Answers to Problems for Chapter 1

Q1

(i) Winter pole, $25km$ for the stratosphere but Summer pole, $80km$ for the mesosphere. The lower stratosphere is also very cold year round at the equator, and this reflects the deep penetration of convective motions and the tendency to follow a moist adiabatic structure at low latitudes (this will be discussed in Chapter 3).

(ii) Summer pole near the stratopause for the stratosphere, but Winter pole for the mesosphere (particularly warm right at the stratopause but hot throughout the depth of the mesosphere). This surprising feature, and the cold Summer pole in (i), reflect a planetary circulation extending from the Summer to the Winter poles. In the ascending branch (Summer pole), air cools by doing work of expansion against its surroundings, hence the low temperature there. Likewise, in the descending branch, air is compressed and warms up, hence the high temperature there.

(iii) For both stratosphere and mesosphere, the strongest westerly winds (i.e., from west to east) are found in the Winter Hemisphere (the "polar vortex" for the stratosphere, and much closer to the equator for the mesosphere –near 30° of latitude).

Q2 This is because of the descending motion associated with the Hadley cell in the sub-tropics. This brings dry air from high levels to the surface which prevents any cloud and rain formation.

Q3

(i) The mass mixing ratio of ozone (r_{O_3}) is the ratio of the mass of ozone in a sample of air to the mass of dry air in the sample, $r_{O_3} = m_{O_3}/m_d$. The density of ozone is $\rho_{O_3} = m_{O_3}/V$ where V is the volume of the sample. We can neglect the presence of water vapour in the stratosphere and treat the sample as "dry air", hence $P_o = \rho_d R_d T_o$ where ρ_d is the density of dry air. From $\rho_{O_3} = r_{O_3}\rho_d$, we get $\rho_{O_3} = r_{O_3} P_o / R_d T_o \approx 4 \times 10^{-7} kgm^{-3}$.

(ii) The volume mixing ratio is the ratio of the number of molecules of ozone to the number of molecules of dry air in a given volume of air, i.e., the ratio of the number densities. This is simply $r_{O_3} \times \mu_d/\mu_{O_3} \simeq r_{O_3} \times (0.8 \times 28 + 0.2 \times 32)/48 \simeq 0.6 r_{O_3}$. (As in the lecture notes for Chapter 1, masses of molecules are denoted by μ).

(iii) Simply multiply the latter by the number density for dry air n_d, i.e., $n_{O_3} = 0.6r_{O_3} \times n_d = 0.6r_{O_3}\rho_d/m_d = 0.6r_{O_3}P_o/(R_dT_om_d) \simeq 5\times10^{18}m^{-3}$ (using $m_d \approx (0.8\times28+0.2\times32)\times 1amu = 4.6\times10^{-26}kg$).

(iv) Using the ideal gas law, $P_{O_3} = n_{O_3}k_BT_o \approx 0.015Pa$.

Q4 At a given temperature T and volume V, one has $e/P_d = N_v/N_d$ using the ideal gas law and $N_v/N_d = 10/1000 = 0.01$. Hence water vapour molecules account for $N_v/(N_d+N_v) = 0.01/(1+0.01) = 0.0099\% \approx 1\%$ of air molecules in the sample. Rewriting $1\% = 10^{-2} = 10^4\times10^{-6} = 10^4\ ppm$, we see that this fraction is $10^4/400 = 25$ times that of CO_2.

Q5

(i) Using the hydrostatic equation $\partial P/\partial z = -\rho g$ in which ρ is density. For an isothermal atmosphere, the ideal gas law reads $P = \rho(k_B/m)T_o$ so that $\partial P/\partial z = -Pmg/k_BT_o$. This proves $p \propto e^{-z/H_s}$ with $H_s = k_BT_o/mg$ as required.

(ii) One needs to estimate first the value of gravity at the planet's surface, which can be done using Newton's aw of gravitation, $g \approx GM/R^2$ in which M is the planet's mass and R its radius. Then it is just a matter of plugging in the numbers. One gets: $g = 8.87(V), 9.81(E), 3.71(M), 24.8(J)ms^{-2}$ and the resulting scale heights are, $16(V), 8.2(E), 12.7(M), 20.6(J)km$.

Q6 Start from $\alpha = (V_g + V_l + V_i)/(m_d + m_v + m_l + m_i)$ (in which V_g is the volume occupied by the gas phase, V_l, V_i by the liquid and ice phases, respectively) factorize by m_d to obtain $\alpha = (V_g/m_d+V_l/m_d+V_i/m_d)/(1+r_t)$ in which $r_t = (m_v + m_l + m_i)/m_d$ is the total mass mixing ratio of water substance. Acknowledging that $\alpha_d = V_g/m_d$, this can be rewritten as, $\alpha = (\alpha_d + V_l/m_d + V_i/m_d)/(1 + r_t)$. Introducing $\alpha_l = V_l/m_l$ and $\alpha_i = V_i/m_i$, one gets, $\alpha = \alpha_d(1 + r_l\alpha_l/\alpha_d + r_i\alpha_i/\alpha_d)/(1 + r_t)$. Both r_l and r_i are very small compared to unity, as are α_l/α_d and α_i/α_d, so $\alpha_d \approx \alpha_d/(1 + r_t)$. Since $1 + r_t = 1/(1 - q_t)$, the result follows.

Q7 Higher pressure is found on the western than on the eastern side of the Rockies. The atmosphere is thus "pushing" the Rockies towards the east. From Newton's 3rd law, the Rockies must thus "push" the atmosphere towards the west. The Rockies thus contribute to a loss of angular momentum (same sign as the loss due to surface friction). NB: The pressure pattern is somewhat easier to see for the Andes on the ERA website. This probably reflects the Taylor column effect partly at work in the Rockies (the wind going around the mountain), but less so over the Andes because they occupy a broader range of latitudes (air parcels have "nowhere to go" but over the mountain).

Q8 Assuming $u \approx 0$ at the equator, the angular momentum of the ring as it is about to go upward is $L = \Omega R^2$. Assuming L is conserved all the way during the ascent and the poleward motion, we have, $\Omega R^2 = R\cos\phi(\Omega R\cos\phi + u)$. This provides a formula for u as a function of latitude ϕ, namely, $u = \Omega R\sin^2\phi/\cos\phi$. At $30°N$ this provides, $u \approx 134ms^{-1}$! Such velocities are not observed because the ring breaks up in waves (the storms) before reaching this value

Q9

(i) heating due to the hydrological cycle is quoted here as a "latent heating" of $88Wm^{-2}$. Multiplying this by the surface area of the Earth, one gets $88 \times 4\pi R^2 \approx 45PW$.

(ii) Likewise, absorption of solar radiation is quoted as $75Wm^{-2}$, hence a heating rate of $75 \times 4\pi R^2 \approx 38PW$. This $75Wm^{-2}$ is the result of "what enters the atmosphere" minus "what leaves the atmosphere" $= 340 - (165 + 23 + 47 + 27)$.

(iii) The net cooling rate due to infrared radiation is quoted as $188Wm^{-2}$ hence $188 \times 4\pi R^2 \approx 95PW$. Again this results from the difference between what enters minus what leaves the atmosphere, $= 398 - (345 + 239)$.

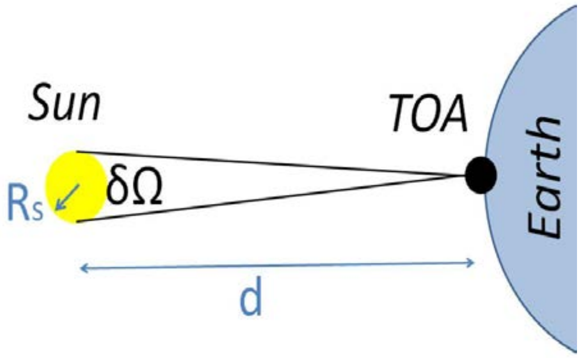

Figure 1: Approximate calculation of solid angle for Q2.

Solution to Chapter 2's problems

Q1. For the southward facing side, the angle δ between the normal to the slope and the beam is $\delta = \alpha - \theta_S$. The intensity reaching the slope is then $I\cos\delta = I\cos(\alpha - \theta_S)$ (notice that it makes sense, no light reaching the slope when it is parallel to the slope, i.e., when $\theta_S = \pi/2 - \alpha$). For the northward facing side, $\delta = \alpha + \theta_S$ so that the ratio sought is $\cos(\alpha - \theta_S)/\cos(\alpha + \theta_S)$. This is 1.53 for $\theta_S = 30°$ and 4.4 for $\theta_S = 60°$.

Q2. The irradiance and intensity are related through $F = \int I\cos\theta d\Omega$. Here we consider the case of zero zenith angle so $\theta = 0$. The rays reaching the TOA do not originate equally from all directions. Rather they originate from a "radiation pencil" whose solid angle is $\delta\Omega$ (Fig. 1). The fraction of the hemisphere of solid angle (i.e., "the sky") that is occupied by the Sun is the same as the fraction of the area of the hemisphere of radius d, centered on Earth, occupied by the Sun: $\delta\Omega/4\pi = \pi R_s^2/(4\pi d^2) = 6.84\times10^{-5} sr^{-1}$. Hence, $I \approx F/\delta\Omega = 2\times10^7 Wm^{-2}sr^{-1}$.

Q3. From Beer's law, the ratio of radiation intensity is $e^{-\alpha_\lambda \int_0^s \rho_a ds}$. (i) We have $\int_0^s \rho_a ds = 1kgm^{-2}$ so the ratio is $e^{-0.01\times1} = 0.99$, i.e., very little absorption. (ii) Just rearrange Beer's law to get $\int_0^s \rho_a ds = \ln 2/\alpha_\lambda = 69.3kgm^{-2}$.

Q4. (i) From the definition of optical depth, $\tau(z) = \int_z^{+\infty} \rho q k_\lambda dz$ with

$k_\lambda = \alpha_\lambda = 0.01 m^2 kg^{-1}$ (from the previous question). Using the hydrostatic equation, $dP/dz = -\rho g$, and noticing that k_λ and q are constants, we get, $\tau(z) = qk_\lambda \int_0^{P_s} dP/g = qk_\lambda P/g$. (ii) To reach an optical depth of unity, P must be equal to $g/(qk_\lambda) = 9.81/(0.01 \times 10^{-3}) = 9.81.10^5 Pa$. This is about ten times larger than P_s so cannot be reached. The optical depth increases with pressure, with maximum value at the surface, $\tau(z = 0) = qk_\lambda P_s/g = 0.1$.

Q5. Write $\rho = \rho_s e^{-z/H_s}$ in which H_s is the scale height and ρ_s the surface density. From the definition of optical depth,

$$\tau_\lambda(z) \equiv \int_z^{+\infty} \rho q_a k_\lambda dz = q_a k_\lambda \rho_s H_s e^{-z/H_s} \tag{1}$$

assuming uniform mixing ratio q_a. From the notes, $Q_\lambda \propto \rho F_\lambda^\downarrow \propto \rho e^{-\tau(z)}$. Hence the heating rate is maximum when

$$\frac{dQ_\lambda}{dz} = 0 \quad \text{i.e.,} \quad \frac{d\rho}{dz} e^{-\tau} = \rho \frac{d\tau}{dz} e^{-\tau} \tag{2}$$

This can be rewritten as $1 = q_a k_\lambda \rho_s H_s e^{-z/H_s} = \tau$, hence the result.

Q6. Simply divide the previous answer by ρ to express the heating rate in $W kg^{-1}$ rather than in $W m^{-3}$,

$$Q_\lambda/\rho \propto e^{-\tau_\lambda(z)} \tag{3}$$

which is indeed maximum at the top-of-the-atmosphere. This simply reflects the fact that, per unit mass, there will be more absorption of radiation at upper levels (more intensity of radiation incoming) than at low levels (not much radiation left to absorb).

Q7. (i) Denote by I_o the intensity emitted by the surface, I_1 that impinging at the height where $\tau = 0.2$ and I_2 the intensity at the height where $\tau = 4$. What we are asked to compute is $(I_1 - I_2)/I_o$. Using Beer's law (or the first term on the right hand side in Schwarzchild equation), this is simply $e^{-0.2} - e^{-4} = 0.8$, i.e., 80 % of the radiation emitted by the surface is absorbed by this layer. (ii) Denote by B_o the Planck function for the atmosphere and Earth's surface at temperature T_o, i.e., $B_o = B_\lambda(T_o)$. The total infrared radiation is obtained from Schwarzchild equation. At a fixed path distance s, with $s_o = 0$ (the Earth's surface), we have:

$$I_\lambda(s) = B_o e^{-\tau(0,s)} + \int_0^{\tau(0,s)} B_o e^{-(\tau(0,s) - \tau')} d\tau' = B_o \tag{4}$$

This shows that for the special case considered (isothermal atmosphere at the same temperature as the Earth's surface), the upward infrared flux is constant with height. Thus at the TOA, $I_\lambda = B_o$. The contribution ΔI_λ to the OLR from the layer sandwiched between $\tau(0,s) = 0.2$ and $\tau(0,s) = 4$ is simply,

$$\Delta I_\lambda = \int_{0.2}^{4} B_o e^{-(\tau_\infty - \tau')} d\tau' = B_o e^{-\tau_\infty}(e^4 - e^{0.2}) \approx 0.36 B_o \tag{5}$$

where we have used $\tau_\infty = 5$. The contribution is thus $0.36B_o/B_o = 36$ %.

Q8.

(i) The radiative balance is $4\pi R^2 \sigma T_e^4 = \pi R^2 (1-\alpha_P) S_o$ so $T_e = ((1-\alpha_P)S_o/4\sigma)^{1/4}$.

(ii) Differentiating the radiative balance with respect to T_e, one obtains, $16\sigma T_e^3 \delta T_e = -S_o \delta\alpha_P$ where $\delta\alpha_P$ is the change in albedo and δT_e the resulting change in emission temperature. Rearranging, one gets $\delta T_e / T_e = \delta(1-\alpha_P)/4(1-\alpha_P)$. This shows that a 10 % change in $(1-\alpha_P)$ leads to a $10/4 = 2.5$ % change in emission temperature. Expressed as $\delta T_e/\delta\alpha_P$ the sensitivity is $-91K$ per unit change in planetary albedo. The negative sign reflects that a more reflective planet is colder.

(iii) Assuming $\delta T_s \approx \delta T_e$ where $\delta T_s = 0.2K$ is the change in global surface temperature, one gets $\delta T_s \approx (\delta T_e/\delta\alpha_P)\delta\alpha_P$. Using $\alpha_P = f\alpha_c + (1-f)\alpha_s$ in which f is the fraction of the Earth's surface covered by clouds and $\alpha_c = 0.8, \alpha_s = 0.1$, one then obtains, $\delta T_s \approx (\delta T_e/\delta\alpha_P)(\alpha_c - \alpha_s)\delta f$, or $\delta f = \delta T_e / [(\delta T_e/\delta\alpha_P)(\alpha_c - \alpha_s)] = 0.2/(-91 \times (0.8 - 0.1)] = -0.03$. The negative sign is consistent with a reduction in cloud cover leading to a surface warming.

(iv) The question is a bit unclear because it does not specify the wavelength (shortwave or longwave) of the transmissivity increase. Considering the previous questions, I am assuming the shortwave is considered here, and I am interpreting the increase in transmissivity by 10 % as a decrease in cloud albedo by 10 %. This corresponds to a change in planetary albedo of $\delta\alpha_P = f\delta\alpha_c = -0.1f\alpha_c$. To estimate f, use $\alpha_P = f\alpha_c + (1-f)\alpha_s$, leading to $f = (\alpha_P - \alpha_s)/(\alpha_c - \alpha_s) = 0.2/0.7 = 0.28$. As a result, $\delta\alpha_P = -0.1 \times 0.28 \times 0.8 = 0.022$. The change in surface temperature is then, assuming again $\delta T_s \approx \delta T_e = (\delta T_e/\delta\alpha_P)\delta\alpha_P = -91 \times (-0.02) = +2K$. This is an order of magnitude larger than the assumed change in (iii) so this effect is too big to explain the surface temperature change.

The reduction in cloud cover of 0.03 corresponds to a relative change of $0.03/0.28 = 10\ \%$ as well and thus is more likely to be relevant to the surface warming.

(v) The reason why the observed surface temperature larger than the effective temperature is the greenhouse effect. The transmittance associated with solar radiation is larger than the transmittance for thermal radiation. This means that the atmosphere allows solar radiation to heat the surface but traps thermal radiation emitted by the surface, which yields further heating of the surface.

(vi) The Earth's surface cools by evaporation and also because it is warmer than the air above (thermals). As a result, there must be a net downward radiative flux (surface energy gained) to oppose this cooling if an equilibrium is to be observed.

Q9. The total energy radiated at the Sun's surface is $\pi r_{sun}^2 \sigma T_{sun}^4$. At a distance of $1AU$ the same total energy is radiated by the Sun, neglecting the weak absorption of shortwave radiation as it travels through space. So $\pi r_{sun}^2 \sigma T_{sun}^4 = \pi d^2 S_o$, with $d = 1AU$. Plugging numbers one finds $S_o = 1362 W m^{-2}$. This is indeed close to the observed values displayed in the figure.

Q10. (i) From $I_\lambda = I_{\lambda,\infty} e^{-\tau_\lambda \sec\theta}$, one gets $(I_\lambda)_1/(I_\lambda)_2 = e^{-\tau_\lambda(\sec\theta_1 - \sec\theta_2)}$. The answer follows after taking the logarithm and rearranging. (ii) If we make the assumption that the atmosphere is not changing much in terms of k_λ, ρ and q_a between the two solar paths considered, then τ_λ can be interpreted as a measure of total optical depth for the atmosphere. Plugging numbers, one gets $\tau_\lambda = \ln 1.12/(\sec 40 - \sec 20) = 0.47$.

Answers to Problems for Chapter 3

Q1

(i) Start from the definition of entropy for dry air, yielding $ds = c_{p,d}dT/T - R_d dP/P$. The atmosphere being in hydrostatic balance, $\alpha dP/dz = -g$ where α is the specific volume. Hence $ds = c_{p,d}dT/T + gR_d dz/(\alpha P)$. Using the ideal gas law, $P\alpha = R_d T$, this is also, $ds = c_{p,d}dT/T + gdz/T$. For the adiabatic case, $ds = 0$, so $c_{p,d}dT + gdz = 0$. The result follows. Numerically, $\Gamma_d = 9.81/1005 \approx 9.8K/km$.

(ii) Starting from the definition of potential temperature $\theta = T(P_{ref}/P)^{R_d/c_{p,d}}$, and differentiating the ln of this expression with respect to z, we get $\frac{T}{\theta}\partial\theta/\partial z = \partial T/\partial z - \frac{R_d T}{Pc_{p,d}}\partial P/\partial z$. Using the hydrostatic equation $\partial P/\partial z = -\rho g$ and the ideal gas law $P = \rho R_d T$, this can be rewritten as, $\frac{T}{\theta}\partial\theta/\partial z = \partial T/\partial z + g/c_{p,d}$. The result follows.

(iii) A dry atmosphere with $N^2 \propto \partial\theta/\partial z < 0$ is unstable to vertical displacements. Thus, it might happen that sporadically the lapse is larger than Γ_d (i.e., the temperature decreases with height more rapidly than about $9.8K/km$) but the atmosphere would then quickly overturn and reach $\Gamma = \Gamma_d$. The opposite case $N^2 > 0$ is stable and associated with a lapse rate weaker than the dry adiabatic value. In the stratosphere absorption of ultraviolet radiation from the Sun leads to high entropies there and a lapse rate even opposite in sign to Γ_d!

Q2 The parcel undergoes a moist adiabatic ascent over $4km$. Its temperature thus decreases by $4 \times 6.5 = 26K$. It then experiences a dry adiabatic compression over $3km$ and thus an increase in temperature of $3 \times 10 = 30K$. The parcel is thus warmer by $4K$ on the plateau than when it started above the sea.

Q3 (i) At room temperature and pressure the phase diagram of water shows that only the vapour phase should exist at equilibrium. Thus, if there is no liquid water in the room at all, thermodynamic equilibrium has been reached. If however there is liquid water (say someone has just mopped the floor), then the situation is not in thermodynamic equilibrium: evaporation will occur until all the liquid water remains in the vapour phase (when that occurs, thermodynamic equilibrium has been restored). (ii) Not in thermodynamic equilibrium: the rain will evaporate into the dry air mass.

Q4. At the top-of-the-atmosphere, the equilibrium condition is: $\sigma T_e^4 = \sigma T_2^4$, hence $T_2 = T_e = 255K$. For layer 1, equilibrium requires $\sigma T_2^4 + \sigma T_s^4 = 2\sigma T_1^4$, while for the surface, it reads $\sigma T_e^4 + \sigma T_1^4 = \sigma T_s^4$ (you can read these two equations as simply saying that heating equals cooling). Substracting the previous two equations lead to $\sigma T_1^4 = 2\sigma T_e^4$ hence $T_1 = 2^{1/4} T_e = 303K$. Replacing this value in any of the two equilibrium conditions above then yield $T_s = 3^{1/4} T_e = 336K$.

Q5.

(i) Using a reference pressure $P_o = 1000hPa$, the potential temperatures are $\theta_1 = 335K$ (at $700hPa$ and using $T_1 = 303K$), and $\theta_2 = 331K$ (at $400hPa$ and using $T_2 = 255K$), while, at the surface $\theta_s = T_s = 336K$. This is clearly an unstable situation, with high entropy air below low entropy air, from the surface to upper levels.

(ii) Convection carries heat upwards. So the surface temperature should go down.

(iii) Consideration of the TOA radiative budget leads again to $T_2 = T_e$. Hence $T_1 = T_2 + \Delta T$ and $T_s = T_e + 2\Delta T$.

(iv) The surface energy balance is $\sigma T_s^4 + F_s = \sigma T_1^4 + \sigma T_e^4$ while energy conservation for layer 2 reads: $\sigma T_1^4 + F_c = 2\sigma T_2^4$. Using the expressions in (iii), we obtain: $F_s = \sigma T_e^4[1 + (1+x)^4 - (1+2x)^4]$ and $F_c = \sigma T_e^4[2 - (1+x)^4]$.

(v) The pressure difference between the layers is $300hPa$ which is about $3km$ thick. For a moist adiabat, the temperature decreases with height at about $7K/km$, hence a plausible choice for ΔT is $3 \times 7 = 21K$. This has a value $x = 21/255 = 0.08$. (NB: at constant entropy, temperature always increases with pressure so ΔT, and as a result x, must be positive). A plot of F_s and F_c as a function of x shows that these are positive and decrease with increasing x, up to a point where they cross the zero line. The solution physically requires that convection carries heat upward so that both F_s and F_c must be positive. This will be satisfied as long as $0 < x < 0.14$ (the zero crossing with the smallest value of x).

Finally, without convective fluxes, the surface temperature was $T_s = 3^{1/4} T_e$. With convective fluxes, $T_s = T_e(1+2x)$ and so equals the radiative equilibrum value when $x = (3^{1/4} - 1)/2 = 0.158$. Thus, over the range of values for the which is physical ($0 < x < 0.14$, based on $F_s, F_c > 0$) the surface temperature is smaller than the radiative equilibrium value, as expected from (ii).

Q6. It is just a matter of finding values for N^2, Δy and μ_θ. Inspection of the θ distribution from the ERA40 atlas suggests (at $45°N$) that $N^2 \approx (g/285K)20K/400hPa \approx (g/285)(20/4km) = 1.7 \times 10^{-4}s^{-2}$. At the same latitude, following the $\theta = 285K$ surface, one gets $\mu_\theta \approx 4km/5000km = 8 \times 10^{-4}$. The latitudinal extent of the storm can be obtained from the global infrared picture in Chapter 1, typically $\Delta y = 15°$ of latitude, i.e., $\Delta y = 6371 \times 15 \times \pi/180 \approx 1700km$. Plugging those in we get,

$$KE_{max} = \frac{N^2(\Delta y)^2}{8}{\mu_\theta}^2 \approx 40J/kg \tag{1}$$

This is consistent with observations, with implied velocities on the order of $\sqrt{2KE_{max}} = 10m/s$. It is remarkable how such a simple view of the storms seems to work. (NB: This view was pioneered by Prof Eric Eady at Imperial College in the 1950s and it still provides the basic theoretical understanding behind the complicated numerical simulations carried out in climate centers across the world.)

Q7.

(i) Simple harmonic motion at angular frequency N, i.e., $z(t) = A\cos Nt + B\sin Nt$. From the initial conditions $z(0) = 0$ and $dz/dt(0) = w_o$, we get $z(t) = (w_o \sin Nt)/N$. For the values given, the period of the motion is $2\pi/N \approx 10mn$. After $1mn$, it reaches a height of $5.6m$ above its initial position (and is thus just in the initial ascending part of its oscillatory trajectory).

(ii) The motion is unstable and of the form $z(t) = Ae^{Nt} + Be^{-Nt}$. Using the initial conditions, we get $z(t) = w_o(e^{Nt} - e^{-Nt})/2N$. This provides $z = 6.4m$ after $1mn$.

(iii) In this case the parcel's equation predicts $dw_p/dt = 0$, and hence that the parcel will keep rising with the same vertical momentum (the buoyancy force is zero and so there is no upward or downward acceleration provided by the environment). After one minute the height reached is $10 \times 10^{-2} \times 60 = 6m$.

Solution to Chapter 4's problems

Q0. Calculation of the Rossby number indicates that the system is not in geostrophic balance since $R_o = 10/(10^{-4} \times 10km) = 10 \gg 1$. For the hydrostatic approximation, we must check the magnitude of $\overline{\alpha}\partial P'/\partial z$ compared to the vertical acceleration. Using $\overline{\alpha} \approx 1kgm^{-3}$, one gets $\overline{\alpha}\partial P'/\partial z \approx 10hPa/10km = 0.1ms^{-2}$. Likewise, the vertical acceleration is $w\partial w/\partial z \approx 1/10km = 10^{-4}ms^{-2}$ but it also has the contribution $u\partial w/\partial x \approx 10 \times 1/10km = 10^{-3}ms^{-2}$. So the hydrostatic approximation holds for this system.

Q1. The plane is moving along a latitude circle, eastward, and at constant pressure. Since we're told that the pressure surface $P = 100hPa$ slopes downward along the motion ($\Delta z = 6000 - 5750 = 250m$), it means that there is a pressure difference at fixed height (high to the west, low to the east). From geostrophic balance, this pressure difference must, in the Northern Hemisphere, drive a motion towards the equator hence there will be a drift towards lower latitudes.

To make quantitative statements, we need to make a few additional assumptions. I am going to assume that the east-west pressure gradient is on the order of $10hPa/1000km$ (typical of large scale systems in midlatitudes). At a height of $6km$, the density is roughly the surface value ($\approx 1kg/m^3$) times $e^{-6/8}$ (assuming a scale height of $8km$) so the specific volume is roughly $e^{6/8} = 2m^3/kg$. Hence,

$$v = \frac{\alpha}{f}\frac{\partial P}{\partial x} \approx \frac{2}{2\Omega \sin 45^\circ}\frac{10hPa}{1000km} \approx 20m/s \tag{1}$$

In a time $\Delta t = 1h$, the equatorward drift is $v\Delta t = 72km$, i.e., a drift in latitude slightly less than a degree.

NB: I haven't actually used the value $\Delta z = 250m$. A student (Adrian) suggested this could be included by using the equality used in section 4.2.3 (thermal wind),

$$\alpha \left(\frac{\partial P}{\partial x}\right)_z = \alpha \left(\frac{\partial z}{\partial x}\right)_P \left(\frac{\partial P}{\partial z}\right)_x = -g \left(\frac{\partial z}{\partial x}\right)_P \tag{2}$$

We know Δz so it is just a matter of guessing how far a plane moves in one hour (Δx) to estimate the pressure gradient term as $\alpha\partial P/\partial x \simeq g\Delta z/\Delta x$. A value of $1000km/h$ is not unreasonable, yielding $\Delta x = 1000km$ and a velocity

$v \simeq 24m/s$.

Q2.

(i) The parcel goes around in circles of radius $R\cos\phi$, with angular velocity $\Omega + u/R\cos\phi$. It thus experiences a centrifugal force $(\Omega + u/R\cos\phi)^2 R\cos\phi$. This force has components $\cos\phi\boldsymbol{k} - \sin\phi\boldsymbol{j}$ in the local coordinate system. Hence the particle feels a northward acceleration $-\Omega^2 R\cos\phi\sin\phi - u^2\tan\phi/R - 2\Omega u\sin\phi$. The first term depends only on position and is a contribution to the apparent gravity $\boldsymbol{g}'$. The second and third terms are readily seen in eq. (4.31). Conversely, projecting onto the vertical direction, we obtain an upward acceleration $\Omega^2 R\cos^2\phi + 2\Omega u\cos\phi + u^2/R$. The first term is again contributing to $\boldsymbol{g}'$ while the second is identified with the Coriolis acceleration. The third term is indeed present in (4.27).

(ii) (a) Angular momentum is the product of azimuthal velocity and radius hence the angular momentum of the parcel at rest is $\Omega R^2\cos^2\phi$.

(b) As the parcel moves north it gets closer to the axis of rotation ($R\cos\phi$ decreases). If it conserves its angular momentum, the parcel must thus increase its azimuthal velocity, implying that an eastward acceleration must have taken place.

(c) This is a consequence of the new angular velocity $\Omega + u/R\cos(\phi + \delta\phi)$ and the new radius $R\cos(\phi + \delta\phi)$.

(d) Conservation of angular momentum yields $\Omega R^2\cos^2\phi = [\Omega + u/R\cos(\phi+\delta\phi)][R\cos(\phi+\delta\phi)]^2$. Using $\cos(\phi+\delta\phi) \approx \cos\phi - \delta\phi\sin\phi$, we obtain, to leading order in small quantities $(\delta u, \delta\phi)$, $0 = -\Omega R^2 2\delta\phi\sin\phi\cos\phi + \delta u R\cos\phi$. From this, $\delta u = 2\Omega R\sin\phi\delta\phi$ (sign makes sense since poleward motion $v > 0$ must lead to faster cyclonic azimuthal flow $u > 0$), so that, since $v \equiv R\delta\phi/\delta t$ (in the limit of very small $\delta\phi, \delta t$), $\delta u/\delta t = 2\Omega\sin\phi v$.

(iii) If the parcel is initially accelerated upwards, it will gain a height δz in a time δt. Note that it will still increase its latitude by $\delta\phi$ so the calculation is the same as before, but with a slightly different version of the conservation of angular momentum: $\Omega(R+\delta z)^2\cos^2\phi = [\Omega + u/(R+\delta z)\cos(\phi+\delta\phi)][R\cos(\phi+\delta\phi)]^2$. Noting that $w = \delta z/\delta t$ we get, $\delta u/\delta t = 2\Omega v\sin\phi - 2\Omega w\cos\phi$.

Q3.

(i) Start by rewriting (4.42) as,

$$u_2 - u_1 = \frac{g}{f}\frac{\partial}{\partial y}(z_1 - z_2) \tag{3}$$

Then, approximate $z_1 - z_2$ as,

$$z_1 - z_2 = \frac{R_d}{g}\left(\ln\frac{P_2}{P_1}\right)\overline{T} \tag{4}$$

The result follows.

(ii) Applying the formula with $f = 2\Omega\sin 45°$ yields $u_1 - u_2 \approx 90m/s$. Assuming the winds to be small near the Earth's surface, we get $u_1 \approx 90m/s$. This is a very strong jet! (most likely due the deep extent assumed $1000 - 200hPa$).

Q4.

(i) Using the right hand rule, an increase with height of wind going towards the $+\boldsymbol{i}$ direction has a vorticity along the $+\boldsymbol{j}$ direction. Its magnitude is simply $\partial u/\partial z = 5/1000 = 0.5 \times 10^{-2}s^{-1}$. Note that this is very large compared to the ambient planetary vorticity in the $\boldsymbol{j}$ direction (assuming this coincides with the local North-South direction) which has a magnitude $2\Omega\cos\phi \simeq 10^{-4}s^{-1}$ where ϕ is latitude. Hence the total vorticity in the $\boldsymbol{j}$ direction is dominated by the relative vorticity in that direction.

(ii) For the vertical component of vorticity, we get,

$$[(\boldsymbol{\zeta}_a.\boldsymbol{\nabla})\boldsymbol{u}_R].\boldsymbol{k} = (\boldsymbol{\zeta}_a.\boldsymbol{i})\frac{\partial w}{\partial x} + (\boldsymbol{\zeta}_a.\boldsymbol{j})\frac{\partial w}{\partial y} + (\boldsymbol{\zeta}_a.\boldsymbol{k})\frac{\partial w}{\partial z} \tag{5}$$

Since the updraft is idealised as a line along $y = 0$, independent of the x-direction, the first term on the l.h.s vanishes ($\partial w/\partial x = 0$) hence,

$$[(\boldsymbol{\zeta}_a.\boldsymbol{\nabla})\boldsymbol{u}_R].\boldsymbol{k} = (\boldsymbol{\zeta}_a.\boldsymbol{j})\frac{\partial w}{\partial y} + (\boldsymbol{\zeta}_a.\boldsymbol{k})\frac{\partial w}{\partial z} \tag{6}$$

(iii) Right at the start of the process the vertical component of the absolute vorticity (ζ_a) is only due to the ambient planetary vorticity $\zeta_a \approx f \simeq 10^{-4}s^{-1}$. This is nearly two orders of magnitude smaller than the relative vorticity along the $\boldsymbol{j}$ (call this η) direction computed in (i). Likewise the $\partial w/\partial y \approx 1/100 \gg \partial w/\partial z = 1/1000$. Hence initially,

$$[(\boldsymbol{\zeta}_a.\boldsymbol{\nabla})\boldsymbol{u}_R].\boldsymbol{k} = \eta\frac{\partial w}{\partial y} + \zeta\frac{\partial w}{\partial z} \approx \eta\frac{\partial w}{\partial y} \quad (7)$$

This term reflects the bending of the vortex initially lying along the $\boldsymbol{j}$ direction into a vortex becoming progressively more aligned with the vertical. (You can convince yourself of this effect by simply rotating your pen along an horizontal axis and lifting it upward while keeping the rotation). It would take a time $\Delta t = \zeta/[\eta(\partial w/\partial y)] = 10^{-2}/[0.5 \times 10^{-2}(1/100)] = 200s$ to reach a value $\zeta = 10^{-2}s^{-1}$.

(iv) We can estimate ζ for the mature tornado using the circulation theorem. If u_θ denotes the azimuthal velocity,

$$\oint u_\theta dl = \iint \zeta dS \quad (8)$$

leading to $2\pi r u_\theta = \zeta\pi r^2$ or $\zeta = 2u_\theta/r \simeq 200/200 = 1s^{-1}$. This is two orders of magnitude larger than the vorticity generated by the bending in (iii). The way the tornado acquires even larger ζ is through the stretching term $\zeta\partial w/\partial z$. This term is initially small compared to $\eta\partial w/\partial y$ but once the vortex has been tilted fully in the vertical $\eta \approx 0$ so that,

$$[(\boldsymbol{\zeta}_a.\boldsymbol{\nabla})\boldsymbol{u}_R].\boldsymbol{k} \approx \zeta\frac{\partial w}{\partial z} \quad (9)$$

This stretching effect leads to an increase in vorticity,

$$\frac{\partial \zeta}{\partial t} \approx \zeta\frac{\partial w}{\partial z} \quad (10)$$

i.e., of the exponential type. For $\partial w/\partial z = 1/1000s^{-1}$, $\zeta_o = 10^{-2}s^{-1}$ (the vorticity generated through bending) and $\zeta = 1s^{-1}$, it would take a time $t \simeq \ln(\zeta/\zeta_o)/(\partial w/\partial z) = \ln(1/0.01)/(1/1000) \simeq 76mn$ to reach the mature tornado stage.

NB: This question is certainly very hard. To reassure you, you would not get anything as difficult at the exam.

Q5.

(i) Vorticity is a vector defined as the curl of the velocity field. Its vertical component is particularly important as the geostrophic flow makes an important contribution to it. It is important to understand and predict the weather because, unlike the geostrophic approximation which is purely diagnostic, the vorticity equation predicts evolution through time.

(ii) Start with conservation of the vertical component of vorticity ($\zeta_a = f + \zeta$) for a 2D (horizontal) flow,

$$\frac{D(f+\zeta)}{Dt} = 0 \tag{11}$$

where $D/Dt = \partial/\partial t + u\partial/\partial x + v\partial/\partial y$. We are interested here in the linearised version of this equation, with $u = U + u'$ and $v = 0 + v'$ (no background flow in the North South direction),

$$\frac{\partial \zeta'}{\partial t} + U\frac{\partial \zeta'}{\partial x} + v'\frac{df}{dy} = 0 \tag{12}$$

where $\zeta' = \partial v'/\partial x - \partial u'/\partial y$. In addition, we assume no y dependence for U, u' and v', hence,

$$[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}]\frac{\partial v'}{\partial x} + \beta v' = 0 \tag{13}$$

This starts to look like the answer. To get there, consider solutions of the form $v' = e^{ik(x-ct)}$, leading to $\partial v'/\partial t = -c\partial v'/\partial x$. Hence the formula given in the exam.

To obtain the value of c, expand the second derivative as a function of the zonal wavenumber k,

$$(U-c)(-k^2)v' + \beta v' = 0 \tag{14}$$

This leads to $c = U - \beta/k^2$.

(iii) For a stationary wave $c = 0$, leading to $k = \sqrt{\beta/U}$. At $50°N$, $\beta = 2\Omega\cos(50°)/R = 1.46 \times 10^{-11} m^{-1}s^{-1}$, leading, for $U = 50ms^{-1}$ to $k = 2\pi/(11,590km)$. The parameter β would be larger at a lower latitude, so the curve $c(k)$ would shift towards the right: this wave would appear to propagate westward ($c < 0$).

Q6. This statement is wrong. First, as expressed by the thermal wind relation, horizontal temperature gradients drive the vertical gradient of the wind, not the wind itself. For example, applying the relation in **Q3**(i),

$$u_2 - u_1 \approx \frac{R_d}{f}\left(\ln\frac{P_2}{P_1}\right)\frac{\partial \overline{T}}{\partial y} \tag{15}$$

where level 1 is taken as that where the jet is strongest (tropopause) and level 2 is taken as the Earth's surface. This shows that changes in $\partial\overline{T}/\partial y$ will affect $u_2 - u_1$, not u_1.

Let's nevertheless assume that one could ignore the latter (maybe because they could be small compared to those occurring higher up) and estimate by how much one would change the jetstream velocity u_1. From the formula,

$$\delta u_1 \approx -\frac{R_d}{f}\left(\ln\frac{P_2}{P_1}\right)\delta\left(\frac{\partial\overline{T}}{\partial y}\right) \tag{16}$$

in which δ indicates the change as a result of increasing greenhouse gas concentrations. Indeed, if the poles warm up quicker than the equator, $\delta(\partial\overline{T}/\partial y) > 0$ (remember that $\partial T/\partial y < 0$ since temperature decreases poleward in the troposphere) and so the jet weakens ($\delta u_1 < 0$). The larger warming ($\simeq 1K$) of the Arctic is confined to a surface layer extending from $1000hPa$ to about $600hPa$. So, the vertically averaged change in temperature is weighted by a factor 600/1000 compared to that occurring in this layer. Taking the gradient from equator to pole, f at $45°N$, $P_1 = 200hPa$ (tropopause pressure) at that latitude, $P_2 = 1000hPa$, one degree of latitude $\approx 100km$, we get,

$$\delta u_1 \approx -\frac{600}{1000}\times\frac{287}{10^{-4}}\times\left(\ln\frac{1000}{200}\right)\left(\frac{1K}{90\times 100km}\right) \approx -0.3ms^{-1} \tag{17}$$

In reality, the change would be even smaller because the equator-to-pole temperature gradient actually increases at upper levels (we'll see this when we talk about climate change). Considering the average jetstream speed is $\approx 30ms^{-1}$, the change caused by Arctic warming is negligible.

NB: A word of caution. The vertical shear of the wind seems indeed not to be changed too much by Arctic warming. But it might be that the wind itself decreases more substantially as a result of Arctic warming (the latter might perturb the path of the storms and the excitation of Rossby waves, possibly leading to less momentum transport into the jetstream). We would need a climate model to look into this interesting possibility...