat{k} + \cos \phi \hat{j})$$

$$= r \cos \phi (u + \Omega r \cos \phi)$$

Atmospheric angular momentum has thus a contribution from the Earth’s solid body rotation, or planetary contribution ($\Omega r^2 \cos \phi$), and a contribution from relative motions ($ur \cos \phi$). In practice, the former dominates over the latter. For example at the latitude of the subtropical Jet Stream ($\approx 30^\circ$), one has $u \approx 30 ms^{-1}$ while $\Omega r \cos(30^\circ) \approx 400 ms^{-1}$.

Integrated over the whole mass of the atmosphere $L$ is approximatively constant, i.e.,

$$\frac{\partial}{\partial t} \int \int \rho L dV \approx 0$$

This implies an intriguing compensation between the Tropics, in which the surface winds are westward (Trade winds) and thus where the atmosphere is gaining angular momentum (friction accelerates low levels in the sense of the Earth’s rotation), and higher latitudes, where the surface winds are eastward and thus where the atmosphere is loosing angular momentum. As
we shall see in Chapter 4, the Tropics and extra-tropics are coupled through the propagation of a certain type of waves called Rossby waves. The latter are excited mostly from midlatitudes by the storm we experience daily and as they propagate equatorward they transport angular momentum poleward. Thus, the storms we experience daily across the UK provide the mechanical coupling between vastly distant parts of the Earth.

Finally, note that this compensation between surface easterlies and westerlies can only be approximate since torques are also exerted as a result of the pressure contrast across the major mountains of the Earth like the Himalayas and the Rockies.

1.5 Heat

The atmosphere can be thought of a giant thermodynamic system, experiencing heating and cooling (absorption and emission of electromagnetic radiation; heat exchange with the Earth’s surface), and performing work with its surroundings (the Earth’s surface and Space),

\[ \frac{\partial}{\partial t} \iiint \rho u dV \approx \dot{Q} + \dot{W} \quad (1.30) \]

in which \( u \) (in this section only, not to be mistaken with the zonal velocity!) is the specific internal energy of the atmosphere and \( \dot{Q} \) and \( \dot{W} \) refer to the rate of heating and work received by the atmosphere, respectively. The previous equation would be a correct application of the 1st law of thermodynamics if the atmosphere was a closed system. However, as we’ve seen in section 1.3, there is exchange of water with the Earth’s surface. To include this, we rewrite (1.30) as,

\[ \frac{\partial}{\partial t} \iiint \rho u dV \approx \dot{Q} + \dot{W} + \bar{E} u_{v,s} - \bar{P} u_{l,r} \quad (1.31) \]

in which \( u_{v,s} \) is the specific internal energy of the water vapour added to the atmosphere by surface evaporation, and \( u_{l,r} \) is that of the liquid water lost as rain. Since from (1.18), \( \bar{E} \approx \bar{P} \), and approximating \( u_{v,s} - u_{l,r} \approx l_v(T_s) \equiv l_{v,s} \) (\( l_v \) being the latent heat of vaporization), we have, in steady state,

\[ 0 \approx \dot{Q} + \dot{W} + l_{v,s} \bar{E} \quad (1.32) \]

The typical rate of surface evaporation is \( \bar{E} \approx 3.5 \times 10^{-5} \text{kg s}^{-1} \text{m}^{-2} \) (for this and other numbers used in this section, see Fig. 1.6 and Q9 below) while
\[ l_{e,s} \approx 2.5 \times 10^6 \text{Jkg}^{-1}. \] The last term is thus on the order of \( 3.5 \times 10^{-5} \times 2.5 \times 10^6 \times 4\pi R^2 \approx 45 \text{PW} \) (1PW \( \equiv 10^{15} \text{W} \)),

\[ l_{e,s} \bar{E} \approx +45 \text{PW} \quad (1.33) \]

To check whether this could be an important energy source for the atmosphere, let’s think about the heating term \( \dot{Q} \). The Earth’s atmosphere receives a total amount of radiation from the Sun equal to \( S_0(1 - \alpha_P)\pi R^2 \) in which \( S_0 = 1361 \text{Wm}^{-2} \) is the Solar constant (see sidenote below) and \( \alpha_P \approx 0.3 \) is the planetary albedo (the fraction of solar radiation reflected back to Space). About 70% of this radiation leaves the atmosphere at the Earth’s surface so that the net atmospheric heating due to solar radiation is on the order of \((1 - 0.7)S_0(1 - \alpha_P)\pi R^2 \approx 38 \text{PW} \) (as we shall see in Chapter 2, this is mostly through UV absorption by ozone molecules in the stratosphere and shortwave radiation by water vapour in the troposphere).

**Technical sidenote:** why \( \pi S_0 R^2 (1 - \alpha_P) \)? It makes sense intuitively that the Sun intercepts the Earth’s disk so that the net flux \( S_0(1 - \alpha_P) \) has to be multiplied by the disk area \( (\pi R^2) \). To see this from 1st principle, it is just a matter of integrating on the sphere \( \int \int S_0(1 - \alpha_P)G(\lambda, \phi)r^2 \cos \phi d\phi d\lambda \) with \( r \approx R \) and where \( G(\lambda, \phi) \) is a geometrical factor including the projection of the incident solar radiation (assumed to be uniform and striking the Earth’s equator at right angle) on the local normal at latitude \( \phi \) and longitude \( \lambda \). This factor is \( G = \cos \phi \cos \lambda \) so that, after integrating from \(-\pi/2\) to \(\pi/2\) for latitude (south to north pole) and likewise for longitude (integrating over the illuminated hemisphere only), one gets \( S_0 R^2 (1 - \alpha_P)[\sin(\pi/2) - \sin(-\pi/2)][\pi/4 + \pi/4] = \pi R^2 S_0(1 - \alpha_P) \).

Absorption of solar radiation and the hydrological cycle thus contribute with a similar order of magnitude to add energy to the atmosphere. One can rule out the work done by the atmosphere \( (W) \) as the significant energy sink required to satisfy (1.32). Indeed, it would require an unrealistically high thermodynamic efficiency for the work to be of the same order of magnitude as the ultimate energy source (the Sun). Likewise, it cannot be heat exchange with the ocean since, as a rule of thumb, the sea is warmer than the lower atmosphere, and the exchange would thus also contribute to increase the heat content of the atmosphere.

The only viable cooling mechanism allowing (1.32) to be satisfied is emission of infrared radiation to Space. Indeed, if the lower atmosphere behaved like a black body (at temperature \( T_b \approx 285K \)) and all the infrared radiation it emitted could reach Space, the cooling rate would be on the order of \( 4\pi R^2 \sigma T_b^4 \approx 190 \text{PW} \)!! A more realistic view of emission of infrared radiation
Figure 1.6: Recent observations of the Earth’s energy budget. All quoted numbers represent $Wm^{-2}$. Solar fluxes are in beige and infrared fluxes in pink. From Stephens et al. (2012).
by the atmosphere will be presented in the next chapter (the observed num-
ber for the cooling rate is about half this value, i.e. $-95PW$, as discussed in
Q9 below), but this simple calculation clearly suggests that infrared emission
is the key cooling mechanism for the atmosphere.

In summary, radiative processes add up to a net cooling effect

$$\dot{Q} \approx +38 - 95 = -57PW \quad (1.34)$$

which opposes the gain of energy ($+45PW$) associated with adding high
energy water vapour by surface evaporation and removing low energy liquid
water through rainfall (the remaining $12PW$ can be accounted for by heating
from the Earth’s surface, e.g., heat exchange with the oceans). The 1st law
of thermodynamics primarily reflects this cancellation, with a negligible role
played by the work exchanged between the atmosphere and its surroundings
(oceans, land, Space).

1.6 Further reading

- Stephens et al., 2012: An update on Earth’s energy balance in light of the
  latest global observations, Nature Geosciences, 5, 691-696.
  This book (and the many sequels) offers a fascinating discussion of atmo-
spheric composition and demonstrates the broadness of the subject.

1.7 Problems

Q1 In the (zonal mean) stratosphere and mesosphere, where (latitude, alti-
tude, season) are found: (i) the coldest temperature (ii) the warmest tem-
perature (iii) the strongest westerly wind? You might find useful to refer to
the slides for Lecture 1.

Q2 Why are deserts more likely to be present in the sub-tropics than in
either the tropics or mid-latitudes?

Q3 At 25 km altitude, where atmospheric pressure is $P_o \approx 25hPa$ and tem-
perature is $T_o \approx 220K$, the mass mixing ratio of ozone is 10 parts per million.
What is its: (i) density (ii) volume mixing ratio (iii) number density and (iv)
partial pressure?

Q4 Express the fraction of water vapour molecules in $ppm$ (part per million)
in a sample of air where the partial pressure of water vapour is $e = 10hPa$
Table 1.1: Data for Q5. The number in parentheses refer to the molecular mass of the main atmospheric constituents.

<table>
<thead>
<tr>
<th>Planet</th>
<th>Major atmospheric component</th>
<th>Mean lower temperature (K)</th>
<th>Mass (kg)</th>
<th>Radius (km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Venus</td>
<td>CO2 (44)</td>
<td>750</td>
<td>4.87 × 10^{24}</td>
<td>6,051</td>
</tr>
<tr>
<td>Earth</td>
<td>N2, O2 (29)</td>
<td>280</td>
<td>5.97 × 10^{24}</td>
<td>6,371</td>
</tr>
<tr>
<td>Mars</td>
<td>CO2 (44)</td>
<td>250</td>
<td>6.42 × 10^{23}</td>
<td>3,397</td>
</tr>
<tr>
<td>Jupiter</td>
<td>H2 (2)</td>
<td>123</td>
<td>1.90 × 10^{27}</td>
<td>71,490</td>
</tr>
</tbody>
</table>

and that of dry air \( P_d = 1000hPa \). Compare the number obtained with the current fraction of carbon dioxide molecules (400ppm).

Q5 (i) Show that in an isothermal atmosphere \((T = T_o)\) the pressure decays exponentially with height with a scale \( H_s = k_B T_o/mg \) (the scale height), in which \( g \) is the gravity of the planet and \( m \) is the mass of an atmospheric molecule. (ii) Estimate \( H_s \) in the lower atmosphere for each of the planet listed in Table 1.1.

Q6 Show that the specific volume \( \alpha \) of a sample of air can be approximated as,

\[
\alpha \approx \alpha_d (1 - q_t)
\]  

in which \( \alpha_d = R_d T/P_d \) is the specific volume of dry air (\( R_d \) being the gas constant for dry air, \( P_d \) the partial pressure of dry air and \( T \) temperature). You might want to start by writing that \( \alpha = (V+V_l+V_i)/(m_d+m_v+m_l+m_i) \) in which \( V \) is the volume of the sample occupied by the gas phase, \( V_l \) and \( V_i \) that of the liquid and ice phases, respectively.

Q7 By inspection of a surface pressure map (for example from the on-line ERA40Atlas\(^3\)), work out whether the Rocky mountains tend to increase or decrease atmospheric angular momentum.

Q8 A simple view of the Tropics is that it is primarily independent of longitude, i.e., rings of air flow upward at the equator in the ascending branch of the Hadley cell, and flow poleward at upper levels, conserving their angular momentum. Estimate the implied zonal velocity \( u \) at 30° of latitude. How does it compare with the observed velocity? Hint: you may assume that the

\(^3\)http://www.ecmwf.int/research/era
relative velocity $u$ of the ring is very small near the ground.

Q9 Figure 1.6 gives a summary of recent observations of the Earth energy budget. Use the numbers from this figure to justify the choices made in section 1.5 regarding the rates of: (i) heating as a result of the hydrological cycle (ii) heating due to absorption of solar radiation in the atmosphere (iii) infrared cooling. The Earth’s radius is $R = 6371 \, km$. 
Chapter 2

Radiative heating and cooling of the atmosphere

Key concepts: solid angle, irradiance, radiation balance, greenhouse effect, Beer’s law, Schwarzschild’s equation, infrared cooling to Space.

2.1 Concepts and definitions

2.1.1 Intensity of radiation

The intensity of electromagnetic radiation at wavelength $\lambda$ measures the number of photon of energy $E = hc/\lambda$ crossing a unit area perpendicular to the direction of propagation per unit time per unit solid angle (Fig. 2.1). The “per unit area” and “per unit time” are familiar, but the “per unit solid angle” is less so. This is added to reflect that the radiation reaching $P$ in Fig. 2.1 in a given direction is made of an infinitesimal cone of rays, or “radiation pencil”, filling a certain amount of space. In three dimensions, the amount of space is measured by solid angle $\Omega$ which is a measure of area on a unit radius sphere, exactly like an angle is a measure of length on a unit radius circle (Fig. 2.2). Solid angle is expressed in steradians ($sr$) and the maximum solid angle attainable (the amount of space filled by the sky if we were floating in the air) is $4\pi \approx 12.5sr$.

Note that there is nothing special about point $P$ in Fig. 2.1 so that, in absence of scattering or absorption of photons, the intensity is the same at any point along a ray. In the following we will denote intensity of radiation at wavelength $\lambda$ by $I_\lambda$ (in $W m^{-2}$ per wavelength per solid angle).
Figure 2.1: The intensity of radiation at wavelength $\lambda$ is denoted by $I_\lambda$ and measures the energy flow along the direction of propagation per unit area and time, and unit solid angle $d\Omega$.

Figure 2.2: From vector Calculus (year 1), the area element on a sphere of unit radius $r = 1$ is $d\Omega = \sin \theta d\theta d\phi$. This is the infinitesimal solid angle which we will use throughout in this chapter.
2.1. CONCEPTS AND DEFINITIONS

2.1.2 Blackbody radiation
As discussed in Year 2 (Thermodynamics and Statistical Physics), blackbody radiation is the radiation emitted by a body in thermal equilibrium. It is isotropic and only depends on the temperature $T$ characterizing the equilibrium, not on the nature of the material making the body. Its intensity, at a given wavelength $\lambda$ is given by the Plank function,

$$ B_\lambda(T) = \frac{2hc^2}{\lambda^5} \left( \frac{e^{hc/\lambda kT} - 1}{e^{hc/\lambda kT} - 1} \right) $$

which has units of W m$^{-2}$ per wavelength per solid angle.

2.1.3 Shortwave and longwave radiation
Because of the small overlap of the Planck functions associated with terrestrial (infrared) and solar (visible) emissions, it is common practice to separate “longwave” and “shortwave” radiations. Typically, 4$\mu$m is used as the separation between the two (Fig. 2.3). We will use this distinction constantly in the following. As we shall see the difference in wavelength will lead to differences in the importance of scattering and emission of shortwave and longwave radiation by atmospheric molecules and aerosols.

The electromagnetic waves propagate at the speed of light $c$, with frequency $\nu$ (in s$^{-1}$) and wavelength $\lambda$ (in m) satisfying $c = \nu \lambda$. It is sometimes convenient to consider the wavenumber $\tilde{\nu}$ instead of $\lambda$,

$$ \tilde{\nu} = \nu / c = 1 / \lambda \quad \text{(wavenumber)} $$

2.1.4 Irradiance
The energy of radiation passing through an horizontal plane, per unit area of that plane, per unit wavelength is called the monochromatic irradiance $F_\lambda$. It requires integrating $I_\lambda$ over solid angle ($2\pi$ at most for either upward or downward hemispheres) and taking into account the angle between the beam and the normal to the horizontal plane.

Consider for example the geometry in Fig. 2.4 and take the horizontal plane to be the $x, y$ plane. The net downward radiation across the horizontal plane is made of several “radiation pencils”, each coming from different angles $\theta$ with the $z$ direction and the polar angle $\phi$ in the horizontal plane. It is thus a matter of summing over all these pencils, each of infinitesimal solid angle $d\Omega$, and projecting onto the vertical $I_\lambda \rightarrow I_\lambda \cos \theta$. Thus,

$$ F_\lambda = \int I_\lambda \cos \theta d\Omega $$
Figure 2.3: The electromagnetic spectrum. Picture taken from Wallace and Hobbs’ textbook. Note that the magnitude of the curves have been rescaled to allow a clearer comparison (in absolute terms the solar curve has much higher values).
Figure 2.4: Geometry for the calculation of irradiance, given a pencil of monochromatic radiation (green cone) of intensity $I_\lambda$ propagating downward (green dashed arrow) across the horizontal plane at an angle $\theta$. 
or using $d\Omega = \sin \theta d\theta d\phi$,

$$F_\lambda = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} I_\lambda \cos \theta \sin \theta d\theta d\phi$$

(2.4)

If the radiation is isotropic (i.e., independent of $\theta$ and $\phi$), then the integral simplifies to

$$F_\lambda = 2\pi I_\lambda \int_{\theta=0}^{\pi/2} \cos \theta \sin \theta d\theta = \pi I_\lambda$$

(2.5)

This result is exact for Blackbody radiation, for which $I_\lambda = B_\lambda$ in (2.1), and so $F_\lambda = \pi B_\lambda$. NB: if one were to integrate the latter over all wavelengths, one would obtain $\int \pi B_\lambda d\lambda = \sigma T^4$ in which $\sigma$ is Stefan-Boltzmann’s constant.

### 2.1.5 Kirchoff’s law

In practice, most components of the climate system do not behave like black bodies, or only do so over a limited range of wavenumbers (for example the oceans in the infrared). This means that they do not absorb all the radiation impinging on them, and that they do not emit according to $B_\lambda(T)$.

To account for this, define the emissivity of a particular body at temperature $T$ (e.g., sample of air) according to,

$$\epsilon_\lambda \equiv \frac{I_\lambda(\text{emitted})}{B_\lambda(T)}$$

(2.6)

and define its absorptivity as,

$$\alpha_\lambda \equiv \frac{I_\lambda(\text{absorbed})}{I_\lambda(\text{incident})}$$

(2.7)

Kirchoff’s law states that, remarkably, irrespective of what the body is made, and irrespective of the nature of the radiation (isotropic or not),

$$\epsilon_\lambda = \alpha_\lambda \quad \text{(Kirchoff’s law)}$$

(2.8)

NB: For a blackbody, $\epsilon_\lambda = \alpha_\lambda = 1$ for all wavelengths. Note also that Kirchoff’s law only applies to systems in thermodynamic equilibrium with their immediate surroundings (“local” thermodynamic equilibrium). This condition is satisfied in the troposphere and the stratosphere but less so above these regions.
2.1. CONCEPTS AND DEFINITIONS

Figure 2.5: Radiation balance of the Earth. The incoming solar radiation is treated as a parallel beam impinging on a spherical Earth. The outgoing infrared radiation is isotropic.
2.1.6 Radiation balance and emission temperature

We define the “Solar constant” $S_o$, as the power per unit area crossing a surface normal to the solar beam at a distance of 1AU (Fig. 2.5, see also Chapter 1). This is about 1361Wm$^{-2}$. The total power reaching the Earth (radius $R$) is,

$$P_{\text{sun}} = \int \int_{\text{Earth's area}} S_o \cdot dS = \pi R^2 S_o$$

in which $S_o$ has the horizontal direction in Fig. 2.5 and $dS$ is normal to the Earth (see calculation in Chapter 1, section 1.5). A fraction $\alpha_P$ (the planetary albedo) of the Solar radiation is reflected. At equilibrium, the same amount of power must be lost by the Earth. The emission temperature is defined as the temperature required to achieve this, were the Earth a perfect blackbody in the infrared:

$$\pi R^2 S_o (1 - \alpha_P) = 4\pi R^2 \sigma T_e^4$$

leading to

$$T_e \equiv \left( \frac{S_o(1 - \alpha_P)}{4\sigma} \right)^{1/4}$$

NB: note the factor of 4 coming from the geometry of the problem (plane radiation impinging a sphere as opposed to radial emission).

The idea of radiative balance at the TOA is an idealization. Global conservation of energy requires that any imbalance be reflected in a change in heat content. This, in practice, is dominated by oceanic heat storage (largest heat capacity in the climate system) which fluctuates on very long timescales (decades and longer) because of the slow ocean dynamics. Thus the TOA net radiative fluxes are not expected to vanish on timescales shorter than at least a few decades.

For Earth annual average, $\alpha_P = 0.3$ so that $T_e = 255K$ or $-18^\circ C$ (very cold!). The Earth’s surface temperature is about 288K, or about +15$^\circ C$. Thus, by contrast with the present model which omits entirely the atmosphere, we can say that the atmosphere is responsible for a $\approx 288 - 255 = 33K$ increase in surface temperature. The way this works is disentangled in the simple model coming next. Before doing so, it is worth mentioning a couple of other interesting aspects of this model:

- it suggests that the bulk of the infrared radiation seen from Space originates from the atmosphere itself rather than from the surface because 255K is found typically at an altitude of 5km above the Earth’s surface.

- the Planck function for a blackbody at 255K is centered near 15$\mu$m (Fig. 2.3). This happens to be a wavelength that the CO$_2$ molecule absorbs strongly, hence the strong “leverage” of CO$_2$ on climate.
2.2 A simple model of the greenhouse effect

We consider a 0D model of radiative balance (averaged over the whole Earth’s surface area and expressed in \( Wm^{-2} \)) and go a little beyond the previous section by explicitly introducing the surface temperature \( T_s \) (Fig. 2.6).

In the shortwave, the solar flux impinging at the TOA is still \( S_o(1-\alpha_P)/4 \) which we denote by \( F_o \). We further assume that only a fraction \( T_{sw} \) of this radiation reaches the surface, to account for absorption by atmospheric molecules and aerosols. (we’ll call \( T_{sw} \) the transmissivity of the atmosphere in the shortwave in the following). From Kirchoff’s law, if some radiation is absorbed it must also be emitted. However, at terrestrial temperature the emission in the range of wavelengths where the bulk of solar radiation resides is negligible, and \( \epsilon_\lambda B_\lambda(T) \ll \alpha_\lambda I_{incoming} \). As a result, even though \( \alpha_\lambda = \epsilon_\lambda \) for wavelengths \( \lambda \) in the shortwave part of the spectrum, we can safely neglect the emission of shortwave radiation by the atmosphere.

In the longwave, we take the atmosphere to be at constant temperature \( T_a \) and denote by \( F_a \) the longwave flux it emits upward and downward. The surface is treated as a blackbody, thus emitting \( F_s = \sigma T_s^4 \) upward. A fraction \( T_{lw} \) of this radiation reaches the top-of-the-atmosphere (we’ll call this fraction the transmissivity of the atmosphere in the longwave). Fig. 2.6 summarizes the energy flows.

We can predict what the surface temperature should be solely from energy conservation arguments. At equilibrium, the net flow of energy across any surface must be zero. Applying this at the Earth’s surface yields (see Fig. 2.6):

\[
F_o T_{sw} + F_a = F_s \tag{2.12}
\]

while, at the “top-of-the-atmosphere”, it produces:

\[
F_o = F_a + F_s T_{lw} \tag{2.13}
\]

By assumption \( F_s = \sigma T_s^4 \) while, using (2.11), \( F_o = \sigma T_e^4 \). As a result, after elimination of \( F_a \) from the above two energy conservation equations, we obtain,

\[
T_s = T_e \left( \frac{1 + T_{sw}}{1 + T_{lw}} \right)^{1/4} \tag{2.14}
\]

This equation is remarkably simple and shows that the surface temperature and the emission temperature differ only by a factor proportional to the transmissivities in the shortwave and the infrared. If there were no atmosphere, \( T_{sw} = T_{lw} = 1 \) and \( T_s = T_e = 255K \). With an atmosphere, and if \( T_{sw} > T_{lw} \), the surface temperature will then exceed \( T_e \). In the Earth atmosphere, \( T_{lw} \approx 0.2 \) while \( T_{sw} \approx 0.9 \), leading to \( T_s \approx 286K \).
28 CHAPTER 2. RADIATIVE HEATING AND COOLING OF THE ATMOSPHERE

Figure 2.6: A simple model of the greenhouse effect. TOA denotes the “top-of-the-atmosphere” where pressure vanishes. The shortwave fluxes are indicated in black, the longwave ones in blue.
2.3. BEER’S LAW

The agreement of this prediction with the observed \( T_s = 288K \) is fortuitous because of the extreme simplicity of the model (isothermal atmosphere, surface treated as a blackbody). The key point though is that because the atmosphere is more transparent to shortwave than it is to longwave \( (T_{sw} > T_{lw}) \), there is a “recycling” of energy towards the surface:

\[
F_o T_{sw} + F_o = F_o \frac{1 + T_{sw}}{1 + T_{lw}} > F_o \quad \text{(surface heating)} \quad (2.15)
\]

The added heating leads to a larger surface temperature –this effect is called the greenhouse effect.

2.3 Beer’s law

NB: Absorption and scattering of radiation by atmospheric molecules and aerosols is discussed in the ppt slide for this chapter.

Consider a monochromatic beam of wavelength \( \lambda \) and of intensity \( I_\lambda \). We want to derive the change in intensity \( (dI_\lambda) \) of this beam along its direction of propagation (measured by the coordinate \( s \)). This change is caused either because some air molecules scatter the radiation (net loss of radiation intensity along the path but no change in radiation intensity integrated over all directions), or absorb it (net loss of radiation intensity along the path but no change in radiation intensity in other directions). If we denote by \( q_a \) the mass of these molecules per unit mass of air, \( \rho q_a \) is the mass of these molecules per unit volume and \( \rho q_a ds \) is the mass of those molecules per unit area perpendicular to the direction of propagation.

Beer’s law states that,

\[
dI_\lambda = -I_\lambda k_\lambda \rho q_a ds \quad (2.16)
\]

in which \( k_\lambda \) (in \( m^2 kg^{-1} \)), called the extinction coefficient, measures the intensity of absorption or scattering of radiation of wavelength \( \lambda \),

\[
k_\lambda = \beta_\lambda (\text{absorption}) + \sigma_\lambda (\text{scattering}) \quad (2.17)
\]

Equation (2.16) can be integrated along the path of the beam as,

\[
I_\lambda(s) = I_\lambda(s_o) e^{-\int_{s_o}^s k_\lambda \rho(s') q_a(s') ds'} \quad (2.18)
\]

In this expression \( s_o \) is a starting point where the radiation intensity is \( I_\lambda(s_o) \). It is convenient to introduce the following quantities,

\[
T_\lambda(s_o, s) = e^{-\int_{s_o}^s k_\lambda \rho(s') q_a(s') ds'} \quad \text{(transmittance)} \quad (2.19)
\]
Figure 2.7: Schematic of a radiation beam being either scattered or absorbed by an infinitesimal layer of atmosphere of thickness $dz$. The beam is making an angle $\theta$ with the vertical (if the beam is solar, then $\theta$ is called the Solar zenith angle). Picture taken from Wallace and Hobbs’ textbook.

\[
\tau_\lambda(s_o, s) = \int_{s_o}^{s} k_\lambda \rho(s') q_a(s') ds' \quad \text{(optical depth)} \tag{2.20}
\]

Note that both $\tau_\lambda > 0$ and $0 \leq T_\lambda \leq 1$ are non dimensional functions (i.e., they are just numbers, without SI units). One measures the opacity of the atmospheric layer over a given path ($\tau_\lambda$, the larger the more opaque the layer; hardly any radiation escapes a layer with optical depth much greater than unity), while the other measures its transparency ($T_\lambda$, the closer to unity the more transparent the layer –see for example the discussion of the greenhouse effect earlier in this chapter).

Sometimes one wishes to relate the direction of the beam to the local vertical (Fig. 2.7). If $\theta$ denotes the angle the beam makes with the vertical then $ds = dz / \cos \theta = dz \sec \theta$ where $z$ denotes height. For example for a solar beam coming from the top-of-the-atmosphere where the radiation intensity is $I_{\lambda, \infty}$, we have,

\[
I_\lambda(z) = I_{\lambda, \infty} e^{-\sec \theta \int_{z}^{\infty} k_\lambda \rho(z') q_a(z') dz'} \tag{2.21}
\]
2.4 Schwarzschild’s equation

2.4.1 Derivation

Beer’s law only considers removal of radiation from a beam. Radiation can however also be added to the beam due to the emission from the layer or to radiation incident from another direction being scattered into the beam. The additional elements also depend on the amount of radiatively active constituent \( \rho_q \, ds \) so Beer’s law can be modified to include them by introducing a source function \( J_\lambda \),

\[
dI_\lambda = -k_\lambda \rho_q \, ds \left[ I_\lambda(s) - J_\lambda(s) \right] \tag{2.22}
\]

This can be simplified by introducing again the optical depth –see eq. (2.20) in which \( s_o \) is simply taken as a reference location,

\[
d\tau_\lambda = k_\lambda \rho_q \, ds \tag{2.23}
\]

Doing so allows a change of variables \( (s \to \tau) \) and (2.22) can be rewritten as,

\[
d(I_\lambda e^{\tau_\lambda})/d\tau_\lambda = J_\lambda e^{\tau_\lambda} \tag{2.24}
\]

Integrating from \( s_o \) to \( s \), and noting that \( \tau_\lambda(s_o, s_o) = 0 \), we obtain,

\[
I_\lambda(s) = I_\lambda(s_o)e^{-\tau} + \int_0^\tau J_\lambda(\tau')e^{-(\tau-\tau')}d\tau' \tag{2.25}
\]

with \( \tau' = \tau_\lambda(s_o, s') \) and \( \tau = \tau_\lambda(s_o, s) \) (to simplify notations).

Let us step back a little from these calculations. The first term on the r.h.s. of the previous equation is readily interpreted as the radiation emitted at \( s_o \), a fraction \( e^{-\tau} \) making to \( s \), in agreement with Beer’s law. The second term, the integral, reflects the contribution to the radiation at \( s \) emitted from all layers between \( s_o \) and \( s \). To recover this term more directly, consider the following. Measuring the location of any of these layers by \( s' \) with \( s_o \leq s' \leq s \), we know from (2.22) that they emit an amount of radiation \( J_\lambda(s')k_\lambda \rho_q \, ds = J_\lambda(s')d\tau' \). From the definition of transmittance, only \( T_\lambda(s', s)J_\lambda(s')d\tau' \) reaches \( s \). This is nothing else than \( J_\lambda(s')e^{-(\tau-\tau')}d\tau' \) since \( T_\lambda(s', s) = T_\lambda(s_o, s)/T_\lambda(s_o, s') \). Thus we can as well rewrite (2.25) as,

\[
I_\lambda(s) = I_\lambda(s_o)e^{-\tau_\lambda(s_o, s)} + \int_0^{\tau_\lambda(s_o, s)} T_\lambda(s', s)J_\lambda(\tau')d\tau' \tag{2.26}
\]

The integral in (2.25) can be rewritten more concisely by acknowledging that,

\[
dT_\lambda(s', s) = e^{-(\tau-\tau')}d\tau' \tag{2.27}
\]
We finally obtain,

\[ I_\lambda(s) = I_\lambda(s_o)T_\lambda(s_o, s) + \int_{T_\lambda(s_o, s)}^1 J_\lambda(s')dT_\lambda(s', s) \quad (2.28) \]

This form can sometimes have some advantage – although I prefer the more physically transparent eq. (2.26). NB: You might be confused by the integral going from \( T_\lambda(s_o, s) \) to 1. This is because the change of variable is from \( \tau_\lambda(s_o, s') \) to \( T_\lambda(s', s) \). The latter quantity goes to 1 when \( s' = s \) while the former reaches its maximum \( \tau_\lambda(s_o, s') \) when \( s' = s_o \). Conversely, when \( s' = s_o \), \( T_\lambda(s', s) \) goes to \( T_\lambda(s_o, s) \) while \( \tau_\lambda(s_o, s') = 0 \). Figure 2.8 illustrates this.

### 2.4.2 Simpler form when scattering can be neglected

If scattering can be neglected, i.e., \( \sigma_\lambda = 0 \) in (2.17) (a good approximation for infrared radiation), the Schwarzschild’s equation takes a particularly simple form. Indeed, in that case, the absorptivity of the layer of thickness \( ds \) is,

\[ \alpha_\lambda = I_\lambda\beta_\lambda\rho q_a ds/I_\lambda = I_\lambda k_\lambda\rho q_a ds/I_\lambda \quad (2.29) \]

while its emissivity is,

\[ \epsilon_\lambda = J_\lambda k_\lambda\rho q_a ds/B_\lambda \quad (2.30) \]
2.5. RADIATIVE HEATING AND COOLING RATES

Using Kirchoff’s law (2.8), Schwarzschild’s equation can then be simply rewritten as,

\[ dI_\lambda = -d\tau_\lambda (I_\lambda - B_\lambda) \] (2.31)

which shows that one simply have to make the substitution \( J_\lambda \rightarrow B_\lambda \) when considering infrared radiation (and no scattering).

2.4.3 A simple application

Consider the case of an isothermal atmosphere and neglect scattering. The downward infrared radiation measured at a distance \( s \) from the top-of-the-atmosphere (\( s = s_o \)) can be computed from (2.26) as follows.

First, acknowledge that for downward infrared radiation, \( I_\lambda(s_o) = 0 \), so that the first term on the r.h.s of (2.26) vanishes. For the integral term, one can take \( J_\lambda = B_\lambda \) outside since the atmosphere is isothermal (say \( B_\lambda = B_o \)). We thus have,

\[ I_\lambda(s) = B_o e^{-\tau_\lambda(s_o,s)}[e^{\tau_\lambda(s_o,s)} - 1] = B_o(1 - e^{-\tau_\lambda(s_o,s)}). \] (2.32)

This expression shows that the amount of infrared radiation emitted increases with downward distance, which we can interpret as reflecting the accumulation of radiation emitted by each layer from the top-of-the-atmosphere to the location considered. It recovers \( I_\lambda(s) = 0 \) at \( s = s_o \). Interestingly, the result above shows that if the atmosphere is opaque, i.e., \( \tau_\lambda(s_o, s) >> 1 \), then \( I_\lambda \approx B_o \). This reflects the fact that there can then be no accumulation of radiation as one goes downward, each layer absorbing all the radiation coming from above and emitting at \( B_o \); the line (or wavelength \( \lambda \)) is “saturated”. This makes sense: as our isothermal atmosphere becomes more opaque, it becomes closer to a blackbody and so radiates as \( B_\lambda(T_o) \).

2.5 Radiative heating and cooling rates

The difference between the radiation incoming and outgoing from the sides of a given volume of air is, by conservation of energy, a heating rate. Although conceptually simple, the calculation of this heating rate is made difficult in practice because of the need to integrate the radiation coming/outgoing from all sides of the sample. In the case of the global atmosphere, the sides in question are spheres of constant radius, or, in the Cartesian geometry adopted here for simplicity, planes of constant height. In other words we will concentrate in this section on the heating of a layer sandwiched between height \( z \) and \( z + dz \).
Figure 2.9: Schematic of the various terms in the heat budget of an infinitesimal layer sandwiched between \( z \) and \( z + dz \). The heat gained per unit time, area and wavelength is \( F^\uplambda_{\lambda}(z) + F^\uplambda_{\lambda}(z + dz) \). Likewise, the heat lost per unit time, area and wavelength is: \( F^\downlambda_{\lambda}(z + dz) + F^\downlambda_{\lambda}(z) \). Thus the net heat gained per unit area, time and wavelength is: \( F^\uplambda_{\lambda}(z + dz) + F^\downlambda_{\lambda}(z) \). Therefore, the net heat gained per unit area, time and wavelength is \( F^\uplambda_{\lambda}(z) - F^\downlambda_{\lambda}(z + dz) + F^\downlambda_{\lambda}(z + dz) - F^\uplambda_{\lambda}(z) = (dF^\downlambda_{\lambda}/dz - dF^\uplambda_{\lambda}/dz)dz \). Dividing this expression by \( dz \) gives the heating per unit volume, wavelength and unit time.

From section 2.1.4, the total flux of radiation across an horizontal plane due to a beam of intensity \( I\lambda \) was defined as the irradiance \( F\lambda \). To distinguish between radiation coming from above and below, we will separately consider \( F^\uplambda_{\lambda} \) and \( F^\downlambda_{\lambda} \). The heating rate \( Q\lambda \) of an infinitesimal layer of air sandwiched between height \( z \) and \( z + dz \) is thus (Fig. 2.9),

\[
Q\lambda = \frac{d}{dz}(F^\downlambda_{\lambda} - F^\uplambda_{\lambda})
\]  \hspace{1cm} (2.33)

Because \( F\lambda \) is in units of \( Wm^{-2} \) per wavelength, \( Q\lambda \) is in units of \( Wm^{-3} \) per wavelength. The total heating rate due to radiation is thus \( Q_{rad} \),

\[
Q_{rad} \equiv \int_0^{+\infty} Q\lambda d\lambda
\]  \hspace{1cm} (2.34)

which has units of \( Wm^{-3} \). As we shall see, the heating due to shortwave absorption is more than offset by radiative cooling due to longwave emission so that in the net, \( Q_{rad} < 0 \) (cooling).
2.5. **RADIATIVE HEATING AND COOLING RATES**

### 2.5.1 Shortwave heating

Consider a downward beam of solar radiation of intensity $I_\lambda$. The direction of this beam with respect to the local vertical (angle $\theta$) varies with time of day and season. Applying (2.21) we obtain,

$$I_\lambda(z) = I_{\lambda,TOA}e^{-\tau_\lambda(z)/\cos \theta} \quad (2.35)$$

where

$$\tau_\lambda(z) = \int_z^{+\infty} k_\lambda \rho(z')q_a(z')dz' \quad (2.36)$$

and $I_{\lambda,TOA}$ refers to the incoming shortwave radiation at the top-of-the-atmosphere. Integrating (2.35) over a downward hemisphere, we get,

$$F_\lambda^\downarrow(z) = \int_{2\pi} I_{\lambda,TOA}e^{-\tau_\lambda(z)/\cos \theta} \cos \theta d\Omega \quad (2.37)$$

For simplicity consider a solar beam centered near $\theta = 0$, covering a small solid angle $\delta \Omega$. As a result we approximate the integral as,

$$F_\lambda^\downarrow(z) \approx F_{\lambda,TOA}e^{-\tau_\lambda(z)} \quad (2.38)$$

where $F_{\lambda,TOA} = I_{\lambda,TOA}\delta \Omega$.

Neglecting the scattering of solar radiation and its reflection at the Earth surface, we set $F_\lambda^\uparrow = 0$ in (2.33) so that,

$$Q_\lambda = \frac{d}{dz}[F_\lambda^\downarrow e^{-\tau_\lambda(z)}] \quad (2.39)$$

$$= (-F_\lambda^\downarrow e^{-\tau_\lambda(z)})(-\rho q_a k_\lambda) \quad (2.40)$$

$$= F_\lambda^\downarrow \rho q_a k_\lambda \quad (2.41)$$

in which we have used (2.20). Fig. 2.10 gives a schematic of the vertical variations of $F_\lambda^\downarrow$ and $\rho q_a = \rho_a$, as well as a scale for the optical depth. The downward radiation decreases monotonically as we go downward, as expected, and the density of absorber is assumed to be exponential-like.

As can be seen, $Q_\lambda$, the product of these two\(^1\), peaks at a height where the optical depth is close to unity. Physically, well above the level of unit optical depth, the incoming beam is virtually undepleted, but the density is so low that there are too few molecules to produce significant absorption and heating. Likewise, well below the level of unit optical depth, there are a lot of molecules to produce absorption and heating, but there is not much left to absorb as the beam has been mostly depleted. You are invited to prove this result mathematically in Q5 below.

\(^1\)We are neglecting here the temperature and pressure dependence of $k_\lambda$. 
Figure 2.10: Schematic of heating rate due to solar absorption of radiation of intensity $I_\lambda$. See text for explanations. Figure from the Wallace and Hobbs’ textbook.
2.5. RADIATIVE HEATING AND COOLING RATES

Figure 2.11: Global mean longwave (left panel) and shortwave (right panel) heating rates in $K/day$ as a function of altitude showing contributions of the major gases. After D. Andrews’ textbook.

Detailed calculations, using a “line-by-line radiation code” are shown in Fig. 2.11 (focus here on the right hand side, i.e., shortwave heating rates). The heating rates are given in units of $K/day$, i.e., $Q_\lambda/(\rho c_p)$ is plotted rather than $Q_\lambda$. Absorption by ozone in the stratosphere dominates the heating rate, on the order of $5 - 10 K/day$. After this and $O_2$ at high altitude (above the mesopause), the next most important absorber of solar radiation is water vapour, which contributes to a relatively uniform heating of the troposphere on the order of $0.5 K/day$.

2.5.2 Longwave cooling

Unlike shortwave radiation, it is impossible to neglect the upward irradiance of longwave radiation (this is the physical mechanism allowing the Earth to cool to Space, see Chapter 1). Nor is it possible to neglect the downward infrared irradiance responsible for the greenhouse effect. To compute both terms, we start from the version of Schwarzschild’s equation relevant for
infrared radiation, namely (2.28) with the substitution \( J_\lambda \rightarrow B_\lambda, \)

\[
I_\lambda(s) = I_\lambda(s_o)T_\lambda(s_o, s) + \int_{T_\lambda(s_o, s)}^1 B_\lambda(s')dT_\lambda(s', s) \quad (2.42)
\]

To obtain the irradiance we need to integrate this equation over solid angle. Unlike in the case of shortwave radiation where the direction of the beam is a well defined function of season and time of the day, beams of infrared radiation come from many directions (Fig. 2.12) so the procedure is a little more complicated. We will accept the result that this can be approximately accounted for by simply increasing the optical depth (and thus lowering the transmissivity) of horizontal layers. To remember that we are using this “diffuse” approximation, we add a subscript \( \star \) to the transmissivity and write,

\[
F_\lambda(z) = F_\lambda(z_o)T_\lambda(z_o, z) + \int_{T_\lambda(z_o, z)}^1 \pi B_\lambda(z')dT_\lambda(z', z) \quad (2.43)
\]

To ease the notation a bit, and also to highlight more clearly the contribution of different atmospheric layers to \( F_\lambda(z) \), we rewrite the latter equation as,

\[
F_\lambda(z) = F_\lambda(z_o)T_\lambda(z_o, z) + \int_{z_o}^z \pi B_\lambda(z') \frac{dT_\lambda(z', z)}{dz'} dz' \quad (2.44)
\]
Both upward and downward irradiance can now be computed from (2.44) so as to estimate the heating rate (2.33). In the upward direction, take $z_o = 0$ (the Earth’s surface) and $F_\lambda(z_o) = \pi B_{\lambda,s}$,

$$F_\lambda^\uparrow(z) = \pi B_{\lambda,s} T^\star_\lambda(0,z) + \int_0^z \pi B_\lambda(z') \frac{dT^\star_\lambda(z',z)}{dz'} dz'$$  \hspace{1cm} (2.45)

This clearly shows that at a given height $z$, the upward infrared radiation is the sum of the radiation emitted by the Earth’s surface and that emitted by all atmospheric layers below $z$.

This equation is the basis for remote sensing of the atmosphere by satellites. For example, for remote sensing of temperature (say $z > TOA$), at a wavelength where there is strong absorption by an atmospheric layer at $z' = z_a$, the $dT^\star_\lambda(z',z)/dz'$ will peak at $z = z_a$ and the integral term above will reflect mostly emission by the layer, and hence the temperature $T(z_a)$. The surface term would not contribute since all the radiation emitted by the Earth’s surface would have been absorbed by the layer. As a result the radiation at that wavelength received by a satellite will allow an estimate of $T(z_a)$.

In the downward direction, take $z_o \to +\infty$ with $F_\lambda(z_o) = 0$,

$$F_\lambda^\downarrow(z) = - \int_z^{+\infty} \pi B_\lambda(z') \frac{dT^\star_\lambda(z',z)}{dz'} dz'$$  \hspace{1cm} (2.46)

At a given height $z$ the downward flux of infrared radiation is simply the accumulation of the radiation emitted by all layers above $z$.

The net heating rates will reflect the competition between heating due to absorption of the radiation emitted by other atmospheric layers and the ground, and the cooling due to the emission of longwave radiation. Detailed calculations of the resulting heating rates, integrated over wavelengths, are shown in Fig. 2.11 (left panel). The first thing to note is that, except for $O_3$ near 25 km, longwave heating rates are negative, on the order of several K/day. Infrared radiation thus cools the atmosphere globally, which opposes the heating due to absorption of shortwave radiation (previous subsection).

The primary reason why cooling is found in the infrared is that the radiation that a given atmospheric layer exchanges with the ground and other layers tend to cancel out, but there is no cancellation when it comes to exchange of radiation with Space. Indeed, a given atmospheric layer emits upwards and downwards towards layers above and below, but it also receives radiation from them. Likewise, it emits towards the Earth’s surface but also receives upwards radiation from the Earth’s surface. Thus each of these exchanges approximately cancel out. In addition, the atmospheric layer in
question also emits infrared radiation to Space but it does not receive such radiation in return. Hence a net cooling is expected, the so-called “cooling to Space” approximation. As Fig. 2.11 shows, this is mostly due to water vapour in the troposphere, and carbon dioxide in the stratosphere and the mesosphere.

2.5.3 Net radiative heating rates

The compensation between infrared cooling and shortwave heating is the weakest in the troposphere, where a net cooling of about 1K/day is suggested in Fig. 2.11. As discussed in Chapter 1, this net cooling is required to balance out the gain of energy associated with the cycling of water through the atmosphere (more energy gained through surface evaporation than lost through rain). The stratosphere experiences little energy gain from exchange of mass with the troposphere and, as a result, the compensation between infrared cooling and shortwave heating is close to perfect: the stratosphere is close to a radiative equilibrium.

One way to visualize this competition is proposed in Fig. 2.13. In the right panel is shown the downward net shortwave flux: largest at the TOA and decreasing slightly towards the Earth’s surface as a result of absorption by atmospheric gases, aerosols and clouds. On the left panel is shown the net longwave flux, which is upward at all levels: at the TOA it must be nearly equal in length to the solar flux since the Earth is approximatively in radiative balance. At the surface its length must be less than the downward solar flux since the Earth’s surface itself is in equilibrium and cools by surface evaporation and heat transfer, in addition to emitting infrared radiation. At a given level, the air cools through radiative exchanges in the infrared (the upward arrows diverge in the direction of propagation) and heat up through radiative exchanges in the shortwave (the downward arrows converge in the direction of propagation).

2.6 Problems

Q1 Calculate the ratio of the solar radiation incident on northward and southward facing slopes, each angled $\alpha = 20^\circ$ to the horizontal, if the solar zenith angle $\theta_S$ (the angle between the vertical and the line of sight to the Sun) is: (i) 30$^\circ$ (ii) 60$^\circ$.

Q2 Calculate the intensity $I$ of solar radiation at the top-of-the-atmosphere
Figure 2.13: Schematic of the compensation between longwave cooling and shortwave heating. The arrows represent the net fluxes, i.e., the sum of upward and downward radiative fluxes in each frequency band. The fluxes are shown at the top-of-the-atmosphere (TOA), the tropopause and the Earth’s surface. For the magnitude of arrows shown, the portion of air above the tropopause is in radiative equilibrium while the troposphere cools in the net. This is a good approximation. See text for explanations.
(TOA), given the irradiance $F = 1361 \text{Wm}^{-2}$ at zero zenith angle. The radius of the Sun is $R_s = 7 \times 10^8 \text{m}$ while the Earth-Sun distance is $d = 1.5 \times 10^{11} \text{m}$.

**Q3** Consider monochromatic radiation passing through a gas with absorption coefficient $\alpha_\lambda = 0.01 \text{m}^2 \text{kg}^{-1}$. (i) What fraction of the beam is absorbed in passing through a layer containing 1kgm$^{-2}$ of the gas? (ii) What mass per unit area would the gas layer have to have in order to absorb half the incident radiation?

**Q4** Suppose the gas in the previous question is present with uniform mass mixing ration $q = 10^{-3}$ in an atmosphere in hydrostatic equilibrium. Take surface pressure to be $P_s = 1000 \text{hPa}$. (i) Show that the optical depth measured from the top of the atmosphere is linearly proportional to pressure. What is the constant of proportionality? (ii) Estimate the pressure of the level that is one optical depth from the top of the atmosphere.

**Q5** Show that, for a beam of monochromatic solar radiation $F_\lambda$ incident vertically on an atmosphere in which the mixing ratio of the radiatively active gas is independent of height and the density decreases exponentially with height, the heating rate per unit volume is greatest at unit optical depth.

**Q6** Show that, for the situation in the previous question, the heating rate per unit mass is greatest near the top of the atmosphere.

**Q7** Now consider infrared radiation traveling upwards (optical depth measured from Earth’s surface) in an atmosphere where the total optical depth at the wavelength of the radiation considered is $\tau_\infty = 5$. (i) What fraction of the monochromatic intensity emitted by the ground is absorbed in passing through the layer of atmosphere extending from optical depth 0.2 to 4.0? (ii) What fraction of the intensity emitted to space is emitted by the layer between 0.2 and 4.0 optical depth? You may assume the atmosphere to be isothermal for this part, and at the same temperature as the Earth’s surface.

**Q8** Past exam question (2005, No 3).

(i) Derive an expression for the emission (or effective) temperature of the Earth as a function of solar constant and albedo.

(ii) Estimate numerically the sensitivity of the effective temperature to fluctuations in albedo. (Solar constant $S_\odot = 1370 \text{Wm}^{-2}$, planetary albedo $\alpha_P = 0.3$ and Stefan-Boltzmann constant $\sigma = 5.67 \times 10^{-8} \text{Wm}^{-2} \text{K}^{-4}$).
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(iii) Over the last 10 years the global mean temperature has risen by about 0.2°C. If clouds have an albedo of 0.8 and the surface has an albedo of 0.1, what change in the cloud percentage cover could account for this temperature rise? State any assumption you make.

(iv) If the transmissivity of clouds increased by 10% but the cloud cover stayed the same what would the new equilibrium temperature be?

(v) Why is the observed surface temperature larger than the effective temperature?

(vi) In which direction must the observed net radiation at the surface be and why?

Q9 Figure 2.14 shows an estimate of the flux of solar radiation impinging on the Earth (the solar constant $S_o$). Show that it is in good agreement with a pen and paper prediction assuming only knowledge of the radius of the Sun $r_{sun} = 6.96 \times 10^8 m$, the Sun’s emission temperature $T_{sun} = 5780K$, and the mean Earth-Sun distance $d = 1AU = 1.5 \times 10^{11} m$.

Q10 A sunphotometer is an instrument designed to measure the optical thickness of the atmosphere due to scattering and absorption of solar radiation by air molecules and aerosols. At the ground, two measurements of the
incident solar radiation at a wavelength $\lambda = 1\mu m$, denoted by $I_\lambda$, are made. These are made at two different solar zenith angles $\theta_1 = 20^\circ$ and $\theta_2 = 40^\circ$.

(i) Show that the ratio $r_\lambda = (I_\lambda)_1/(I_\lambda)_2$ of the two intensities satisfy,

$$\ln r_\lambda = \tau_\lambda (\sec \theta_2 - \sec \theta_1)$$

in which $\tau_\lambda = \int_0^{+\infty} k_\lambda \rho q_a \, dz$ is the “column” optical depth.

(ii) The ratio of the two intensity measurements is $r_\lambda = 1.12$. Compute the column’s optical depth using the data above, stating any assumptions made.
Chapter 3

Radiative-convective equilibrium

Key concepts: radiative equilibrium, fluid parcel, environment, buoyancy, Brunt-Vaisala frequency, radiative-convective equilibrium.

The preceding chapters showed that the atmosphere cools through (net) radiative processes (Chapter 2) and is heated from below through surface evaporation and sensible heat fluxes (Chapter 1). This situation destabilizes the atmosphere because, at same pressure warm air is lighter than cold air, and thus rises, while cold air sinks. The effect is to generate a convective cell which restores the equilibrium by carrying heat upward. In this chapter we study simple models of this interaction between radiation (destabilizing) and convection (stabilizing).

3.1 Radiative equilibrium

Mathematically, the state of radiative equilibrium is such that the net radiative heating (i.e., the sum of shortwave and longwave parts) is zero at all heights. Using the result from Chapter 2, this reads,

\[ Q_{\text{rad}} = \int_0^{+\infty} Q_\lambda d\lambda = \frac{d}{dz} \int_0^{+\infty} (F_\lambda^\uparrow - F_\lambda^\downarrow) d\lambda = 0 \quad (\text{for all } z) \quad (3.1) \]

Since the longwave emission depends on temperature, there must be a particular choice of this variable allowing (3.1) to be satisfied. It is referred to as the “radiative equilibrium” temperature.

Figure 3.1 (left panel) provides an example of a radiative equilibrium temperature calculation. The input for such calculation is the distribution
CHAPTER 3. RADIATIVE-CONVECTIVE EQUILIBRIUM

Figure 3.1: Approach to radiative (left) and radiative-convective (right) equilibrium in a time dependent calculation (Manabe and Strickler, 1964). Each curve represents temperature ($x$-axis) vs pressure ($y$-axis). Dashed curves refer to a cold start while continuous lines refer to a warm start. The thick lines indicate the equilibrium temperature profile in each case. In the right panel, they also locate the region of minimum temperature, the model’s “tropopause”.

of radiatively active constituents as a function of height, and the incident solar radiation at the TOA. (note that in this example, clouds were omitted and water vapour as well as ozone and carbon dioxide distributions were specified). When starting for example from a cold isothermal state at $T = 160K$ (dashed lines), one observes a systematic warming at all levels as time increases, reaching a steady state after about 300 days (thick continuous curve). This warming is pronounced near the surface and at upper levels. Note that the same equilibrium is reached if one starts from a warm state $T = 360K$ (continuous lines).

What happens in these experiments is that the shortwave part of the heating rate is essentially fixed (it does not depend on temperature). The longwave cooling, however strongly depends on temperature through Schwarzchild’s equation (Chapter 2). In the cold start for example, a given layer of air does not emit enough longwave radiation to offset the shortwave heat gain and,
3.2. CONVECTION

as a result, it warms. By warming up, it emits more in the infrared (the blackbody radiation term in Schwarzchild’s equation) and so the net warming decreases. When the equilibrium temperature is reached, the layer cools as much by infrared than it gains heat from shortwave absorption. No further temperature change can arise.

3.2 Convection

If we were to perturb slightly the radiative temperature equilibrium, say by allowing a small upward velocity at low level, and wait long enough, would the same temperature profile be found? In other words, is the radiative temperature equilibrium stable to “dynamical” as opposed to thermal changes?

Unfortunately, the answer is a clear no, and this is the main reason why weather and climate predictions are difficult to make! The radiative temperature profile is not stable to displacements of the layers: convective motion will develop and change the final temperature. We first study the basics of this instability and return to the problem of the radiative temperature equilibrium later in the chapter.

3.2.1 The “parcel’s equation”

The key concept to understand convection is that of buoyancy whom you might remember from Archimede’s principle. In Atmospheric sciences this concept is tied to that of an air parcel, a small enough sample of air that it describes locally what happens in the atmosphere, but made of a large enough number of molecules that we can apply thermodynamics to it.

Consider such parcel initially at a reference height \( z_o \) in the atmosphere. We ignore horizontal variations. From Newton’s second law, the motion of the parcel obeys,

\[
\frac{dw_p}{dt} = \sum_i F_i
\]  

where \( F_i \) refers to any force per unit mass acting on the parcel of upward velocity \( w_p \). The forces in question here are simple gravity and the pressure gradient force,

\[
\frac{dw_p}{dt} = -g - \alpha_p \frac{dP}{dz}
\]  

in which \( \alpha_p \) is the volume per unit mass (or specific volume) of the parcel and \( P \) is the pressure. We wish to establish whether, once displaced initially, the parcel will come back to \( z_o \) (stable case) or move away from it (unstable case).
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The key concept to answer this question is that of the “environment”, i.e., the atmosphere before we displaced the parcel. In the question at hand (stability of the radiative equilibrium state), the environment is simply a state of rest (no motion) with the temperature equal to the radiative equilibrium temperature. The environment (subscript $e$) thus satisfies,

$$0 = -g - \alpha_e \frac{dP_e}{dz}$$  \hspace{1cm} (3.4)

Note that this equilibrium is the “hydrostatic balance” (1.7) introduced in Chapter 1.

To make further predictions regarding the parcel’s fate, we will make two assumptions:

(i) the pressure field is not modified by the parcel’s motion, so the $P$ in (3.3) and in (3.4) are equal

(ii) the parcel does not acquire any heat as it moves and the ascent does not involve irreversibilities such as mixing with surrounding air, friction, etc. As a result the entropy of the parcel is conserved during the ascent (second law of Thermodynamics).

As a result of (i), we can rewrite (3.3) as,

$$\frac{dw_p}{dt} = g \left( \alpha_p - \alpha_e \right) \frac{\alpha_e}{\alpha_e}$$  \hspace{1cm} (3.5)

This equation shows that if the specific volume of the parcel is greater than that of the environment, the parcel will exhibit upward acceleration ($dw_p/dt > 0$). Likewise, if the parcel has a smaller specific volume than the environment, it will be accelerated downwards. The r.h.s in (3.5) is called the buoyancy force $B$,

$$B = g \left( \frac{\alpha_p - \alpha_e}{\alpha_e} \right)$$  \hspace{1cm} (3.6)

3.2.2 Potential temperature

Up to now the derivation has been fairly general. We restrict from now on the discussion to the case of dry air so that the pressure $P = P_d$ and the equation of state for the parcel is simply the ideal gas law $P_d\alpha_d = R_dT$ (see Chapter 1). This allows to rewrite as,

$$\frac{dw_p}{dt} = -g \left( \frac{T_e - T_p}{T_e} \right)$$  \hspace{1cm} (3.7)
Let us now introduce the second assumption above (isentropic ascent). From Thermodynamics (year 2), we know that the specific entropy of the parcel is,

\[ s = s_{\text{ref}} + c_{p,d} \ln \left( \frac{T}{T_{\text{ref}}} \right) - R_d \ln \left( \frac{P}{P_{\text{ref}}} \right) \]  

(3.8)
in which \( T_{\text{ref}}, P_{\text{ref}} \) denote a reference state in which the entropy is \( s_{\text{ref}} \). At a given height \( z \) where the pressure is \( P_e(z) \) the specific entropy of the parcel is,

\[ s_p = s_{\text{ref}} + c_{p,d} \ln \left( \frac{T_p}{T_{\text{ref}}} \right) - R_d \ln \left( \frac{P_e}{P_{\text{ref}}} \right) \]  

(3.9)
while that of the environment is,

\[ s_e = s_{\text{ref}} + c_{p,d} \ln \left( \frac{T_e}{T_{\text{ref}}} \right) - R_d \ln \left( \frac{P}{P_{\text{ref}}} \right) \]  

(3.10)
Taking the difference shows that,

\[ \frac{T_e}{T_p} = e^{(s_e - s_p)/c_{p,d}} \]  

(3.11)

It is common practice in Atmospheric Sciences to rewrite entropy as a “potential temperature” \( \theta \), using the definition,

\[ s \equiv s_{\text{ref}} + c_{p,d} \ln \left( \frac{\theta}{T_{\text{ref}}} \right), \]  

(3.12)
with,

\[ \theta = T \left( \frac{P}{P_{\text{ref}}} \right)^{-R_d/c_{p,d}} \]  

(potential temperature)  

(3.13)
and choose the reference state as that typical of the surface \( P_{\text{ref}} = 1000hPa \). In this way \( \theta \) can be thought of as the temperature a parcel of air at \( T, P \) would have if brought adiabatically towards the surface at \( P = P_{\text{ref}} \) (hence the name “potential temperature”). For example, if we were to move adiabatically a parcel of air initially from \( P_0 = 250hPa, T_0 = 210K \) (i.e., near the tropopause) to the surface \( P_1 = P_{\text{ref}} = 1000hPa \), we can simply predict its temperature from \( \theta_0 = T_0(P_0/P_{\text{ref}})^{-R_d/c_{p,d}} = \theta_1 = T_1(P_1/P_{\text{ref}})^{-R_d/c_{p,d}} = T_1 \). Numerically, we find \( T_1 = 312K \). So although it is true that the in-situ temperature high up above our head is colder, bringing these parcels to us would raise their temperature tremendously because of the work of compression they would experience during the adiabatic descent.

Figure 3.2 illustrates the zonal mean (i.e., averaging along a latitude circle) and time mean distribution of \( \theta \). Stratospheric air is seen as the region of the atmosphere where \( \theta \) is very high (\( \theta \approx 500K \)), that is very buoyant. The downward doming of \( \theta \) surfaces near low latitudes reflect, at a given pressure level, the higher temperature in these regions compared to higher latitudes.
3.2.3 Stability of temperature profiles to vertical displacements of air parcels

All these maths are not useless because they allow us to simplify greatly (3.14) into,

\[
\frac{dw_p}{dt} = -g \frac{\theta - \theta_p}{\theta_e} \tag{3.14}
\]

At first sight, not much is gained. Upon reflection though we see that we have replaced \(T\), a non conserved variable during the ascent, by \(\theta\) which is conserved (since it is proportional to entropy). In particular, if we start the parcel at \(z_o\) with the same temperature as the environment, \(\theta_p(z_o) = \theta_e(z_o)\) while at \(z\), \(\theta_e(z) = \theta_e(z_o) + d\theta_e/dz(z - z_o)\). This shows that, since the parcel conserves its entropy during the ascent, i.e., \(\theta_p(z) = \theta_p(z_o)\),

\[
\frac{d^2z_p}{dt^2} + \frac{g}{\theta_e} \frac{d\theta_e}{dz} (z_p - z_o) = 0 \tag{3.15}
\]

in which we have used \(w_p = dz_p/dt\), \(z_p\) being the height of the parcel.

This equation shows that the fate of the parcel is governed by the sign of the quantity \(N^2\), defined as,

\[
N^2 \equiv \frac{g}{\theta_e} \frac{d\theta_e}{dz} \quad \text{(the Brunt-Vaisala frequency)} \tag{3.16}
\]
When $N^2 > 0$, (3.23) is an harmonic oscillator with angular frequency $N$: the parcel undergoes stable oscillations around $z = z_o$. When $N^2 < 0$, the parcel displacement is unstable: if initially displaced upwards, it will keep going upwards, being constantly accelerated upwards by the buoyancy force (and likewise if initially displaced downwards). The Brunt-Vaisala frequency is clearly a very important variable describing the atmosphere (see section 3.5).

A couple of important comments:

(i) From Fig. 3.2 we see that the atmosphere has, in the mean, $N^2 > 0$. So it is stable to vertical displacements of air parcels.

(ii) From the plot, one can estimate roughly that $N = [(9.81/315) \times (330 - 300)/5 km]^{1/2} \approx 10^{-2} s^{-1}$. This corresponds to a period of oscillation of about $10 mn$.

(iii) The fact that the time mean and zonal mean picture has $N^2 > 0$ does not mean that the atmosphere is always everywhere, and at all times, stable. Indeed the deep anvil clouds that one sometimes experiences on a summer day (Fig. 3.3) are the manifestation of a region of $N^2 < 0$. This arose because parcels at low levels increased their entropy tremendously through heating with the hot Earth’s surface while, aloft, they cooled through infrared radiation. This state of affair generates high $\theta_p$ at low levels and low $\theta_e$ at upper level and thus a positive buoyancy for air parcels from (3.14).

3.2.4 CAPE

The previous comment just made about the existence of sporadic regions with $N^2 < 0$ call for a measure of the strength of the instability developing in these regions. The concept of CAPE (Convective Available Potential Energy) was introduced for that purpose.

The work $W_B$ (per unit mass) done by the buoyancy force $B$ to move the parcel from $z_1$ to $z_2$ is simply,

$$W_B(z_1, z_2) = g \int_{z_1}^{z_2} (\alpha_p - \alpha_e) dz$$

in which it is understood that the parcel conserves its entropy during the displacement. If there is an interval $[z_1, z_2]$ for which $W_B(z_1, z_2) > 0$, then kinetic energy is gained by the parcel. This energy is referred to as Convective Available Potential Energy (CAPE, in units of $J/kg$) and reflects a state of
Figure 3.3: An intense “anvil cloud”. Surface heating throughout the day increased the entropy of air parcels at low levels while infrared radiative cooling decreased that of air parcels at upper levels (this latter effect is admittedly weaker than the heating due to heat exchange with the Earth’s surface). As a result CAPE was being generated. The cloud developed and used this CAPE. Note that air parcels seem “stuck” at the top and spread laterally. This reflects that even though they acquire large buoyancy through surface heating, this buoyancy was still less than that of the environmental air at upper levels (the tropopause here). Thus in this example the spreading allows you to “see” the tropopause!

instability (in the oscillatory case discussed above, $W_B(z_1, z_2) < 0$ for all $z_1, z_2$). To give an order of magnitude, big thunderstorms occurring over continents in the Summer (Fig. 3.3) have $CAPE$ of several $kJ/kg$ (if all this energy went into the parcel’s downdraft/updraft speed, the velocities involved (= $\sqrt{2CAPE}$) would exceed hurricane winds at the surface!). Besides these events, CAPE is generally found to be much lower (a few $100 J/kg$).

3.3 Radiative-convective equilibrium

3.3.1 Stability of the radiative equilibrium temperature profile

We now return to Fig. 3.1 and the stability of the radiative temperature equilibrium. At the surface, the temperature is $T \approx 330 K$ with a pressure
3.3. RADIATIVE-CONVECTIVE EQUILIBRIUM

$P = 1000hPa$. Higher up, there is a minimum temperature of about $180K$, reached at a pressure $\approx 300hPa$. Application of (3.13) with $P_{ref} = 1000hPa$ shows that the surface potential temperature is $330K$ while at $300hPa$ it is $\approx 254K$. Hence $\theta$ decreases with height ($N^2 < 0$) and the radiative equilibrium temperature profile is unstable!

What this says is that radiative processes destabilize the atmosphere and that consideration of radiation only cannot provide a realistic prediction for the temperature structure of the atmosphere. The right panel in Fig. 3.1 is an attempt to represent the effect of convection, by bringing artificially the atmosphere into a state with $N^2 = 0$ (i.e., neutral to vertical displacements of air parcels). We will refer to this state as one of radiative-convective equilibrium.

The radiative - convective equilibrium state has a weaker change of temperature with height, i.e., a weaker “lapse-rate” $\Gamma$ in the right compared to the left panel,

$$\Gamma \equiv -\frac{dT}{dz} \quad \text{ (definition of lapse-rate)} \quad (3.18)$$

It also has colder surface temperature (about $290K$), and higher temperature aloft. This reflects that by carrying high $\theta$ parcels upwards and replacing them by cold $\theta$ parcels going downward, convection carries heat upward.

3.3.2 The Tropopause

An interesting feature in Fig. 3.1 is the presence of a clear discontinuity in the lapse-rate (from being positive to negative) around $300hPa$. This discontinuity is very pronounced in the radiative equilibrium (left panel), with a sharp transition between the lower ($\Gamma > 0$) and upper atmosphere ($\Gamma < 0$). In the radiative - convective equilibrium state, the discontinuity is still seen, but is less pronounced, the layer between 10 and $25km$ being nearly isothermal ($\Gamma \approx 0$).

Returning to Chapter 1 (Fig. 1.1), we identify this discontinuity in lapse rate as the tropopause. The previous paragraph makes it clear that it arises primarily as a result of radiative constraints: the absence of significant absorption of solar radiation below about $10km$ leads, in pure radiative equilibrium, to a region of positive lapse-rate (see the simple calculations in Q4 below). Conversely, in a region where there is absorption of solar radiation, pure radiative equilibrium leads to negative lapse-rate (so that this region can cool by emitting more infrared radiation at its top than at its bottom). The “dynamics” (convective adjustment here) merely modulates the exact value of the height at which this transition happens.
Returning to Fig. 3.1, the fact that the two temperature profiles (left and right panels) differ below the tropopause is expected from the fact that the radiative calculation has $N^2 < 0$ there. Above the tropopause, the radiative equilibrium temperature is very stable (a region of $\Gamma < 0$ has a very large $N^2$), so no instability is expected and the difference between the two temperature profiles is more surprising. What we are seeing in action here is the interaction of convection with radiation: changing the tropospheric temperature through convection affects the upward infrared radiation reaching the stratosphere, and thus perturbs radiative heating rates in this region. The calculation shows that above $\approx 25\, km$ this effect becomes negligible and the radiative and radiative-convective equilibrium temperature profiles are the same.

### 3.4 Dynamical effects of moisture*

We have managed to get a long way by simply considering the buoyancy of dry air. However, as a buoyant air parcel rises, its internal energy (temperature) decreases because of the work of expansion done by the parcel on the surrounding air. As a result, phase change can occur. As we shall see, the latent heat released at this stage increases significantly the buoyancy of the parcel. Thus, in addition to important radiative effects, moisture also plays a key role in the dynamics.

To isolate clearly this role we are going to consider the case of cloudy, as opposed to pure dry, air. We will restrict ourselves to a mixture of dry air, water vapour and liquid water, the two latter phases being in thermodynamic equilibrium. This means that the vapour pressure $e$ is solely a function of temperature, given by the Clausius-Clapeyron equation (Thermodynamics, year 2):

$$ e = e_{eq}(T) \quad \text{with} \quad \left( \frac{de_{eq}}{dT} \right)_{p.b} = \frac{\Delta s}{\Delta \alpha} \quad (3.19) $$

Note that in this equation, $\Delta$ refers to a change across the phase boundary (subscript $p.b$), i.e., $\Delta s = s_v - s_l = l_v/T$ ($l_v$ being the latent heat of vaporization introduced in Chapter 1) and $\Delta \alpha = \alpha_v - \alpha_l \approx \alpha_v$ (a given mass of water vapour occupies a much larger volume than the same mass of liquid water). So we can approximate it as,

$$ \left( \frac{de_{eq}}{dT} \right)_{p.b} \approx \frac{l_v}{\alpha_v T} \quad (3.20) $$

Our goal in this section is to see how the Brunt-Vaisala frequency (and hence the stability of a temperature profile to vertical displacements of air parcels)
is modified by the presence of phase change. Before doing so a few important comments:

- The assumption that gas and liquid phases are in equilibrium is very important. This can occur at the core of a cloud but, in general, we must acknowledge that the two phases do not coexist in a state of equilibrium (net evaporation or net condensation occurs). So we are considering the effect of moisture in a very idealised case (cloudy air).

- The phase change occurring as a parcel ascends is adiabatic. What is said here is that latent heating is internal to the parcel. It is not heat taken from the environment, as occurs for example when a cold continental air mass flows over a warm ocean. So we can still assume that the parcel conserves its entropy during ascent.

We will start from (3.5) and, to simply this expression further, we use the fact that $\alpha$ is a state function, i.e., a mathematical function of any two thermodynamic variables (this is only true because of our assumption of thermodynamic equilibrium). Because of the additional assumption of isentropic ascent, entropy $s$ is a natural choice. We’ll take pressure $P$ (the total pressure, $P = P_d + e$) as the other variable,

$$\alpha = \alpha(P, s)$$  \hspace{1cm} (3.21)

As a result, and assuming small perturbations,

$$\alpha_e - \alpha_p \approx \left( \frac{\partial \alpha}{\partial s} \right)_P (s_e - s_p)$$  \hspace{1cm} (3.22)

(the term involving pressure changes drops because of our assumption that the environment and the parcel are at the same pressure). Using one of Maxwell’s relations, this can be rewritten as,

$$\alpha_e - \alpha_p \approx \left( \frac{\partial T}{\partial P} \right)_s (s_e - s_p)$$  \hspace{1cm} (3.23)

At this stage we haven’t actually used that $s_p = \text{cst}$. Following the same procedure used in the dry case when we considered how $\theta$ is conserved by a parcel moving upwards or downwards, we write,

$$s_e - s_p \approx \frac{ds_e}{dz} (z_p - z_o)$$  \hspace{1cm} (3.24)

As a result, the parcel’s motion obeys,

$$\frac{d^2 z_p}{dt^2} = -\frac{g}{\alpha_e} \left( \frac{\partial T}{\partial P} \right)_s \frac{ds_e}{dz} (z_p - z_o)$$  \hspace{1cm} (3.25)
showing that the Brunt-Vaisala frequency is,

\[ N^2 = \frac{g}{\alpha_e} \left( \frac{\partial T}{\partial P} \right)_s \frac{ds_e}{dz} \tag{3.26} \]

Stepping back from all this Thermodynamic, we realize that we have actually not explicitly introduced moisture here—the above derivation is entirely general. For example, in the case of pure dry air \((P = P_d)\), we recover (3.16) by using \( s = c_{p,d} \log \theta \), and \((\partial T/\partial P)_{s_d} = \alpha_d/c_{p,d}\) (since for dry air, \(ds = c_{p,d}dT/T - R_d dP/P\) and \(P_d \alpha_d = R_d T\) from the ideal gas law).

Acknowledging that at constant entropy, temperature always increases with pressure \((\partial T/\partial P)_s > 0)\), (3.26) provides a fairly simple rule to test the stability of a temperature profile (for the case of either pure dry air, or cloudy air in thermodynamic equilibrium): the profile is stable only if the entropy increases with height.

To see how moisture changes the buoyancy of parcels undergoing adiabatic ascent or descent, let’s simply compare the temperature profiles corresponding to neutrality \((N^2 = 0)\) for the dry and moist cases. In the case of dry air, we readily obtain,

\[ c_{p,d}dT/T - R_d dP/P = 0 \tag{3.27} \]

which, using hydrostatic equilibrium and the ideal gas law can be rewritten as,

\[ \Gamma_d = \frac{g}{c_{p,d}} \text{ (“dry adiabatic lapse - rate”)} \tag{3.28} \]

In the case of moist air, we accept the result that the entropy of cloudy air is approximately given by,

\[ s = s_{ref} + c_{p,d} \ln(T/T_{ref}) - R_d \ln(P_d/P_{ref}) + \frac{l_v q_v}{T} \tag{3.29} \]

in which \(q_v\) (specific humidity) was introduced in chapter 1. Using again the hydrostatic equilibrium, applying \(ds = 0\) to this equation leads to,

\[ \left( 1 + \frac{\alpha_d}{\alpha_v c_{p,d} T} \right) \frac{dT}{dz} = - \left( \frac{\alpha_d}{\alpha_v} \right) \frac{g}{c_{p,d}} - \frac{T}{c_{p,d}} d \frac{d(l_v q_v / T)}{dz} \tag{3.30} \]

The first term on the r.h.s is similar to the case of dry air, the factor in parenthesis being close to unity. The second term on the r.h.s is new and clearly reflects the effect of phase change. In a realistic range of temperature, this term can be approximated as,

\[ - \frac{T}{c_{p,d}} d \frac{d(l_v q_v / T)}{dz} \approx - \frac{l_v}{c_{p,d}} \frac{dq_v}{dz} \tag{3.31} \]
Likewise, the second term in parenthesis on the l.h.s is small compared to unity so that,

\[ \Gamma_m \approx \frac{g}{c_{p,d}} + \frac{l_v}{c_{p,d}} \frac{dq_v}{dz} \]  

(“moist adiabatic lapse - rate”) \hspace{1cm} (3.32)

Because specific humidity decreases with height (water vapour condensing to liquid water in ascending air, liquid water evaporating in descending air), the second term on the r.h.s is negative so that the moist adiabatic lapse rate is weaker (typically 6.5\(K/km\)) than the dry adiabatic one.

### 3.5 Radiative-convective equilibrium and the real world

The main message from all the above is that there is a competition between radiative processes, which tend to destabilize the atmosphere, and dynamical processes, which tend to stabilize the atmosphere. The 1D case illustrated in Fig. 3.1 showed that radiation acts relatively slowly (it took several months to achieve radiative equilibrium on the left panel). Convection is expected to be much faster (a moderate updraft of 10\(cm/s\) would reach the tropopause in typically 10\(km/10cms^{-1}\) \(\approx\) 1\(day\)). As a result, the prediction is that convection should maintain the atmosphere close to a state of near neutrality \((N^2 = 0)\), with a temperature lapse rate close to the moist adiabat.

Figure 3.4 displays the observed ratio \((\Gamma_m - \Gamma)/\Gamma_m\). It is clearly seen that in the Tropics (equatorward of 30\(^\circ\) of latitude), and above the near surface layer \((P \leq 900hPa)\), observed lapse rates are within 10% of the moist adiabatic value. This is suggestive of a strong role of convective processes at low latitudes and has deep implications for a variety of problems. For example, it suggests that any change in surface temperature will be quickly felt throughout the atmosphere through adjustment to a new (warmer) moist adiabatic profile. This is indeed one of the prediction of the Intergovernmental Panel on Climate Change (IPCC) to which we will come back later in the course.

In mid and high latitudes, observed lapse rates are weaker than moist adiabatic values. Thus in addition to the vertical convection discussed above, something else must be stabilizing the atmosphere. We all know this something else: the weather systems we experience daily at the latitude of England. We discuss next briefly that these motions can also be thought of as resulting from a convective instability. We’ll go deeper in their dynamics in Chapter 4.
3.6 Sloping convection

The distribution of potential temperature $\theta$ (Fig. 3.2) shows a striking distinction between the Tropics and higher latitudes. To within a few tens of degree of latitude off the equator, $\theta$-surfaces are nearly flat while they slope upward in the extra-tropics. This distribution reflects the presence of marked horizontal temperature gradients away from the equator (we will see in Chapter 4 that this difference in slopes of $\theta$-surfaces between low and high latitudes arises because of the effect of rotation on motions).

If we schematize the extra-tropical state of affair as in Fig. 3.5, and imagine swapping point $A$ with either point $B$, $C$ or $D$, we note that:

- $A \leftrightarrow B$: this is the situation discussed in section 3.2 when the Brunt-Vaisala frequency is positive ($\theta$ increases upward, $\theta(A) < \theta(B)$). Thus work would have to be done against the buoyancy force to achieve this displacement. This is a stable situation.

- $A \leftrightarrow C$: this displacement is along a $\theta$ surface so nothing happens.

- $A \leftrightarrow D$: this time $\theta(A) > \theta(D)$ so the displacement would be unstable, i.e., the buoyancy force would do work on the parcels.

- $A \leftrightarrow E$: again $\theta(A) > \theta(E)$ but no work by the buoyancy force can be generated since the displacement is horizontal. This situation will thus not occur spontaneously.
3.6. SLOPING CONVECTION

This suggests that if the displacement of fluid parcels follows a sloping path shallower than the isentropic slope, kinetic energy can be gained in the process: an instability occurs.

To rationalize this, consider the “parcel’s equation” (3.5) in the form,

\[ \frac{dw_p}{dt} = g \left( \frac{\theta_p - \theta_e}{\theta_e} \right) \quad (3.33) \]

with \( \theta_p = \theta(A) \). The gain in kinetic energy after a time \( t \) has elapsed since the swap is,

\[ \Delta \frac{1}{2} (w_p^2) = \int_0^t \frac{dw_p}{dt} dt = \int_0^t g \left( \frac{\theta_p - \theta_e}{\theta_e} \right) w_p dt = \int_{z_A}^{z_D} g \left( \frac{\theta_p - \theta_e}{\theta_e} \right) dz \quad (3.34) \]

where the integral is along the slanted path \( A \leftrightarrow D \).

Before going further, a few important comments are needed:

- The paths shown in Fig. 3.5 are imposed, and all we do is compute the work done by buoyancy forces which is associated with a given path. We have to proceed in this way because we do not have yet the artillery to include an equation for the horizontal motion (upcoming in Chapter 4).

- The calculation below only considers what happens in the branch \( A \rightarrow D \), but clearly not all particles can go upward (in a statistically steady state, as many would have to go downward). You can check by yourself that the exact same mathematics would describe the \( D \rightarrow A \) branch.

- The parcels cannot cross during the swap (if this occurred, the velocity field would have two different values at the crossing point, which is impossible) so the swap must occur along different paths. For example one could imagine \( A \) and \( D \) to have the exact same trajectory in the \( y - z \) plane but be at different values of \( x \) (this is what happens in midlatitude storms, \( y \) being the North-South direction and \( x \) the East-West direction –the motion is thus fully 3D). The swap could also reflect a motion independent of \( x \), in which case it would consist of 2D rolls or cells in the \( y - z \) plane.

Define \( \Delta z = z_D - z_A \) and rewrite the integral as,

\[ \Delta \frac{1}{2} (w_p^2) = g \Delta z < \frac{(\theta_p - \theta_e)}{\theta_e} > \quad (3.35) \]
CHAPTER 3. RADIATIVE-CONVECTIVE EQUILIBRIUM

where the bracket $<>$ denotes an average along the path. Approximate the latter using a Taylor expansion, as,

$$< \frac{(\theta_p - \theta_e)}{\theta_e} > \approx -\frac{1}{2} \left( \frac{\partial \theta}{\partial z} \Delta z + \frac{\partial \theta}{\partial y} \Delta y \right) / \theta_e \tag{3.36}$$

in which $\Delta y = y(D) - y(A) > 0$ is the gain in latitude of the parcel and we have introduced the vertical ($\partial \theta / \partial z > 0$) and meridional ($\partial \theta / \partial y < 0$) temperature gradients. Note that the implicit assumption used here is that the parcel conserves its $\theta$ from $A$ to $D$ ($\theta_p - \theta_e = 0$ at $A$ while $\theta_e - \theta_p = \Delta z \partial \theta / \partial z + \Delta y \partial \theta / \partial y$ at $D$, hence the factor $1/2$ and the negative sign).

Inserting this result, and after introducing the slope of the displacement $\mu = \Delta z / \Delta y$ and the isentropic slope $\mu_\theta = -\left( \frac{\partial \theta}{\partial y} \right) / \left( \frac{\partial \theta}{\partial z} \right) > 0$, we have,

$$\Delta \frac{1}{2} \left( w^2_p \right) \approx \frac{N^2 (\Delta y)^2}{2} \mu (\mu - \mu_\theta) \tag{3.37}$$

For a fixed displacement $\Delta y$, we recover the discussion at the beginning of this section:

- $A \leftrightarrow B$: $\mu > \mu_\theta$ and the kinetic energy gained is negative!
- $A \leftrightarrow C$: $\mu = \mu_\theta$ and the kinetic energy gained is zero.
- $A \leftrightarrow E$: $\mu = 0$ and the kinetic energy gained is zero.
- $A \leftrightarrow D$: $0 < \mu < \mu_\theta$ and the kinetic energy gained is positive. The maximum possible kinetic energy gained is when $\mu = \mu_\theta / 2$,

$$KE_{max} = \frac{N^2 (\Delta y)^2}{8} \mu_\theta^2 = \frac{g (\Delta y)^2}{8 \theta_e} \left( \frac{\partial \theta}{\partial y} \right)^2 / \left( \frac{\partial \theta}{\partial z} \right) \tag{3.38}$$

The bottom line is that we can understand the extra-tropical storms in Fig. 1.2 (the “wavy bits”) in the same way as we understand their tropical counterparts (the “spotty bits”). They result from a convective instability, “sloped”, as opposed to vertical. The distortion of $\theta$ surfaces brought about by the swapping of air parcels at a shallow angle ($\mu \leq \mu_\theta$) lead to an enhancement of the local Brunt-Vaisala frequency in the extra-tropics, as seen in Fig. 3.4.

3.7 Summary

Figure 3.6 offers a graphical summary of the various ideas discussed in this chapter:
Figure 3.5: Sloping convection. Kinetic energy can be gained in certain swapping of parcels while they conserve their potential temperature $\theta$.

- Above the tropopause, the atmosphere can be thought to be in radiative equilibrium: the sum of shortwave and longwave irradiances (grey and black arrows, respectively), does not converge or diverge in the vertical.

- Below the tropopause, radiative effects alone lead to a strong lapse-rate. Displacements of air parcels either purely upward (section 3.2) or in a slanted way (section 3.6) are unstable and convective motions develop. The troposphere as a whole (Tropics, midlatitudes and high-latitudes) can be thought of being in a state of convection.

- Convection carries heat upwards (blue arrows, converging upward) from the surface to the tropopause. As such it cools the surface and heat the upper levels, leading to a weaker (but still positive) lapse-rate.

- Convective heating is opposed by a net radiative cooling, seen in the vertical divergence of the net longwave irradiances (black arrows).

### 3.8 Problems

**Q1.** In this question we treat the Earth’s atmosphere as dry and in hydrostatic equilibrium.
Figure 3.6: A summary schematic of the radiative - convective equilibrium view of the atmosphere. Black upwards arrows denote longwave irradiances while grey downward arrows denote shortwave irradiances (see Chapter 2). Blue upwards arrows indicate the heat flux driven by convective motions. Note that absorption of solar radiation above the tropopause was entirely neglected in this picture.
3.8. PROBLEMS

(i) Show that the adiabatic lapse rate is simply $\Gamma = \Gamma_d = g/c_{p,d}$ where $c_{p,d} = 1005 J/kgK$ is the specific heat capacity of dry air at constant pressure and $g = 9.81 ms^{-2}$ is gravity. Compute its numerical value in K/km.

(ii) Show that the actual lapse rate is $\Gamma = \Gamma_d - \frac{T \partial \theta}{\sigma \partial z}$.

(iii) In light of the result in (ii) discuss whether a dry atmosphere can have a lapse rate greater (or lower) than $\Gamma_d$.

Q2. Air flows over the ocean (so it is well-supplied with moisture), across the coast, over a mountain which is 4 km high and down to a plateau on the far side at 1 km above sea level. Estimate the difference in temperature between sea level on the windward side and at the surface of the plateau. [Take $\Gamma_d = 10 K/km$ and $\Gamma_m = 6.5 K/km$].

Q3. Relative humidity ($RH$) is defined as the ratio of the vapour pressure found in a sample of air at temperature $T$ to that found when vapour and liquid water are in thermodynamic equilibrium at the same temperature: $RH = e/e_{eq}(T)$. Discuss whether the situations below correspond to a state of thermodynamic equilibrium: (i) Room temperature, $RH = 0.7$ (ii) precipitation falling into a dry air mass, $RH = 0.5$ (as can happen on the edge of a cloud).

Q4. We study a simple model of radiative equilibrium (Fig. 3.7) by treating the atmosphere as a two-layer system. Each of these layers is assumed to radiate like a blackbody in the infrared. The Earth’s surface is also treated like a blackbody. The atmosphere is assumed to be completely transparent to solar radiation, so that at each level, the solar flux is $S_o(1-\alpha_P)/4$. Compute the radiative equilibrium temperatures $T_s$ (surface), $T_1$ (lower atmosphere) and $T_2$ (upper atmosphere).

Q5. The temperature profile predicted in the previous question is not stable to vertical displacements of air parcels. The latter lead to convective motions and a new temperature distribution close to an adiabat. To represent this simply, we fix the temperature difference $\Delta T = T_1 - T_2 = T_s - T_1$ in the previous model, and also acknowledge the presence of additional heating terms: a surface heat exchange $F_s$ and a convective heat flux $F_c$ (Fig. 3.8).

(i) Taking the two atmospheric layers to be at a pressure of 700hPa and 400hPa respectively, and using your answers in Q3, check that indeed the radiative equilibrium temperature profile is unstable. You may take the surface pressure to be 1000hPa.
Figure 3.7: Simple radiative equilibrium model. The surface receives solar radiation \( \sigma T_e^4 \equiv S_o (1 - \alpha_P)/4 \) and emits infrared radiation upward. The atmosphere consists of two layers of longwave emissivity unity and shortwave absorptivity zero.

(ii) Before doing any calculation, do you expect the surface temperature to decrease or increase when the fluxes \( F_s \) and \( F_c \) are included in the model?

(iii) Solve for the temperature \( T_s, T_1 \) and \( T_2 \) as a function of \( T_e \) and \( \Delta T \).

(iv) Solve for the convective flux \( F_c \) and the surface heat exchange \( F_s \). Express your answer as a function of \( T_e \) and the non dimensional parameter \( x = \Delta T/T_e \).

(v) Find a plausible value for \( x \) and discuss the range of values of this parameter for which the model makes sense. Also check that for this range of values the model agrees with your answer in (ii).

Q6. Using Figs. 1.2 and 3.2, estimate the kinetic energy gain in sloping convection —eq. (3.38). How does this compare with your guess for the kinetic energy in storms?

Q7. A parcel of air has the same potential temperature than the environment at \( t = 0 \) and \( z = 0 \), and an upward velocity \( w_o = 10 \text{cm/s} \). Find its height after 1mn if (i) \( N^2 = 10^{-4} \text{s}^{-2} \) (ii) \( N^2 = -10^{-4} \text{s}^{-2} \). (iii) \( N^2 = 0 \). Neglect the effects of moisture on the motion.
Figure 3.8: Addition of convective fluxes, $F_s$ and $F_c$ to the simple radiative equilibrium model.
Chapter 4
Atmospheric motions

Key concepts: Eulerian and Lagrangian descriptions of motions, material derivative, Coriolis force, Rossby number, geostrophic balance, rotational and divergent flows, “thermal wind”.

The vertical instability discussed in Chapter 3 is at the heart of convective cells and cloud clusters with horizontal scales ranging from a few km up to 100km. Updrafts and downdrafts in such systems are however not the sole motions driven by radiative processes. Indeed, the net radiative loss at high latitudes and the net radiative heat gain at the equator set up a whole range of motions. On scales of $\approx 1000km$, we find the traveling weather systems familiar at our latitudes, and their embedded cold and warm fronts ($\approx 100km$). On even larger scale (10,000km) we find planetary cells organized in east-west (Walker cell) and north-south (Hadley cell) directions—see Fig. 4.1 for a summary. In this chapter we study these motions from first principles (essentially Newton’s second law). As we shall see, the Earth’s rotation has a profound impact on the dynamics, giving a (counter-intuitive) rigid or solid-like behavior to atmospheric motions.

4.1 Equations of motions

These are simply the three components of the “momentum” equation (Newton’s second law). Compared to the classical mechanics you have dealt with so far, the difficulties and novelty lie in:

(i) the “fluid nature” of the motion, air parcels transporting their own momentum (material derivative)

(ii) the Earth rotation, which introduces new forces as seen by an observer
at a fixed location on the planet

(iii) the sphericity of the Earth, which introduces mathematical complexity and a “channel-like” geometry to the study of atmospheric motions.

4.1.1 Forces acting on a parcel of air

Per unit mass, these are simply: the acceleration due to gravity (kept to a constant \( g = 9.81 \text{ms}^{-2} \) for practical purposes, owing to the thinness of the atmosphere compared to the Earth’s radius), the pressure gradient force we’ve seen in Chapter 3 (but now in 3D), and frictional (or viscous) forces,

\[
a = g - \alpha \nabla P + F_{fric}
\]  

(4.1)

Note that in this equation \( a \) is the acceleration vector in an inertial frame (say from an observer looking at the Earth from deep space), and \( \alpha \) is the volume per unit mass used extensively in Chapter 3.

4.1.2 Material derivative

Imagine a parcel of air undergoing an adiabatic ascent. Following the parcel, there is no change in its entropy. At a fixed location, however, an observer might see a change in entropy since the parcel might originate from a warmer region than that in which the observer is sitting. To make it clear that the
change “following the parcel” is a special case of derivative, we will write it as $D/Dt$. In the previous example, we would thus write,

$$\frac{Ds}{Dt} = 0 \quad (4.2)$$

Mathematically, the change following the parcel (also called “Lagrangian derivative”) includes the parcel’s displacement. If we write that entropy is a function of space and time, i.e., $s = s(x, y, z, t)$, a small change in entropy $\delta s$ in a small time interval $\delta t$ is, after Taylor expansion,

$$\delta s = \left(\frac{\partial s}{\partial t}\right)\delta t + \left(\frac{\partial s}{\partial x}\right)\delta x + \left(\frac{\partial s}{\partial y}\right)\delta y + \left(\frac{\partial s}{\partial z}\right)\delta z \quad (4.3)$$

Dividing by $\delta t$ and taking the limit, we thus identify $D/Dt$ as,

$$\frac{Ds}{Dt} \equiv \lim_{\delta t \to 0} \frac{\delta s}{\delta t} = \frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial y} + w\frac{\partial s}{\partial z} \quad (4.4)$$

Note that we have used the fact that the velocity components satisfy $u = \lim_{\delta t \to 0} \delta x/\delta t$, $v = \lim_{\delta t \to 0} \delta y/\delta t$ and $w = \lim_{\delta t \to 0} \delta z/\delta t$.

The local change with time can then be mathematically identified with,

$$\frac{\partial s}{\partial t} = \frac{Ds}{Dt} - u \nabla s \quad (4.5)$$

in which we have introduced the velocity vector $\mathbf{u} = (u, v, w)$. This equation states that the local change in entropy results from the change following the parcel minus that due to advection. The view encapsulated in (4.5) is referred to as “Eulerian” (fixed location), as opposed to the “Lagrangian” (following a parcel) view in (4.2).

We used entropy in the previous example, but the concept applies to any variable, and even vector (applying the material derivative to each of its component). For example, if we use momentum,

$$\mathbf{a} = -\mathbf{g} - \mathbf{a} \nabla P + \mathbf{F}_{fric} = \frac{D\mathbf{u}}{Dt} = \lim_{\delta t \to 0} \frac{\delta \mathbf{u}}{\delta t} \quad (4.6)$$

This is the Navier-Stokes equation for fluid flow in an inertial frame of reference. Note that the $D\mathbf{u}/Dt$ is a vector with components $(Du/Dt, Dv/Dt, Dw/Dt)$. Each of those are non linear expressions, for example, for the third component of momentum,

$$\frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \quad (4.7)$$

showing that the upward motion carries its own upward momentum and that the latter is also carried by the horizontal flow.
4.1.3 Rotating frame of reference

The choice of the Earth as a frame of reference is natural. However, this is not an inertial frame because it is accelerating (rotating) with respect to a coordinate system fixed in space. Newton’s law can be applied in the non-inertial frame if the acceleration of the coordinates is taken into account, introducing “apparent forces” in the equation of motion.

Consider a vector $\mathbf{A}$ in a rotating frame (Fig. 4.2) with constant angular rotation rate $\Omega$. This vector is defined at point P. To start with, let us assume that P is fixed relative to the rotating frame, and that $\mathbf{A}$ is constant in that frame. Although $\mathbf{A}$ does not change in the rotating frame, it changes by $\delta \mathbf{A}$ in a small time interval $\delta t$ in an inertial frame (blue axes centred at $O$ in Fig. 4.2),

$$\mathbf{A}(t + \delta t) = \mathbf{A}(t) + \delta \mathbf{A}$$  \hspace{1cm} (4.8)

To make things more concrete, we take the origin $O$ of the inertial frame to be the Earth’s center. The latter is orbiting around the Sun but, neglecting tidal forces, this can be thought of as an inertial frame (exact same argument than when saying that a free falling lift can be taken as an inertial frame).

In the case considered in 4.2, $\delta \mathbf{A}$ is solely due to the rotation of point P as seen in the inertial frame.
4.1. EQUATIONS OF MOTIONS

P, and it is thus perpendicular to A (circular motion of P), and is also perpendicular to the axis of rotation,

\[ \delta A = \Omega \times A \delta t \]  \hspace{1cm} (4.9)

(to see this, it might be useful to decompose A into a component parallel to \( \Omega \) and a component orthogonal to \( \Omega \). As discussed in the lecture, only the latter is affected). Thus,

\[ \left( \frac{DA}{Dt} \right)_I = \lim_{\delta t \to 0} \frac{\delta A}{\delta t} = \Omega \times A \]  \hspace{1cm} (4.10)

in which the subscript \( I \) is used to make it clear this is a change viewed by an observer in the inertial frame. It is important to realize that the velocity in \( (DA/Dt)_I \) refers to the velocity in the inertial frame, i.e.,

\[ \left( \frac{D}{Dt} \right)_I = \frac{\partial}{\partial t} + u_I \frac{\partial}{\partial x} + v_I \frac{\partial}{\partial y} + w_I \frac{\partial}{\partial z} \]  \hspace{1cm} (4.11)

Consider now the more general case in which P is moving relative to the rotating frame and in which A also changes in the rotating frame (Fig. 4.3). There is still a change in A due to the rotation of the frame, to which we must add the change seen by P in the rotating frame (subscript \( R \)) to obtain the total change in \( \delta t \),

\[ \left( \frac{DA}{Dt} \right)_I = \left( \frac{DA}{Dt} \right)_R + \Omega \times A \]  \hspace{1cm} (4.12)

Applying (4.12) to the vector position \( A = r \) (measured from the centre of the Earth), one recovers the Galilean transformation of velocities between reference frames,

\[ u_I = u_R + \Omega \times r \]  \hspace{1cm} (4.13)

Applying it to \( A = u_I \) provides,

\[ \left( \frac{Du_I}{Dt} \right)_I = \left( \frac{Du_I}{Dt} \right)_R + \Omega \times u_I \]  \hspace{1cm} (4.14)

which, after using (4.13), reads,

\[ \left( \frac{Du_I}{Dt} \right)_I = \left( \frac{Du_R}{Dt} \right)_R + 2\Omega \times u_R + \Omega \times (\Omega \times r) \]  \hspace{1cm} (4.15)

Rearranging, and using (4.6), we finally obtain,

\[ \left( \frac{Du_R}{Dt} \right)_R = g - \alpha \nabla P + F_{frie} - 2\Omega \times u_R - \Omega \times (\Omega \times r) \]  \hspace{1cm} (4.16)
CHAPTER 4. ATMOSPHERIC MOTIONS

Figure 4.3: Same as Fig. 4.2 but in a more general case. The trajectory of P as seen in the inertial frame is the sum of the motion of the frame and the displacement as seen in the rotating frame (red dashed arrows).

4.1.4 Coriolis and centrifugal forces

Comparing (4.16) with (4.6), one sees two new forces on the r.h.s, namely the Coriolis and centrifugal forces. The latter only depends on the position of the parcel (like a conservative force), and in effect acts to reduce gravity. The Earth has adjusted to this force so that its shape is not exactly spherical (the radius at the equator is $\approx 21$ km larger than at the poles—the mean radius is 6371 km so this is a very small effect) but at right angle with the combined acceleration $g'$ defined as,

$$g' \equiv g - \Omega \times (\Omega \times r) \quad (4.17)$$

It is sometimes convenient to introduce a gravitational potential $\Phi$ such that,

$$g' \equiv -\nabla \Phi \quad (4.18)$$

We will in the following approximate surfaces of constant gravitational potential as spheres, with the net gravity given by $g'$.

The Coriolis force ($-2\Omega \times u_R$) is required to explain why an object moving in a straight line in an inertial frame appears to have a curved path in a rotating frame. (As will be apparent throughout this chapter, there is more to it than this.) For now, let’s just analyze its effect on the momentum equation (4.16). Adopting the coordinate system in Fig. 4.4 (see also the
4.1. EQUATIONS OF MOTIONS

Figure 4.4: Local coordinate system \((x, y, z)\). Longitude is denoted by \(\lambda\), latitude by \(\phi\).

sidenote below), \(\mathbf{u}_R = (u, v, w)\) and \(\Omega = \Omega(0, \cos \phi, \sin \phi)\) in which \(\phi\) is latitude and the three axes are \(\mathbf{i}\) (West to East), \(\mathbf{j}\) (South to North) and \(\mathbf{k}\) (anti-parallel with \(\mathbf{g}'\)). Thus the Coriolis force will have the components,

\[-2\Omega \times \mathbf{u}_R = -2\Omega[(w \cos \phi - v \sin \phi)\mathbf{i} + u \sin \phi \mathbf{j} - u \cos \phi \mathbf{k}] \quad (4.19)\]

Thus a parcel of air with a purely eastward velocity \((u > 0, v = w = 0)\) will be accelerated southward in the Northern hemisphere \((-u \sin \phi < 0)\) and northward in the Southern Hemisphere. The general rule is accelerated to the right of the motion in the Northern Hemisphere, and to the left in the Southern Hemisphere. You will also notice an upward acceleration, but this is pretty small in comparison to gravity. We will proceed to a more systematic analysis of the magnitude of each term in (4.16) in section 4.2.

Technical sidenote: local coordinate system. The \((x, y, z)\) coordinate system in Fig. 4.4 is local and cartesian. As such, it has some advantages compared to that of spherical geometry. Specifically,

\[\mathbf{u}_R = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} \quad (4.20)\]

with \(u = r \cos \phi \frac{D\lambda}{Dt}, v = r \frac{D\phi}{Dt}, w = \frac{Dr}{Dt}\). Here \(r = R + z\) in which \(R\) is the (mean) Earth’s radius, and since \(z \ll R\) for practical purposes, one can further simplify and use,

\[u \approx R \cos \phi \frac{D\lambda}{Dt}, v \approx R \frac{D\phi}{Dt}, w = \frac{Dz}{Dt} \quad (4.21)\]
This system of coordinate however has the complication that the \( i, j, k \) vectors vary with location, and as a result, \( (D\mathbf{u}_R/ Dt)_R \neq i(Du/ Dt)_R + j(Dv/ Dt)_R + k(Dw/ Dt)_R \). Rather, \( (D\mathbf{u}_R/ Dt)_R = i \left( \frac{Du}{Dt} \right)_R + j \left( \frac{Dv}{Dt} \right)_R + k \left( \frac{Dw}{Dt} \right)_R \). Rather, \( (D\mathbf{u}_R/ Dt)_R = i \left( \frac{Du}{Dt} \right)_R + j \left( \frac{Dv}{Dt} \right)_R + k \left( \frac{Dw}{Dt} \right)_R \).

We will accept the result that, 
\[
\begin{align*}
\left( \frac{Di}{Dt} \right)_R &= \frac{u \tan \phi}{R} \mathbf{j} - \frac{u}{R} \mathbf{k} \\
\left( \frac{Dj}{Dt} \right)_R &= -\frac{u \tan \phi}{R} \mathbf{i} - \frac{v}{R} \mathbf{k} \\
\left( \frac{Dk}{Dt} \right)_R &= \frac{u}{R} \mathbf{i} + \frac{v}{R} \mathbf{j}
\end{align*}
\]

As a result, 
\[
\begin{align*}
\left( \frac{Du_R}{Dt} \right)_R &= \left[ \frac{Du}{Dt} - \frac{u v \tan \phi}{R} + \frac{u w}{R} \right] \mathbf{i} + \left[ \frac{Dv}{Dt} - \frac{u^2 \tan \phi}{R} + \frac{v w}{R} \right] \mathbf{j} + \left[ \frac{Dw}{Dt} - \frac{u^2 + v^2}{R} \right] \mathbf{k}
\end{align*}
\]

As can be easily checked the extra-terms purely reflect the spherical geometry of the Earth and disappear in the limit \( R \to \infty \).

4.1.5 Mass conservation

Ignoring the loss of mass when it rains, and its gain when evaporation occurs above the Earth’s surface, the conservation of mass can be written as 
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}_R) = 0
\]
in which \( \rho = 1/\alpha \) is the density of an air parcel. This equation is sometimes called the “continuity equation”.

4.2 Scale analysis of the equation of motions

The momentum equation (4.16) is quite complicated and we would not go very far if we were not able to simplify it further. To do so we are going to use a technique similar to dimensional analysis in which we are going to put orders of magnitude on each terms in (4.16). We will ignore the effects of
friction and focus here on scales of motions typical of those seen on weather charts:

Lengthscale (horizontal) \( L \approx 10^6 \text{m} \) (a thousand kilometers)
Lengthscale (vertical) \( H \approx 10^4 \text{m} \) (thickness of the troposphere)
Velocity \( (u, v) \) \( U \approx 10 \text{ms}^{-1} \)
Velocity \( (w) \) \( W \approx 10^{-2} \text{ms}^{-1} \)
Time \( (t) \) \( T = L/U \approx 10^5 \text{s} \) (about one day)
Horizontal Pressure gradient \( \nabla_L P \approx 10 \text{hPa}/1000 \text{km} = 10^{-3} \text{Pam}^{-1} \)

Remember also that \( R = 6371 \text{km} \) and \( 2\Omega = 4\pi/1 \text{day} \approx 2 \times 10^{-4} \text{s}^{-1} \).

4.2.1 Vertical momentum equation

The vertical component of (4.16) is, using (4.26),

\[
\left( \frac{Dw}{Dt} \right)_R = \frac{u^2 + v^2}{R} + 2\Omega u \cos \phi - \alpha \frac{\partial P}{\partial z} - g \tag{4.28}
\]

in which we have dropped \( g' \) for \( g \) and have ignored the friction force. The first two terms on the r.h.s scale respectively as, \( U^2/R \) and \( 2\Omega U \cos \phi \). Their strength relative to the acceleration of gravity are \( U^2/gR \approx 10^{-6} \) and \( 2\Omega U \cos \phi/g \approx 2 \times 10^{-4} \cos \phi \), respectively. The l.h.s has several components since,

\[
\left( \frac{Dw}{Dt} \right)_R = \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \tag{4.29}
\]

The first three have the same scale (since we have chosen \( T = L/U \)), namely \( W/T \), while the last term scales as \( W^2/H \). The ratio of these to gravity is thus \( W^2/gT \approx 10^{-8} \) and \( W^2/gH = 10^{-9} \), respectively. Note that because of the values listed above, \( U/L \approx 10W/H \) so from now on we will simply scale all \( D/Dt \) terms by simply dividing by \( T \) (i.e., \( (Dw/Dt)_R \approx W/T \)).

The bottom line is that all the terms considered are negligible compared to the acceleration of gravity and so, to a very good approximation, we can safely use,

\[
0 \approx -\alpha \frac{\partial P}{\partial z} - g \tag{4.30}
\]

This is the “hydrostatic equation” introduced earlier in Chapter 3, expressing a near cancellation between the acceleration due to gravity and that due to the vertical pressure gradient force.

To some extent the above scaling is misleading. It makes sense that gravity opposes the vertical pressure gradient, but the motions we are considering
consist of pressure fluctuations superimposed on this state of pure hydrostatic equilibrium. (In other words, the pressure fluctuations used above, i.e., $10hPa$ has nothing to do with the much larger pressure changes between surface and tropopause which is $\approx 750hPa$). What we should really check is whether this perturbed state satisfies the hydrostatic approximation. To do this, write

$$\alpha = \bar{\alpha} + \alpha' \quad \text{and} \quad P = \bar{P} + P' \quad (4.31)$$

where the bar denotes the background state in hydrostatic balance and the prime the fluctuations associated with weather systems. One can readily obtain that,

$$\alpha \frac{\partial P}{\partial z} + g = \bar{\alpha} \frac{\partial P'}{\partial z} - g \frac{\alpha'}{\bar{\alpha}} \quad (4.32)$$

Putting orders of magnitude on each term, we find,

$$\frac{\alpha}{\bar{\alpha}} \frac{\partial P'}{\partial z} \approx (1kgm^{-3}) \frac{10hPa}{10km} = 10^{-1}ms^{-2} \quad (4.33)$$

and

$$-g \frac{\alpha'}{\bar{\alpha}} \approx -g \frac{\theta'}{\theta} \approx 9.81 \times \frac{3}{300} = 10^{-1}ms^{-2} \quad (4.34)$$

These terms are still much larger than all the others in (4.28) and so the hydrostatic approximation also applies to dynamic, as opposed to purely static situations,

$$0 \approx \alpha \frac{\partial P'}{\partial z} - g \frac{\alpha'}{\bar{\alpha}} \quad (4.35)$$

Most climate models use this approximation. (Among other things it allows a change of vertical coordinate $z \rightarrow P$ which simplifies considerably the model numerics).

### 4.2.2 Horizontal momentum equation

The two components of the horizontal momentum equations take the form,

$$\left( \frac{Du}{Dt} \right)_R = -\frac{uw}{R} + \frac{uw \tan \phi}{R} + 2\Omega v \sin \phi - 2\Omega w \cos \phi - \alpha \frac{\partial P}{\partial x} \quad (4.36)$$

$$\left( \frac{Dv}{Dt} \right)_R = -\frac{vw}{R} - \frac{u^2 \tan \phi}{R} - 2\Omega u \sin \phi - \alpha \frac{\partial P}{\partial y} \quad (4.37)$$

Scaling these terms, we obtain (from left to right): $U/T \approx 10^{-4}$, $UW/R \approx 10^{-8}$, $U^2/R \approx 10^{-5}$, $\Omega U \sin \phi \approx 10^{-3} \sin \phi$, $\Omega W \cos \phi \approx 10^{-6} \cos \phi$ and $\alpha \nabla_L P \approx 10^{-3}$ (using the surface value for $\alpha \approx 1m^3/kg$) for the zonal
momentum equation. Likewise, for the meridional momentum equation: \( U/T \approx 10^{-4} \), \( UW/R \approx 10^{-8} \), \( U^2/R \approx 10^{-5} \), \( \Omega U \sin \phi \approx 10^{-3} \sin \phi \), and \( \alpha \nabla_L P \approx 10^{-3} \).

Looking at these numbers it becomes clear that, as long as we are not too close to the equator where \( \sin \phi = 0 \), the approximate form of the horizontal momentum equation is,

\[
0 \approx +2\Omega v \sin \phi - \alpha \frac{\partial P}{\partial x} \tag{4.38}
\]

\[
0 \approx -2\Omega u \sin \phi - \alpha \frac{\partial P}{\partial y} \tag{4.39}
\]

This approximation is called the geostrophic approximation, expressing a near cancellation between the acceleration due to the horizontal pressure gradient and Coriolis forces. One way to “visualize” this balance is that air parcels tend to flow along lines of constant pressure (anticlockwise around a low pressure system in the extra-tropics in the Northern Hemisphere).

After these two terms the next larger one is \((Du/Dt)_R\) for the zonal momentum equation and \((Dv/Dt)_R\) for the meridional momentum equation. Both terms scale as \( U/T = U^2/L \). How small is this term compared to the Coriolis force is measured by a non dimensional number called the Rossby number \((R_o)\),

\[
R_o \equiv \frac{U}{2\Omega L} \tag{4.40}
\]

In other words, the smaller the Rossby number, the closer we are to geostrophic balance. For the scales considered at the beginning of section 4.2, \( R_o \approx 0.1 \).

### 4.2.3 The thermal wind relation

The smallness of the Rossby number is the main reason why the Earth rotation has so much influence on atmospheric motions. We have everything at hand to see straight away one of the powerful constraints resulting from \( R_o \ll 1 \), namely a constraint on how motions vary with height.

All we need are the approximate momentum equations (4.30), (4.38) and (4.39). Take the vertical derivative of (4.38), yielding,

\[
f \frac{\partial v}{\partial z} \approx \frac{\partial}{\partial z} \left[ \alpha \frac{\partial P}{\partial x} \right] \tag{4.41}
\]

in which we have introduced the Coriolis parameter,

\[
f \equiv 2\Omega \sin \phi \tag{4.42}
\]
Using (4.30), this can be rewritten as,

\[
f \frac{\partial v}{\partial z} \approx \frac{\partial \alpha}{\partial z} \frac{\partial P}{\partial x} - \frac{\partial \alpha}{\partial x} \frac{\partial P}{\partial z}
\]  

(4.43)

Likewise,

\[
f \frac{\partial u}{\partial z} \approx -\frac{\partial \alpha}{\partial z} \frac{\partial P}{\partial y} + \frac{\partial \alpha}{\partial y} \frac{\partial P}{\partial z}
\]  

(4.44)

or, in vector form,

\[
f \frac{\partial \mathbf{V}}{\partial z} \approx (\nabla \alpha \times \nabla P)_H
\]  

(4.45)

in which \( \mathbf{V} = u \mathbf{i} + v \mathbf{j} \) is the horizontal velocity vector and the subscript \( H \) indicates we only consider the horizontal component.

The terms on the r.h.s of (4.45) vanish if \( \alpha = \alpha(P) \). That is, in such condition, the horizontal velocity field cannot vary with height! This spectacular prediction has been tested and we will come back to it in the subsection on “Taylor’s column”. More realistically, in the atmosphere \( \alpha = \alpha(T, P) \) and the terms on the r.h.s do not vanish. Rather they set a constraint on how much the horizontal wind can vary with height.

A geometrical derivation brings more insight into these relations. Using the triple product rule for mixed derivative and the hydrostatic approximation, one has,

\[
\alpha \left( \frac{\partial P}{\partial x} \right)_z = \alpha \left( \frac{\partial z}{\partial x} \right)_P \left( \frac{\partial P}{\partial z} \right)_x = -g \left( \frac{\partial z}{\partial x} \right)_P
\]  

(4.46)

which relates the pressure gradient to the slope of a pressure surface. As a result, the difference in meridional wind \( (v) \) between two heights \( (z_1 < z_2) \) is, using the geostrophic balance,

\[
v_2 - v_1 = \frac{g}{f} \left( \frac{\partial z}{\partial x} \right)_{P_1} - \frac{g}{f} \left( \frac{\partial z}{\partial x} \right)_{P_2}
\]  

(4.47)

Likewise, for the zonal wind,

\[
u_2 - u_1 = \frac{g}{f} \left( \frac{\partial z}{\partial y} \right)_{P_2} - \frac{g}{f} \left( \frac{\partial z}{\partial y} \right)_{P_1}
\]  

(4.48)

These relations show that variations with height of the geostrophic wind reflect horizontal gradients in the thickness between two pressure surfaces (Fig. 4.5). The thickness is itself directly related to temperature since the hydrostatic equation can be rewritten, using the ideal gas law (and neglecting moisture), as,

\[
\frac{\partial \ln P}{\partial z} = -\frac{g}{R_d T}
\]  

(4.49)
4.3 Available Potential Energy

The thermal wind relation allows us to understand why the potential temperature distribution in Fig. 3.2 is so different at low and at mid-to-high latitudes, and why, in Chapter 3, kinetic energy can be gained by swapping air parcels in slanted motions in mid-to-high latitudes, and by vertical motions at low latitudes.
To see this, write $\alpha = \alpha(\theta, P)$ since any thermodynamic variable can be expressed as a function of two others. Then (4.45) becomes,

$$f \frac{\partial V}{\partial z} \approx (\nabla \theta \times \nabla P)_H \left( \frac{\partial \alpha}{\partial \theta} \right)_P.$$  

(4.51)

As one approaches low latitudes, $f$ becomes smaller and smaller (as latitude $\phi$ goes to zero, so does $f \propto \sin \phi$), and so does the l.h.s of this equation. As a result the $\theta$ surfaces must become more and more aligned with pressure surfaces as one moves from the poles towards the equator – this is in agreement with the distribution in Fig. 3.2.

From an energy point of view, we know from the discussion in section 3.6 (Chapter 3) that such sloping $\theta$-surfaces store potential energy. This is rather counter-intuitive: if you fill a bucket with a wall in the middle with light fluid on one side of the partition, and dense fluid on the other, and then remove the partition, the light fluid will spread over the dense one (Fig. 4.6). The thermal wind relation says no, this will not happen. Instead, light fluid will remain partly over the dense fluid, the wall (although gone) still seeming to be there (Fig. 4.6)! Fluid dynamicists say that rotation confer a rigidity to fluid motions and this is one illustration of this more general statement.

In the thought experiment schematized in Fig. 4.6, the center of mass is higher in the final state with rotation than it is in the final state without rotation. Some potential energy is “available”. In a similar line of thought, the center of mass was even higher in the initial state, so in the presence of rotation, some potential energy has been released. Where has this energy gone? In the kinetic energy of the jets predicted by (4.45).

### 4.4 The vorticity view

#### 4.4.1 The geostrophic flow, vorticity and divergence

The approximations derived in section 4.2 are very useful and allow to understand the basic structure of atmospheric motions. However, you might have noticed that they do not include time derivatives. In other words, they are just diagnostic relationships. Were you to forecast the weather based on them, you would go nowhere!

The breakthrough to achieve this came in the 1950s when it was realized that the geostrophic flow is essentially rotational: it just goes around pressure centers (cyclones or anticyclones) but does not cross much the pressure lines. And so it became clear that what needs to be forecast is not momentum but
4.4. THE VORTICITY VIEW

Figure 4.6: A thought experiment illustrating how rotation “stores” potential energy. Initially (left panel) a bucket is filled with light (blue) and dense (green) fluid. The two are separated by a wall. Then one removes the wall. If the bucket is not rotating, then, as one expects, light fluid spreads over dense fluid (top right panel). If the bucket is rotating however, the spreading is, somewhat counter-intuitively, “halted” (bottom right panel). Some of the initial potential energy has been converted to time mean kinetic energy in the presence of rotation (the “thermal wind indicated on the lower right panel).
vorticity $\zeta$, the rotational (or curl) of the velocity field,

$$\zeta \equiv \nabla \times \mathbf{u}_R$$  \hfill (4.52)

Physically, vorticity measures the amount of spin that a fluid parcel has. It can be visualised using a paddle, with the local rotation vector being equal to $\zeta/2$ (to get this you would need to do a Taylor expansion of the relative velocity between two fluid parcels—we will not go into that).

To see the importance of vorticity, simply consider its vertical component ($\zeta$) in the local coordinate system,

$$\zeta \equiv \zeta \cdot \mathbf{k} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$  \hfill (4.53)

On the scales considered in section 4.2, one can, for practical purposes further simplify the geostrophic balance as,

$$v \approx \frac{\alpha_o}{f_o} \frac{\partial P}{\partial x}, \quad \text{and} \quad u \approx -\frac{\alpha_o}{f_o} \frac{\partial P}{\partial y}$$  \hfill (4.54)

in which $\alpha_o$ and $f_o$ are typical values of the specific volume and the Coriolis parameter over the region of interest. From this,

$$\zeta \approx \frac{\alpha_o}{f_o} \left( \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} \right)$$  \hfill (4.55)

while, the horizontal divergence of the (relative) flow $\delta$ is,

$$\delta \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \approx 0$$  \hfill (4.56)

Thus indeed, the geostrophic is close to being purely rotational, i.e., containing vorticity but no divergence. For such flows a streamfunction $\psi$ can be introduced from the definition,

$$u \equiv -\frac{\partial \psi}{\partial y} \quad \text{and} \quad v = \frac{\partial \psi}{\partial x}$$  \hfill (4.57)

Comparison of (4.54) with (4.57) shows that $\psi \approx \alpha_o P/f_o$: the horizontal flow follows approximately lines of constant pressure.

### 4.4.2 The vorticity equation*

To obtain an equation for the vorticity vector $\zeta$ (all components, not only its vertical component), all you need is to take the curl of (4.16), and use
vector identities. To do this in an efficient way though we need to a little bit of preparatory work. First, we rewrite (4.16) as,

$$\frac{\partial \mathbf{u}_R}{\partial t} + (\mathbf{u}_R \cdot \nabla) \mathbf{u}_R = -\nabla \Phi - \alpha \nabla P - 2\Omega \times \mathbf{u}_R + \mathbf{F}_{fric} \quad (4.58)$$

Then we use the following vector identity,

$$\nabla (\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) \quad (4.59)$$

to obtain,

$$\frac{\partial \mathbf{u}_R}{\partial t} + (2\Omega + \zeta) \times \mathbf{u}_R = -\nabla (\Phi + \mathbf{u}^2/2) - \alpha \nabla P + \mathbf{F}_{fric} \quad (4.60)$$

The quantity $2\Omega + \zeta$ that appears in this equation requires a physical interpretation. It is the sum of the vorticity of the relative flow ($\zeta$) and the vorticity of the solid body rotation at angular velocity $\Omega$. We will call it from now on the absolute vorticity,

$$\zeta_a \equiv 2\Omega + \zeta \quad (4.61)$$

and measures the total local spin (planetary + relative) of a fluid parcel. Typically in a cyclone, the relative and planetary vorticity add up, so the total spin can be quite large. They tend to cancel each other in anticyclones.

You might be surprised that a solid body rotation leads to vorticity. To see this, acknowledge that the velocity of the Earth’s solid body rotation is $\Omega \times \mathbf{r} = \Omega r \cos \phi \hat{i}$, so that it is a function of latitude and radius $r = R + z$. As discussed in the lecture, the gradient (or shear) in velocity associated with the variations with latitude ($\phi$) adds up to the shear introduced by the variations with $r$ to produce a planetary vorticity vector always parallel to $\Omega$. To be more quantitative one needs to do the math properly. A simple derivation is to compute $\nabla \times (\Omega \times \mathbf{r})$ in a Cartesian coordinate system centred at the Earth’s core (since the curl of a vector must be independent of the coordinate system used to compute it, we might as well use the simplest coordinate system –we’ll denote it by $i’, j’, k’$, the latter being parallel to $\Omega$). Using the vector identity,

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A} (\nabla \cdot \mathbf{B}) - \mathbf{B} (\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla) \mathbf{A} - (\mathbf{A} \cdot \nabla) \mathbf{B} \quad (4.62)$$

this becomes $\nabla \times (\Omega \times \mathbf{r}) = \Omega (\nabla \cdot \mathbf{r}) - \mathbf{r} (\nabla \cdot \Omega) + (\mathbf{r} \cdot \nabla) \Omega - (\Omega \cdot \nabla) \mathbf{r}$. The second and third terms are zero because $\Omega$ is a constant vector. Since $\nabla \cdot \mathbf{r} = 3$ and $(\Omega \cdot \nabla) \mathbf{r} = \Omega \partial (zk’)/\partial z = \Omega$, we get $\nabla \times (\Omega \times \mathbf{r}) = 3\Omega - \Omega = 2\Omega$. 

We are now ready to take the curl of (4.60). Note first that the first term
on the r.h.s. of (4.60) will not contribute since \( \nabla \times \nabla = 0 \). So we simply
have,

\[
\frac{\partial \zeta_a}{\partial t} + \nabla \times (\zeta_a \times u_R) = \nabla \times (-\alpha \nabla P + F_{\text{fric}})
\]  (4.63)

where we have also used \( \partial \zeta_a/\partial t = \partial \zeta/\partial t \). This can be simplified further by
using (4.62) to produce,

\[
\left( \frac{D \zeta_a}{Dt} \right)_R + \zeta_a (\nabla \cdot u_R) - (\zeta_a \cdot \nabla) u_R = \nabla \times (-\alpha \nabla P + F_{\text{fric}})
\]  (4.64)

At first sight this looks more complicated, but using the continuity equation
(4.27), the first two terms on the l.h.s combine to produce,

\[
\left( \frac{D \zeta_a}{Dt} \right)_R + \frac{\zeta_a}{\rho} (\nabla \cdot u_R) = \nabla \times (-\alpha \nabla P + F_{\text{fric}})/\rho
\]  (4.65)

The first term on the r.h.s can also be simplified by using the vector identity,

\[
\nabla \times (A \nabla B) = \nabla A \times \nabla B
\]  (4.66)

to produce finally,

\[
\left( \frac{D \zeta_a}{Dt} \right)_R = \frac{\zeta_a}{\rho} \nabla \times \left( \nabla \alpha \nabla P + \nabla \times F_{\text{fric}} \right)/\rho
\]  (4.67)

Let us step back from all this Math. What we have arrived at is a powerful
statement regarding how the absolute vorticity is changed following a fluid
parcel. The first term on the r.h.s shows that any velocity gradient along
\( \zeta_a \) will act to change the absolute vorticity of the parcel: a stretching, i.e.
(\( \zeta_a \cdot \nabla) u_R > 0 \), leading to an increase in vorticity (this is the “ballerina effect”
for fluid motions –see the lab demo at \( t \approx 14m\text{min}.15s \) in the Dave Fultz’s
Youtube video on Blackboard). Interestingly, the effect is proportional to \( \zeta_a \)
itself so cyclones are more likely to get stronger than anticyclones: this is
the main reason why we hear a lot about cyclones in the news but not so
much about anticyclones! The term \( (\zeta_a \cdot \nabla) u_R \) also allows for a bending of
the absolute vorticity vector by relative motions, not solely a stretching, and
we will see that this allows us to recover the “thermal wind” relation.

The second and third terms on the r.h.s show how absolute vorticity can
be created, as opposed to being intensified or damped. This can happen when
surfaces of constant \( \alpha \) and \( P \) cross each other: say warm and cold air on a
given pressure surface. This effect is very intuitive and simple although the
name given to it (“baroclinicity”) is not: just take another look at Fig. 4.6)
where a box with light and dense fluid is separated by a vertical wall; remove the wall; the light fluid spreads over the dense fluid. This is it. Finally, the last term simply reflects the spin introduced by friction. Anyone who has rowed has noticed how vortices are shed from an oar.

Equation (4.67) can be used to shed light on phenomena so diverse as the Hadley cell, tornadoes, dust devils, planetary waves, etc. This is what we will do in the next subsections. Maybe the most striking and most counter-intuitive prediction of (4.67) is also the least well known: the “Taylor column effect”. We will start by this.

4.4.3 Taylor columns

The experiment described in the lecture shows the astonishing result that when a homogeneous fluid \((\rho = c_{st})\) in rotation encounters an obstacle, the whole fluid column “avoids” the obstacle. This rigid behaviour is called “Taylor column” after the fluid dynamicist G. I. Taylor who published the experimental results in 1923 (the mathematical proof had been available for a while but it was so astonishing to Taylor that he did not believe the effect could actually be observed—hence his experimental study).

Consider the case in which we neglect friction, and take a non baroclinic fluid (i.e., one with \(\alpha = \alpha(P)\) so that the baroclinic term vanishes). We also simplify the geometry by thinking about a large rotating tank, whose axis of rotation coincides with the vertical. The rotation rate of the tank is taken as \(\Omega\). The vertical component of eq. (4.67) applied to this case produces,

\[
\frac{D}{Dt}\left(\frac{2\Omega + \zeta}{\rho}\right) = \left(\frac{2\Omega + \zeta}{\rho}\right)\frac{\partial w}{\partial z} \tag{4.68}
\]

Note here that \(\zeta = \zeta \cdot k\) denotes the vertical component of the relative vorticity introduced in section 4.4.1,

\[
\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{4.69}
\]

We see that it scales like \(U/L\) where \(U\) is the typical magnitude of the relative velocity and \(L\) a typical horizontal scale (e.g., the tank size). Hence the ratio of relative to planetary vorticity is the Rossby number (4.40),

\[
R_o = \frac{U}{2\Omega L} \propto \frac{\zeta}{2\Omega} \tag{4.70}
\]

As a result, if the tank rotates fast enough (small Rossby number), (4.68) can be further simplified as,

\[
\frac{D}{Dt}\left(\frac{2\Omega}{\rho}\right) \approx \frac{2\Omega}{\rho} \frac{\partial w}{\partial z} \tag{4.71}
\]
Since $\Omega$ is a constant, we are led to
\[
\frac{\partial w}{\partial z} \approx 0 \quad (4.72)
\]
which, since $w = 0$ at the free surface of the tank, leads further to,
\[
w \approx 0 \quad (4.73)
\]
This means that the motion must be limited to regions where the bottom of
the tank is flat: the fluid above the obstacle is stagnant (a motion above the
obstacle would have $w > 0$). You can see a lab demo of this at $t \approx 12mn40s$
in the Dave Fultz’s Youtube video posted on Blackboard.

4.4.4 The Hadley cell and the subtropical jet

Figure 4.7 shows the presence of several “cells” in the latitude-height plane.
The strongest of these (the Hadley cells –see Chapter 1) extend from about
the equator to $30^\circ$ of latitude. Focusing for example on the one seen in the
Northern Hemisphere, the motion is equatorward at low levels and poleward
at upper levels, indicating the presence of vorticity in the $-\mathbf{i}$ direction (right
hand rule).

The center of the Northern Hemisphere Hadley cell is well within the
troposphere, with a maximum at about $500hPa$. Of the two terms generating
vorticity in (4.67), friction is thus unlikely to be the main source and
the baroclinic generation must play a role. To check this, let’s compute its
component in the $\mathbf{i}$ direction and check that it is negative,
\[
- \left( \frac{\nabla \alpha \times \nabla P}{\rho} \right) \cdot \mathbf{i} = - \left( \frac{\partial \alpha}{\partial \theta} \right)_{\rho} \left( \frac{\nabla \theta \times \nabla P}{\rho} \right) \cdot \mathbf{i} \quad (4.74)
\]
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We can see that this is indeed in the negative $i$ direction by referring to Fig. 3.2 (Chapter 3). In this figure we see that $\nabla P$ is downward while $\nabla \theta$ is upward and equatorward: the cross product is thus towards $+i$ direction and so the baroclinic term will indeed be towards the $-i$ direction since $(\partial \alpha / \partial \theta)_P > 0$ (you can easily convince yourself of this using the ideal gas law and the definition of $\theta$ in Chapter 3).

The main driver of the Hadley cell is thus the baroclinicity associated with high $\theta$ near the equator and lower $\theta$ in the subtropics. This distribution of $\theta$ itself arises as a result of the radiative processes studied in Chapter 2: because of the latitudinal gradient of solar radiation, but also because of the strong “longwave cooling to Space” at work in the subtropics, and the larger greenhouse effect in the ascending branch of the Hadley cell. These last two features reflect the sharp contrast in moisture between the moist ascending branch and dry descending branch of the Hadley cell, so there is an interesting coupling between dynamics and radiation there, mediated by water vapour.

The baroclinic term is constantly generating vorticity in the $-i$ direction so something else must be generating vorticity in the $+i$ direction to maintain a steady state Hadley cell. If we look at other terms in (4.67), the simplest (in the sense of being linear) is the bending of the planetary vorticity $2\Omega$ by the zonal flow,

$$\left(\frac{\zeta}{\rho} \cdot \nabla\right) u_{R,i} \approx \frac{f}{\rho} \frac{\partial u}{\partial z}$$  (4.75)

[You can “experiment” with this term by (i) simply holding a pen upright and rotating it (ii) tilting it forward. In doing so you have transformed upward vorticity into horizontal vorticity.] You might remember from Chapter 1 that indeed in the subtropics one observes Trade winds at low levels ($u < 0$) and a Jet Stream aloft ($u > 0$), hence $\partial u / \partial z > 0$ and vorticity in the $+i$ direction is generated by the bending. There is thus a tight link between the strength of the Trade winds, the subtropical jet and the strength of the Hadley cell.

To finish this section, you might find that a balance between (4.74) and (4.75) reminds you of something. You would be right...this is nothing else than the thermal wind!

4.4.5 Rossby waves

A large class of motions of the atmosphere are loosely defined as being of the “planetary wave” type, to indicate that they are of sufficiently large spatial scale that they are affected by the fact that the Earth rotation vector $\Omega$ has a projection on the local vertical which varies with latitude. An example is given in Fig. 4.8, showing the height of the 500hPa surface for
Figure 4.8: The height (in km) of the 500hPa pressure surface on 10/02/2014. Black contours indicate its absolute values while the colours (in m) denote anomalies compared to the long time mean. From http://www.cpc.ncep.noaa.gov/products/precip/CWlink/MJO/block.shtml.

the Northern Hemisphere on 10 February 2014—this approximately provides a streamfunction for the flow at mid-tropospheric levels from the discussion in section 4.4.1 and the derivative relation (4.46). A wavenumber 2 structure is clearly apparent in the east-west direction, spreading from the North Pole to 30°N.

The Rossby number for these motions is particularly low \( R_o = U/2\Omega L \approx \frac{10}{(2 \times 7.2 \times 10^{-5} \times 14,000 km)} = 0.005 \) (using a wavenumber 2 at 45°N) so that they satisfy the geostrophic balance and are nearly purely rotational (section 4.4.1). A lot of insight can be gained into these motions by considering the vertical component of (4.67) after making a few assumptions:
(i) neglect friction. This is reasonable since we are looking at motions at upper levels, i.e., away from the Earth’s surface.

(ii) neglect the baroclinic generation of vorticity along the vertical axis. This is reasonable because we are looking at motions of very large scale, hence the $|\nabla \alpha|$ and $|\nabla P|$ become small.

(iii) neglect vertical motion. This is reasonable because we are looking at motions near the tropopause, which, because of the large buoyancy contrast across it, tends to act as a rigid lid. As a result, $[(\zeta_a \nabla) \mathbf{u}_R] \cdot \mathbf{k} = (\zeta_a \nabla)w = 0$.

Under these approximations, the vertical component of vorticity $\zeta_a$, satisfy,

$$
\left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) \zeta_a = 0
$$

(4.76)

Note that $\zeta_a$ has not only the relative component introduced in (4.55) but also a planetary component,

$$
\zeta_a = \zeta + f
$$

(4.77)

after using the definition (4.61). Equation 4.76 states that vorticity “sticks” to air parcels: you can tell the origin of a given air mass from looking at its absolute vorticity (e.g., a low value of $\zeta_a$ typically originates from low latitudes)! A very powerful tool.

The equation (4.76) supports linear waves called Rossby waves after the Swedish meteorologist Carl Gustav Rossby who highlighted their dynamics in the 1950s. Take the simplest possible case, i.e., linear motion on a state of rest. Then (4.76) can be rewritten as,

$$
\frac{\partial \zeta'}{\partial t} + \beta v' = 0
$$

(4.78)

in which primes indicate perturbations and

$$
\beta \equiv \frac{df}{dy}
$$

(4.79)

captures the latitudinal variation of the projection of $\Omega$ onto the local vertical. This equation can be solved by using the streamfunction $\psi$ introduced in (4.57) and looking for plane wave solution,

$$
\psi' \propto e^{i(kx+ly-\omega t)}
$$

(4.80)
This provides the dispersion relation,

\[ \omega = -\frac{\beta k}{k^2 + l^2} \]  

(4.81)

and the associated phase velocity \( v_P \),

\[ v_P = -\frac{\beta}{k^2 + l^2} (1, k/l) \]  

(4.82)

The frequency of Rossby waves is typically much lower than that of the weather system we experience nearly daily. For example, for the observations in Fig. 4.8, I found \( \omega \approx 2\pi/43 \text{days}^{-1} \) (using a wavenumber 2 at 45°N for \( k \) and \( l = 2\pi/3000 \text{km} \)). No wonder why these waves are studied tremendously since, owing to their long timescales, they introduce predictability to our weather.

Rossby waves display a myriad of other interesting features:

• their phase speed is always westward \( (\omega/k < 0) \). This follows directly from the conservation of vorticity (4.76), as illustrated in Fig. 4.9.

• they are associated with a transverse motion of air parcels. This is illustrated in Fig. 4.10, where the velocity vector (blue arrows) is aligned along lines of constant pressure (by geostrophy) while the phase speed (black arrow) is at right angle (you can see this from the \( x \) and \( y \) components of \( v_P \), which make this vector at right angle with a line of constant \( kx + ly - \omega t \) at given time \( t \)). This is very different from other types of waves supported by fluids (acoustic, gravity waves, i.e., swell) which involve longitudinal motions. Again, this is a behaviour closer to that of solids (e.g., shearwaves in the solid Earth generated by earthquakes). As you can see, this all has to do with \( \beta \neq 0 \).

• they carry east-west momentum across latitude circles. This property is illustrated in Fig. 4.10. Consider the portion of a latitude circle corresponding to a zonal wavelength (marked by the red \( a-b-c \) line in the figure). At \( a \), a fluid parcel has \( u' > 0 \) and \( v' > 0 \), hence it carries eastward momentum northward. At \( b \), the parcel goes southward \( (v' < 0) \) but it has a negative eastward momentum \( (u' < 0) \), hence again, it carries eastward momentum northward (removing negative eastward momentum from a region is like adding positive eastward momentum to it). At \( c \), we have the repeat of what occurs at \( a \). So averaged over a zonal wavelength, there is a net transport of eastward momentum northward. Mathematically this is expressed as a systematic positive
correlation between $u'$ and $v'$: averaged over a zonal wavelength (east-west direction), the product $u'v'$ has a sign proportional to $-kl$ (simply take $\psi' \propto \sin(kx+ly-\omega t)$ and check that $u'v' \propto -k\cos^2(kx+ly-\omega t)$).

The bottom line is that Rossby waves carry zonal momentum with them: this is the mechanism linking trade winds and westerlies that was mentioned in Chapter 1.

### 4.5 Weather systems in midlatitudes

To finish this chapter we return to the idea that extra-tropical storms feed on available potential energy by “swapping” air parcels between low and high latitudes (sloping convection, Chapter 3, section 3.6). Now armed with basic fluid dynamics of the atmosphere we can explore this idea further and see if it can tell us something about the observed structure (horizontal scale $L$ and vertical scale $H$) of the storms. We’ll ignore entirely the presence of water vapour$^1$.

We saw in section 3.6 that kinetic energy will be gained in the process of swapping low and high latitude air parcels if they are exchanged at an angle shallower than the slope of the mean $\theta$ surfaces (=isentropes). Mathematically this reads,

$$\frac{W}{U} \leq \mu_\theta \equiv \frac{\partial \bar{\theta}}{\partial y}/\frac{\partial \bar{\theta}}{\partial z}$$

(4.83)

where $\bar{\theta}$ refers to the time and zonal (i.e., averaged along a latitude circle) mean potential temperature and $W$ and $U$ denote the scale for upward and horizontal velocities of the storm. To bring in explicitly the horizontal scale $L$ of the storm in this inequality, consider the scaling of the continuity equation (4.27),

$$\frac{\rho}{T} + \frac{\rho U}{L} + \frac{\rho W}{H} = 0$$

(4.84)

The “sloping convection” view from Chapter 3 suggests that the motion of air parcels is fully three dimensional, with both ascent/descent and poleward/equatorward motion. At a fixed location, this motion will lead to changes in density, reflecting the different origin of air masses and the amount

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$^1$Including the effect of latent heat release in a simple theory for extra-tropical storms is a hot topic of current research. The idea for the “dry” derivation below was suggested by Prof Raymond Hide from the Math Department. You may thank him because the standard “dry” derivation, using something called “quasi-geostrophic potential vorticity”, is way harder!
CHAPTER 4. ATMOSPHERIC MOTIONS

Figure 4.9: Schematic of the propagation mechanism for Rossby waves in the Northern Hemisphere ($x-$axis from west to east, $y-$axis from south to north). The horizontal line indicates lines of constant planetary vorticity ($f$), increasing towards the $+y$ direction ($f_1 < f_2 < f_3$). In presence of an anticyclone, these lines are bend and, for the case shown in the figure, low latitudes air parcels move northward to the west of the anticyclone, and high latitudes air parcels move equatorward to the east (black arrows). Assuming that these parcels did not possess significant relative motion before they were disturbed by the anticyclone (i.e., their initial absolute vorticity is simply $f$), the conservation of their absolute vorticity $\zeta + f$ as they move requires that a region of $\zeta < 0$ develops to the west of the anticyclone (blue patch), and that a region of $\zeta > 0$ develops to the east of the anticyclone (red patch). As a result the anticyclone appears to move westward (an anticyclone corresponds to a region of $\zeta < 0$), and creates a cyclone in its lee (a cyclone corresponds to $\zeta > 0$). You can easily see from there why a wave of anticyclone/cyclones would naturally propagate westward.
Figure 4.10: Schematic of the phase lines (blue) for a Rossby wave with $k > 0, l < 0$ in the Northern Hemisphere (same axes as in previous figure). The flow (blue arrows) is clockwise around the high pressure (H) and anti-clockwise around the low pressure (L). The phase velocity (black arrow) is perpendicular to these motions. For the case considered here, the transport of east-west momentum is to the North (green upward arrow).
of work they have done/received with their surroundings. So it is fair to assume that all terms contribute equally, i.e., \( T = U/L = W/H \). Remember that in this equation \( U \) refers to the velocity scale for either \( u \) or \( v \). So at first sight it looks like \( W = UH/L \). Our discussion of the nearly non divergent nature of the geostrophic flow in section 4.4.1 however suggests that \( W \ll UH/L \) since \( \partial u/\partial x + \partial v/\partial y \approx 0 \). We will accept that the appropriate scaling is,

\[
W = R_o UH/L
\]

where the Rossby number \( R_o = U/fL \ll 1 \) was introduced in section 4.2.2. Introducing this scaling into the above inequality, we obtain

\[
L^2 \geq \frac{UH}{f} \frac{\partial \varphi}{\partial y} \frac{\partial \varphi}{\partial z}
\]

We are now just going to manipulate this further by using the thermal wind relation. First introduce the Brunt-Vaisala frequency \( N^2 \) from eq. (3.16) to obtain,

\[
L^2 \geq \left( \frac{NH}{f} \right)^2 \frac{fU}{H} \left( \frac{g}{\theta} \frac{\partial \varphi}{\partial y} \right)
\]

The \( fU/H \) term is nothing else than the scaling for \( f\partial u/\partial z \) from the thermal wind relation. Since this involves horizontal gradients of \( \theta \), there is hope for further simplification. Indeed, assuming that the slope of pressure surface is much smaller than that of isentropes (in agreement with the observation that \( \theta \) decreases poleward on a pressure surface—see Fig. 3.2), the thermal wind in the form (4.51) can be rewritten as,

\[
f \frac{\partial u}{\partial z} \approx -\rho g \left( \frac{\partial \theta}{\partial y} \right) \left( \frac{\partial \alpha}{\partial \theta} \right)_p
\]

Using the definition of \( \theta \) and the ideal gas equation for dry air, one can show that,

\[
\rho \left( \frac{\partial \alpha}{\partial \theta} \right)_p = 1/\theta
\]

As a result, the inequality becomes,

\[
L/H \geq \frac{N}{f}
\]

which relates simply horizontal and vertical scales of a weather system in midlatitudes.

Let’s step back. What we found is that a geostrophic motion can be at a shallower angle than the surface of constant \( \theta \), but only if the ratio of
its horizontal to vertical scales is larger than the quantity $N/f$. Chapter 2 made it clear that the interaction of radiation and convection (either purely upright or sloping) was confined to the troposphere so it seems appropriate to set $H$ equal to the tropopause height $H_t$,

$$H \approx H_t$$

(4.91)

This implies that the horizontal lengthscale of the storm must satisfy,

$$L \geq \frac{NH_t}{f}$$

(4.92)

Were $L$ smaller than that, the upward motion would be too large (recall $W = R_o U H / L = U^2 H / f L^2$) and the parcel swap generated by the storm would occur on too steep a surface (somewhere in between the $A \leftrightarrow B$ and $A \leftrightarrow C$ swaps in section 3.6). Put differently, large horizontal scales have weak upward motion and so offer the most favorable swaps to gain kinetic energy.

The maximum kinetic energy gained in a swap was estimated to be that occurring when the motion occurs at a slope half that of the $\theta$ surface ($\mu = \mu$ in section 3.6), which provides a selection for the scale $L$,

$$L = \sqrt{2}N H_t$$

(4.93)

Using $N \approx 10^{-2}s^{-1}$, $H_t = 10km$ and $f = 10^{-4}s^{-1}$ ($45^\circ N$), one gets $L = 1,400km$. This is about 13° of latitude, and agrees nicely with the size of extra-tropical cyclones (see for example Fig. 1.2 in Chapter 1)!

### 4.6 Problems

**Q0.** A thunderstorm moving in a sheared flow has the following scales: $L = 10km$, $H = 10km$, $U = 10ms^{-1}$, $W = 1ms^{-1}$, $f = 10^{-4}s^{-1}$, $P' = 10hPa$. Determine whether the geostrophic and hydrostatic approximations apply to this system.

**Q1.** An aeroplane is due to fly eastward over the ocean at $45^\circ N$. At some moment it is at an altitude of 6000$m$ and a pressure altimeter indicates a pressure of 100$hPa$. The pilot maintains this pressure altitude but notices that after 1 hour the radar altimeter indicates an absolute altitude of 5750$m$. How far, and in what direction, has the plane drifted off course? Clearly
state any assumptions you make. [Hint: this question is more difficult than it looks.]

Q2. The derivation of the equation of motion (4.16), and its decomposition into components (4.28), (4.36) and (4.37), was quite mathematical. This question aims at re-deriving those equations from a more intuitive perspective.

(i) Particle moving purely east-west, i.e. \( \mathbf{u} = (u, 0, 0) \) in the local frame of reference \((\mathbf{i}, \mathbf{j}, \mathbf{k})\). By thinking about the particle’s circular motion around the Earth at constant latitude \( \phi \), and its associated centrifugal force, show that it must experience an acceleration \( (\Omega^2 R \cos \phi + u^2/R \cos \phi + 2\Omega u)(\cos \phi \mathbf{k} - \sin \phi \mathbf{j}) \). Identify the corresponding terms in (4.28) and (4.37).

(ii) Particle moving purely north-south, i.e. \( \mathbf{u} = (0, v, 0) \).

(a) A particle of unit mass is stationary on the Earth’s surface at latitude \( \phi \). Show that its angular momentum is \( \Omega R^2 \cos^2 \phi \).

(b) Suppose this particle is subject to an impulsive force which sets it moving northward, staying on the surface. It has experienced no torque so its angular momentum must be conserved; why does this imply that it must develop an eastward velocity component?

(c) If in a time \( \delta t \) the particle has reached latitude \( \phi + \delta \phi \) and acquired an eastward velocity component \( \delta u \), show that its angular momentum is now \( [\Omega + \delta u/R \cos(\phi + \delta \phi)] R^2 \cos^2(\phi + \delta \phi) \).

(d) Using conservation of angular momentum, expanding \( \cos(\phi + \delta \phi) \) and neglecting 2nd order terms in small quantities, show that \( \delta u = 2\Omega R \delta \phi \sin \phi \) and hence that \( \delta u/\delta t = 2\Omega v \sin \phi \). Identify this term in (4.36).

(iii) Particle moving purely upward, \( \mathbf{u} = (0, 0, w) \). Repeat part (ii) but consider a particle impelled vertically upwards with speed \( w \).

Q3. The schematic below (Fig. 4.11) represents a front between two air masses at different temperatures. We suppose that the temperature difference is 6 K at all levels, that the front extends over a horizontal distance of 300 km and from the surface \( (P_2 = 1000hPa) \) to a level of \( P_1 = 200hPa \). We wish to estimate the wind at 200hPa at the center of the front.
4.6. PROBLEMS

(i) By using eqs. (4.48) and (4.50), show that the changes in zonal (east-west) wind with height can be approximated as,

\[ u_2 - u_1 \approx R_d \frac{f}{\bar{T}} \left( \ln \frac{P_2}{P_1} \right) \frac{\partial \bar{T}}{\partial y} \]  

(4.94)

where \( \bar{T} \) is the averaged temperature over the 1000 - 200 hPa layer.

(ii) Assuming that the mean latitude of the front is 45°N, estimate the wind at its center at 200 hPa. State any assumptions made.

Q4⋆. Tornadoes developing underneath a convective storm can cause a lot of damage. They also offer a beautiful, if not fearful, illustration of vortex “bending” and “stretching”. Consider the idealised situation in which a line of updrafts develop at \( y = 0 \) along the \( i \) direction, in the presence of a windshear \( u(z) \) along that direction. Cartesian geometry (\( i, j, k \)) is used throughout this question (\( k \) being the unit vector in the vertical).

(i) Compute the vorticity in the \( j \) direction associated with the windshear \( u(z) \). Compute its magnitude if the wind varies by 5 ms\(^{-1} \) every km.

(ii) Develop the vertical component of the \( (\zeta_a \cdot \nabla) \mathbf{u}_R \) term in the vorticity equation (4.67), assuming no \( x \)-dependence. We’ll refer to this term as the bending/stretching term in the following.
(iii) Assuming that the updraft is concentrated over a distance $\delta y = 100 m$ and that it reaches $w = 1 m/s$ at a height of $1 km$, estimate the dominant term in the bending/stretching term. Interpret the physics of this dominant term and estimate how long it would take to reach a value of $\zeta_a k = \zeta_a = 10^{-2} s^{-1}$.

(iv) A mature tornado might be associated with azimuthal winds of $100 m/s$ and have a radius $r = 200 m$. Compute the associated vorticity and discuss how such values could be created using your answers in (ii) and (iii).

Q5. Past Exam Question (2003 no.4).

(i) What is vorticity and why is it useful for understanding the weather?

(ii) Consider a long-wave pattern at the level of non-divergence in a zonal current of uniform constant velocity, $U$. From the principle of conservation of absolute vorticity and making the assumption that the total velocities are independent of latitude show how the Rossby wave equation can be written as:

$$ (U - c) \frac{\partial^2 v'}{\partial x^2} + \beta v' = 0 \quad (4.95) $$

where $c$ is the wave velocity, $v'$ the meridional velocity perturbation and $\beta$ the meridional rate of change of the Coriolis parameter. Find an expression for $c$ in terms of the wavelength and $\beta$.

(iii) Calculate the wavelength of the Rossby wave if it appears stationary at $50^\circ N$ when the zonal wind is $50 m/s$. For the same zonal wind, what would happen to a wave of the same wavelength at lower latitudes? (The radius of the Earth is 6371 km).

Q6. Using the formula in Q3, discuss whether you agree with the following statement taken from a popular science article about the very anomalous 2013-2014 winter weather in the Northern Hemisphere:

“The Jet Stream is driven in part by the temperature difference between cold Arctic air and the warmer air of the middle latitudes. Because the Arctic is warming more rapidly than the rest of the planet, that difference is shrinking. This ought to produce a less potent Jet Stream.”

[Info: the polar warming amplification ($\approx 1 K$) is restricted to a surface layer extending roughly from 1000hPa to 600hPa.]