Foundations of Quantum Mechanics – Problem Sheet 1

1. Consider $\mathbb{C}^3$, the vector space of 3-entry column vectors. Are the following subsets of $\mathbb{C}^3$ vector subspaces? Justify your answers.

   (a) $U \equiv \left\{ \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \in \mathbb{C}^3 : c_1 + c_2 + c_3 = 0 \right\}$

   (b) $V \equiv \left\{ \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \in \mathbb{C}^3 : c_1 + c_2 + c_3 = 1 \right\}$

   (c) $W \equiv \left\{ \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \in \mathbb{C}^3 : c_1 + c_2 + \ldots + c_n = 0 \quad \text{and} \quad c_1 = c_2 \right\}$

   (d) $X \equiv \left\{ \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \in \mathbb{C}^3 : c_1^2 = c_2^2 \right\}$

   (e) $Y \equiv \left\{ y \in \mathbb{C}^3 : (w, y) = 0 \right\}$ where $w$ is an arbitrary, fixed vector in $\mathbb{C}^3$.

   Note 1: to prove that a subset, $X$, is a vector subspace you only need to prove that when you add any two vectors in $X$ it gives a vector in $X$, and when you multiply any vector in $X$ by any complex number, it gives a vector in $X$. Most of the other conditions for being a vector space are automatically satisfied because the addition and scalar multiplication operations are inherited from $\mathbb{C}^3$ and since the conditions hold there, they will hold in any subspace. The existence of the zero vector and inverses ($-u$ for every vector $u$) in $X$ will follow if it is true that when you multiply a vector in $X$ by any complex number, it gives a vector in $X$ (just choose the complex number to be 0 and then $-1$).

   Note 2: the notation $\equiv$ means “is defined to be”, so part (a) reads “The set $U$ is defined to be the set of all vectors in $\mathbb{C}^3$ such that the entries add up to 0”

   Hint: use the linear properties of the inner product.

2. Let $\{|e_1\rangle, \ldots |e_N\rangle\}$ be an orthonormal basis for Hilbert space $\mathcal{H}$. Calculate

   (a) The norm of $|\psi\rangle = 5 |e_1\rangle + (2 + i) |e_2\rangle$
(b) The norm of 
\[ | \psi \rangle = \sum_{i=1}^{N} \psi_i | e_i \rangle, \quad \psi_i \in \mathbb{C} \]

(c) The inner product \( \langle \psi | \phi \rangle \) between the two vectors

\[ | \psi \rangle = i | e_1 \rangle + (3 + 2i) | e_2 \rangle \]

\[ | \phi \rangle = 6 | e_1 \rangle + | e_2 \rangle + i | e_3 \rangle \]

3. Let \( \mathcal{H} \) be a Hilbert space with a basis \( \{ | e_1 \rangle, \ldots | e_N \rangle \} \). Prove that since the vectors in the basis are linearly independent, the expansion of a vector \( | \psi \rangle \) as a linear combination of basis vectors is unique.

Prove that every other basis of \( \mathcal{H} \) also has \( N \) vectors. Hint: prove that if \( \mathcal{H} \) has a basis with \( N \) vectors, then any set of \( N + 1 \) vectors in \( \mathcal{H} \) must be linearly dependent. To do this you may use the fact that \( N \) linear homogeneous equations for \( M \) variables has a non-trivial solution (i.e. a solution in which not all the variables are zero) if \( M > N \).
Foundations of Quantum Mechanics – Problem Sheet 2

First some useful information. The following statements about operators $A$ and $B$ all follow from one another. (Take a little thought to convince yourself of these statements).

(i) $A = B$

(ii) $A^\dagger = B^\dagger$

(iii) $A|\psi\rangle = B|\psi\rangle$ for all $|\psi\rangle$ in the Hilbert space.

(iv) $\langle \phi | A = \langle \phi | B$ for all $|\phi\rangle$ in the dual Hilbert space

(v) $A|e_i\rangle = B|e_i\rangle$ for all $|e_i\rangle$ in a basis of the Hilbert space.

(vi) $\langle \phi | A|\psi\rangle = \langle \phi | B|\psi\rangle$ for all $|\phi\rangle$ and $|\psi\rangle$ in the Hilbert space.

(vii) $\langle e_i | A|e_j\rangle = \langle e_i | B|e_j\rangle$ for all basis vectors $|e_i\rangle$ and $|e_j\rangle$.

So, for example, take the equation in (iii). Because it holds for ANY $|\psi\rangle$ you can “cancel off” the $|\psi\rangle$ to leave $A = B$. Also, if you are asked to prove that operator X equals operator Y, try acting with X on an arbitrary basis vector and show you get the same thing as Y acting on the same basis vector. Or try showing they have the same matrix elements between basis vectors.

1. Show that the adjoint of the adjoint of an operator is the original operator.

2. Let the inverse $A^{-1}$ of an operator $A$ be defined by $AA^{-1} = A^{-1}A = 1$ where 1 is the identity (do nothing) operator (both sides of the definition are needed). Prove that $(AB)^{-1} = B^{-1}A^{-1}$.

3. Let $A$ be a self-adjoint operator with non-degenerate eigenvalues, $\lambda_i$, $i = 1, ... N$. This means that the set of all normalised eigenkets of $A$ ($A|e_i\rangle = \lambda_i|e_i\rangle$) form an orthonormal basis of the Hilbert space.
(i) Show that $A$ can be written in terms of its eigenvalues $\lambda_i$, as

$$A = \sum_i \lambda_i |e_i\rangle \langle e_i|.$$ 

What is the matrix representation of $A$ in the basis of its eigenvectors?

(ii) Show that the eigenvalues of the operator $A^m$ are $(\lambda_i)^m$ where $m$ is a positive integer. Show that the eigenvalues of the operator $A^{-1}$ are $\lambda_i^{-1}$ so long at none of them are zero.

(iii) Let $f(x)$ be a real function that has a power series expansion: $f(x) = \sum_{m=0}^{\infty} a_m x^m$. $f(A)$ is defined to be the operator $f(A) = \sum_{m=0}^{\infty} a_m A^m$ where $A^0$ is defined to be the identity operator. Show that the eigenvalues of the operator $f(A)$ are $f(\lambda_i)$.

(iv) (Projectors) Show that the operator $P_1 \equiv |e_1\rangle \langle e_1|$ satisfies $P_1^2 = P_1$ (any operator $O$ that satisfies $O^2 = O$ is called a “projector” or “projection operator”). Describe the effect of $P_1$ acting on an arbitrary state $|\psi\rangle$. Calculate the eigenvalues and eigenvectors of $P_1$. Do the same for $P_3 \equiv |e_3\rangle \langle e_3|$

(v) Consider the operator $Q = P_1 + P_3$. Prove that $Q$ is a projector. What happens to a state $|\psi\rangle$ when acted on by $Q$?

Not part of Rapid Feedback:

4. A 3-d Hilbert space has orthonormal basis $\{|e_1\rangle, |e_2\rangle, |e_3\rangle\}$. Let operator $A$ act on the basis vectors as

$$A|e_1\rangle = |e_2\rangle + |e_3\rangle$$
$$A|e_2\rangle = |e_3\rangle + |e_1\rangle$$
$$A|e_3\rangle = |e_1\rangle + |e_2\rangle$$

What is the matrix representation of $A$ in this basis? Write down $A$ as a sum of “back-to-back” operators in Dirac notation. Find the eigenvalues and eigenvectors of $A$.

5. We have seen in lectures if a basis $\{|e_i\rangle\}$ is orthonormal then it satisfies the completeness relation

$$\sum_{i=1}^{N} |e_i\rangle \langle e_i| = 1$$

Prove the converse of this result, i.e., that if the basis satisfies the completeness relation then it must be orthonormal. (Hint: use property (v) at the beginning of the sheet).
Foundations of Quantum Mechanics – Problem Sheet 3

1. $\mathcal{H}$ is a Hilbert space with basis $\{|e_1\rangle, |e_2\rangle\}$ and $S$ is an operator on $\mathcal{H}$. $S$ acts on the basis vectors:

\[ S|e_1\rangle = \frac{i}{2}|e_2\rangle, \quad S|e_2\rangle = -\frac{i}{2}|e_1\rangle. \]

Prove that $S$ is Hermitian. Find the eigenvalues and eigenvectors of $S$. Let the normalised state of the system be $|\psi\rangle = \psi_1|e_1\rangle + \psi_2|e_2\rangle$. What are the possible outcomes of a measurement of the observable corresponding to $S$? What is the probability of each of those outcomes? Hint: You may either work entirely in a matrix representation, or use an equivalent operator representation in which

\[ S = \sum_{i,j} S_{ij}|e_i\rangle\langle e_j| \]

and in which the eigenstates are sums of $|e_1\rangle$ and $|e_2\rangle$.

2. Same question as (1) but a 3 dimensional example. $\mathcal{H}$ is a Hilbert space with basis $\{|e_1\rangle, |e_2\rangle, |e_3\rangle\}$ and $A$ is an operator on $\mathcal{H}$. $A$ acts on the basis vectors:

\[ A|e_1\rangle = -i|e_2\rangle + i|e_3\rangle, \quad A|e_2\rangle = i|e_1\rangle - i|e_3\rangle, \quad A|e_3\rangle = -i|e_1\rangle + i|e_2\rangle. \]

Prove that $A$ is Hermitian. Find the eigenvalues and eigenvectors of $S$. Let the normalised state of the system be $|\psi\rangle = 1/2|e_1\rangle + 1/2|e_2\rangle + 1/\sqrt{2}|e_3\rangle$. What are the possible outcomes of a measurement of the observable corresponding to $S$? What is the probability of each of those outcomes?

3. Let $A$ be a self-adjoint operator. Show that its matrix representation in any orthonormal basis is a Hermitian matrix (meaning that taking the transpose and complex conjugate of the matrix gives back the original matrix).

Let $B$ be a unitary operator. Show that its matrix representation in any orthonormal basis is a unitary matrix (meaning that taking the transpose and complex conjugate of
the matrix gives the inverse of the original matrix). Hint: start with $BB^\dagger = B^\dagger B = 1$ and stick a basis bra on the left and a different basis ket on the right. Use the completeness relation $\sum_i |e_i\rangle\langle e_i| = 1$.

Not part of Rapid Feedback:

4. Show that the eigenvalues of a unitary operator are pure phases. Let $\{|e_1\rangle, \ldots |e_N\rangle\}$ and $\{|e'_1\rangle, \ldots |e'_N\rangle\}$ be two orthonormal bases for the Hilbert space $\mathcal{H}$. Show that the operator $B \equiv \sum_i |e'_i\rangle\langle e_i|$ is unitary. Show that it takes $|e_i\rangle$ basis vectors to $|e'_i\rangle$ basis vectors.

5. You may be familiar from earlier courses with the Gram-Schmidt iterative method for orthogonalizing a given set of vectors. This question introduces you to a different method which is more direct and explicitly orthogonalizes the entire set all at once. Let $\{|f_i\rangle\}$ be a non-orthogonal basis. We first define

$$ G = \sum_i |f_i\rangle\langle f_i| $$

(which is not equal to the identity since the set is not orthogonal). Show that $G$ has strictly positive eigenvalues. It follows that the operator $G^{-\frac{1}{2}}$ exists (Why?). We define

$$ |e_i\rangle = G^{-\frac{1}{2}} |f_i\rangle $$

Show that

$$ \sum_i |e_i\rangle\langle e_i| = 1 $$

It follows (from Q5 on Problem Sheet 2) that the set $\{|e_i\rangle\}$ are orthonormal.
Foundations of Quantum Mechanics – Problem Sheet 4

1. \( \mathcal{H} \) is a Hilbert space with basis \( \{ |e_1\rangle, |e_2\rangle \} \) and \( A \) and \( B \) are two observables corresponding to the Hermitian operators that act on the basis vectors as follows:

\[
A|e_1\rangle = \frac{1}{2}|e_1\rangle, \quad A|e_2\rangle = -\frac{1}{2}|e_2\rangle
\]

and

\[
B|e_1\rangle = \frac{i}{2}|e_2\rangle, \quad B|e_2\rangle = -\frac{i}{2}|e_1\rangle.
\]

(i) Show that the eigenvalues of \( A \) are \( \pm \frac{1}{2} \) and the same for \( B \). Find the eigenvectors of \( A \) and \( B \). See problem 1 of the last exercise set.

(ii) Show that operators \( A \) and \( B \) do not commute.

(iii) Let the normalised state of the system be \( |\psi\rangle = |e_1\rangle \). Calculate the probability of obtaining the result \( \frac{1}{2} \) for a measurement of observable \( A \) and then a split second later, obtaining the result \( \frac{1}{2} \) for a measurement of observable \( B \). Calculate the probability of obtaining the result \( \frac{1}{2} \) for a measurement of observable \( B \) and then, a split second later, obtaining the result \( \frac{1}{2} \) for a measurement of observable \( A \).

(iv) Now consider Hermitian operator \( C \) such that

\[
C|e_1\rangle = \frac{1}{2}|e_2\rangle, \quad C|e_2\rangle = \frac{1}{2}|e_1\rangle.
\]

(v) Write out the three matrix representations of \( C \), \( B \) and \( A \). You should recognise them as \( \frac{1}{2} \) the Pauli matrices \( \sigma_x \), \( \sigma_y \) and \( \sigma_z \) respectively. The observables that correspond to them are the spin of a spin-half particle in the \( x \), \( y \) and \( z \) directions (up to a factor of \( \hbar \)). We say that if value \( +\frac{1}{2} \) \((-\frac{1}{2}) \) is measured for the spin in a given direction then the spin is measured to be “up” ("down") in that direction. Prove that if the state is the state of spin up in the \( z \)-direction then the probability of getting spin up (or spin down) when either the \( y \)-spin or the \( z \)-spin is measured is 1/2. (You have already answered some of this in the first part of the question.)
Hence, conclude that if the state is an eigenstate of spin in any one of the three directions x, y or z, then the probability of measuring spin up (or down) in either of the other two directions is $\frac{1}{2}$.

2. Compute the objects $\langle \phi | \hat{p} \hat{x} | \psi \rangle$ and $\langle \phi | \hat{x} \hat{p} | \psi \rangle$ in terms of the wave functions $\phi(x)$, $\psi(x)$ and their derivatives. (Hint: insert a resolution of the identity in terms of the position eigenstates $|x\rangle$). Confirm that the two results are compatible with the commutation relations.

3. Assuming that the position operator $\hat{x}$ is self adjoint and that the operator $\hat{p}$ satisfies $[\hat{x}, \hat{p}] = i\hbar \textbf{1}$ prove that $\hat{p} - \hat{p}^\dagger = if(\hat{x})$ where $f$ is a real function. In other words, one has to make a separate assumption that $\hat{p}$ is hermitian – it doesn’t follow from the hermiticity of $\hat{x}$ and the canonical commutation relation.

4. Using the unitary displacement operators $U(a)$ and $V(b)$ introduced in lectures, show that, by suitable choice of $a$ and $b$, a given state $|\psi\rangle$ may be unitarily transformed to a state $|\phi\rangle$ for which

$$\langle \phi | \hat{x} | \phi \rangle = 0 = \langle \phi | \hat{p} | \phi \rangle$$

(Hint: do this by considering the action of $U$ and $V$ on the state and on operators $\hat{x}$, $\hat{p}$, not by looking at the wave function $\psi(x)$). Show also that the variances $(\Delta x)^2$ and $(\Delta p)^2$ are invariant under the unitary shifts of the state implemented by $U(a)$ and $V(b)$ (for any choices of $a$ and $b$).

**Not part of Rapid Feedback:**

5. The harmonic oscillator Hamiltonian

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 \hat{x}^2$$

has eigenstates $|u_n\rangle$ and eigenvalues $E_n$. Use the unitary displacement operators to determine the eigenvalues and eigenvectors of the Hamiltonian

$$H_{ab} = \frac{(\hat{p} + b)^2}{2m} + \frac{1}{2}m\omega^2(\hat{x} + a)^2$$
in terms of \( |u_n\rangle \) and \( E_n \).

6. The reflection operator (or parity operator) \( R \) is defined by

\[
R = \int_{-\infty}^{\infty} dx |x\rangle \langle -x|
\]

Prove that \( R|x\rangle = |-x\rangle \), \( R^2 = 1 \), and that

\[
R\hat{x}R = -\hat{x}
\]

Show also that \( R \) has precisely the same form in the momentum representation, i.e.,

\[
R = \int_{-\infty}^{\infty} dp |p\rangle \langle -p|
\]

7. (This question is more difficult than most and is non-examinable material). Consider a pair of non-commuting operators \( A, B \) such that

\[
[A, B] = C
\]

where \( C \) commutes with \( A \) and \( B \). The Baker-Campbell-Hausdorff (BCH) formula in this case is

\[
e^{A+B} = e^{A} e^{B} e^{-\frac{1}{2}[A,B]}
\]

This question shows you how to prove this.

Define the operator function

\[
F(\lambda) = e^{\lambda(A+B)} e^{-\lambda B}
\]

Show that

\[
F'(\lambda) = F(\lambda) A(\lambda)
\]

where

\[
A(\lambda) = e^{\lambda B} A e^{-\lambda B}
\]

This equation is to be solved for \( F(\lambda) \) with \( F(0) = 1 \).

Show that \( A'(\lambda) = -C \), and hence \( A(\lambda) = A - \lambda C \). Insert this in the equation for \( F'(\lambda) \) and confirm that it may be solved with the solution

\[
F(\lambda) = e^{\lambda A - \frac{1}{2} \lambda^2 C}
\]

Show that these two different forms for \( F(\lambda) \) yield the BCH formula when \( \lambda = 1 \).
Foundations of Quantum Mechanics – Problem Sheet 5

1. Show that \((A \otimes B)\dagger = A\dagger \otimes B\dagger\) for any pair of operators \(A, B\).

2. The purpose of this question is to show that, for two vectors \(a, b\),

\[
\langle \psi | (a \cdot \sigma) \otimes (b \cdot \sigma) | \psi \rangle = -a \cdot b
\]

in the EPRB state

\[
| \psi \rangle = \frac{1}{\sqrt{2}} (| \uparrow \rangle \otimes | \downarrow \rangle - | \downarrow \rangle \otimes | \uparrow \rangle)
\]

This is an important result in relation to the Bell inequalities. Here, we do a direct calculation (which is a useful test of understanding of the tensor product), but later we will see that there are briefer and more subtle ways of obtaining the result.

(i) Compute the action of the three Pauli matrices on the \(\sigma_z\) eigenstates \(| \uparrow \rangle, | \downarrow \rangle\) and hence compute the action of \(a \cdot \sigma\) on these states

(ii) Compute the four matrix elements of \(a \cdot \sigma\) in these states.

(iii) Use the definition of the tensor product to write out the left-hand side of Eq.(1) in terms of the matrix elements of \(a \cdot \sigma\) and \(b \cdot \sigma\).

(iv) Hence, using (ii) and (iii), prove Eq.(1).

3. Consider a two state system with Hamiltonian

\[
H = \omega \mathbf{n} \cdot \sigma
\]

where \(\sigma_i\) denotes the Pauli spin matrices and \(\mathbf{n}\) is a unit vector. Show that

\[
(n \cdot \sigma)^k = \begin{cases} 1, & \text{for } k \text{ even} \\ n \cdot \sigma, & \text{for } k \text{ odd.} \end{cases}
\]

By expanding the exponential show that the unitary time evolution operator may be written

\[
\exp \left( -\frac{i}{\hbar} H t \right) = \cos \left( \frac{\omega t}{\hbar} \right) \mathbf{1} - i \sin \left( \frac{\omega t}{\hbar} \right) \mathbf{n} \cdot \sigma
\]
Compute the evolution to time $t$ of the initial state $|\uparrow \rangle$, the $+1$ eigenstate of $\sigma_z$.

**Not part of Rapid Feedback**

4. A particle starts in the initial state described by a wave packet

$$\psi(x) = \frac{1}{(2\pi \sigma^2)^{1/4}} \exp \left( -\frac{x^2}{4\sigma^2} \right)$$

Write down (or compute) this state in the momentum representation, $\tilde{\psi}(p)$. The wave packet in position space at time $t$ may be written

$$\psi(x, t) = \langle x | \exp \left( -\frac{i}{\hbar} H t \right) | \psi \rangle$$

$$= \int_{-\infty}^{\infty} dp \langle x | \exp \left( -\frac{i}{\hbar} H t \right) | p \rangle \langle p | \psi \rangle$$

Show that for the free particle, with $H = \frac{p^2}{2m}$, this expression may be evaluated to yield

$$\psi(x, t) = N' \exp \left( -\frac{x^2}{4(\sigma^2 + \hbar t / 2m)} \right)$$

(where $N'$ is a normalization factor which you do not need to compute). You will need the formula

$$\int_{-\infty}^{\infty} dx \ e^{-ax^2 + bx} = \sqrt{\frac{\pi}{a}} e^{b^2 / 4a}$$

By comparing $|\psi(x, t)|^2$ with the standard form of a Gaussian position probability distribution, show that

$$(\Delta x)^2_t = \sigma^2 + \frac{\hbar^2 t^2}{4m^2 \sigma^2}$$

5. Suppose a state $|\psi\rangle$ for some system evolves under the Schrödinger equation for time $t$ to state $|\psi_t\rangle$. Using the unitary time evolution operator, show that

$$|\langle \psi_t | \psi \rangle|^2 \approx \exp \left( -\frac{t^2}{t_z^2} \right)$$

for small $t$, where $t_z = \hbar / \Delta H$. This timescale is often called the Zeno time, and is the timescale on which a state becomes significantly different to its initial value.
1. Compute the averages of the operators $\hat{x}$, $\hat{p}$, $\hat{x}^2$, $\hat{p}^2$ in an energy eigenstate $|n\rangle$ of the simple harmonic oscillator. (Hint: express the operators in terms of $a$ and $a^\dagger$). Hence show that

$$(\Delta x)(\Delta p) = \hbar(n + \frac{1}{2})$$

Show also that the average kinetic and potential energy of the harmonic oscillator are equal in this state.

2. Write down the Heisenberg picture operator for the harmonic oscillator $\hat{x}(t)$ in terms of the initial position and momentum operators $\hat{x}$ and $\hat{p}$. Use it to compute the unequal time commutation relations $[\hat{x}(t), \hat{x}(t')]$, $[\hat{p}(t), \hat{p}(t')]$ and $[\hat{x}(t), \hat{p}(t')]$. (Note that the second two may be obtained by taking derivatives of the first one). Check that these give the right answer for the equal time commutators when $t = t'$. Now write $\hat{x}(t)$ in terms of raising and lowering operators $a^\dagger$ and $a$ and show that

$$\langle 0|\hat{x}(t)\hat{x}(t')|0\rangle = \frac{\hbar}{2m\omega}e^{-i\omega(t-t')}$$

where $|0\rangle$ denotes the ground state. Confirm that this is compatible with your result for the commutator $[\hat{x}(t), \hat{x}(t')]$ computed above, averaged in the state $|0\rangle$.

3. The coherent states $|p, q\rangle$ have wave functions

$$\langle x|p, q\rangle = \psi_{pq}(x) = \frac{1}{(2\pi\sigma^2)^{1/4}} \exp\left(-\frac{(x - q)^2}{4\sigma^2} + \frac{i}{\hbar}px\right)$$

Show that the coherent states satisfy the completeness relation

$$\int \frac{dp dq}{2\pi\hbar} |p, q\rangle \langle p, q| = 1$$

(Hint: sandwich the left-hand side between two position eigenstates $\langle x|$ and $|y\rangle$ and use the formula for the Fourier transform of the delta-function). This property shows that
the coherent states form a basis which can be used to expand any state. The basis is non-orthogonal, and it is also over-complete, meaning that it contains more states than necessary to be a basis.

**Not Part of Rapid Feedback**

4. The purpose of this question is to consider the evolution of a simple wave packet for the free particle and the harmonic oscillator and highlight the key difference between them: for a free particle, the wave packet spreads, and for a harmonic oscillator, there exist wave packet states that do not spread.

We consider the evolution of a Gaussian wave packet with $\langle x \rangle = 0 = \langle p \rangle$. The most general such Gaussian is

$$\psi(x) = \frac{1}{(2\pi\sigma^2)^{1/4}} \exp\left(\frac{(1 + i\beta - \hbar \dot{x}^2)}{4\sigma^2} x^2\right)$$

where $\beta$ and $\sigma$ are real. Show that $\langle \dot{x}^2 \rangle = \sigma^2$ and $\langle \dot{p}^2 \rangle = \hbar^2 (1 + \beta^2) / 4\sigma^2$

We define $\Sigma = \frac{1}{2} (\dot{x} \dot{p} + \dot{p} \dot{x})$. Show that its value in the Gaussian state above is

$$\Sigma = -\frac{\hbar}{2} \beta$$

Now we consider the time evolution of the above state for the harmonic oscillator hamiltonian. Note that

$$\langle \dot{x}^2 \rangle_t = \langle \psi_t | \dot{x}^2 | \psi_t \rangle = \langle \psi | \dot{x}^2(t) | \psi \rangle$$

which means that the time evolution of $\langle \dot{x}^2 \rangle_t$ may be computed using the Heisenberg equations of motion for $\dot{x}(t)$. Using this observation, show that $\langle \dot{x}^2 \rangle_t$, $\langle \dot{p}^2 \rangle_t$ and $\Sigma_t$ obey the differential equations

$$\frac{d}{dt} \langle \dot{x}^2 \rangle_t = \frac{2}{m} \Sigma_t$$

$$\frac{d}{dt} \Sigma_t = \frac{1}{m} \langle \dot{p}^2 \rangle_t - m \omega^2 \langle \dot{x}^2 \rangle_t$$

$$\frac{d}{dt} \langle \dot{p}^2 \rangle_t = -2m \omega^2 \Sigma_t$$

Use these equations to determine the conditions under which $\langle \dot{x}^2 \rangle_t$ and $\langle \dot{p}^2 \rangle_t$ remain constant under time evolution, and express those conditions in terms of $\sigma$ and $\beta$. Is such a solution possible for the free particle?
Foundations of Quantum Mechanics – Problem Sheet 7

1. The angular momentum commutation relations are

\[ [J_i, J_j] = i\hbar\epsilon_{ijk}J_k \]  

(1)

Using Eq.(1), prove that

\[ [J^2, J_i] = 0 \]
\[ [J_z, J_\pm] = \pm\hbar J_\pm \]
\[ J_\mp J_\pm = J^2 - (J_z^2 \pm \hbar J_z) \]

where \( J^2 \) is defined as \( J^2 = J_1^2 + J_2^2 + J_3^2 \) and \( J_\pm = J_1 \pm iJ_2 \).

2. The raising and lowering equations for angular momentum states are

\[ J_+ | j, m \rangle = \hbar \sqrt{(j - m)(j + m + 1)} | j, m + 1 \rangle \]
\[ J_- | j, m \rangle = \hbar \sqrt{(j + m)(j - m + 1)} | j, m - 1 \rangle \]

(a) Prove that for \( j = \frac{1}{2} \) the angular momentum operators \( J_1, J_2 \) and \( J_3 \) are represented, in a basis of eigenstates of \( J_3 \), by \( \frac{\hbar}{2}\sigma_1, \frac{\hbar}{2}\sigma_2 \) and \( \frac{\hbar}{2}\sigma_3 \), where \{\( \sigma_i \)\} are the Pauli spin matrices.

(b) Find the matrix representations of each component of \( J \) for \( j = 1 \), in a basis of eigenstates of \( J_3 \).

3. Show that in the 2-dimensional Hilbert space of a spin-half particle, the rotation operators \( \mathcal{R}(\mathbf{n}, \omega) = \exp(-i\omega \mathbf{n} \cdot \mathbf{S}/\hbar) \) are represented by the matrices

\[ \mathcal{R}(\mathbf{n}, \omega) = \cos(\omega/2)1 - i\sin(\omega/2)\mathbf{n} \cdot \sigma \]

where \( 1 \) is the identity matrix and \( \sigma = (\sigma_x, \sigma_y, \sigma_z) \). (See Q3 on Problem Sheet 5 – it is almost identical). Confirm that a 2\( \pi \) rotation around any axis sends any state in the spin-1/2 system to minus itself.
4. The Pauli spin matrices rotate according to the formula

\[ \mathcal{R}(\mathbf{n}, \omega) \mathbf{a} \cdot \mathbf{\sigma} \mathcal{R}^\dagger(\mathbf{n}, \omega) = (a \cos \omega + \mathbf{n} \times a \sin \omega) \cdot \mathbf{\sigma} \]

where the rotation matrix \( \mathcal{R} \) is given in Question 3. Show that by a suitable choice of the axis vector \( \mathbf{n} \) and angle \( \omega \), the Pauli matrix \( \sigma_z \) may be rotated into \( \sigma_x \) or \( \sigma_y \). Hence deduce the eigenvectors of \( \sigma_x \) and \( \sigma_y \) in terms of the eigenvectors of \( \sigma_z \), \( | \uparrow \rangle \), \( | \downarrow \rangle \). (You may use the properties of the Pauli matrices in Problem Sheet 5, Q2).

Not Part of Rapid Feedback

5. Under a rotation through angle \( \theta \) about an axis with unit vector \( \mathbf{n} \), the position operator \( \hat{x}_i \) rotates according to the formula

\[ \hat{x}_i(\theta) = R^\dagger(\mathbf{n}, \theta)\hat{x}_i R(\mathbf{n}, \theta) \]

where \( R \) is the rotation operator

\[ R(\mathbf{n}, \theta) = \exp \left( -\frac{i}{\hbar} \theta \mathbf{n} \cdot \mathbf{L} \right) \]

By differentiating \( \hat{x}_i(\theta) \) with respect to \( \theta \) and solving the resulting differential equation, show that

\[ \hat{x}(\theta) = \hat{x} \cos \theta + \mathbf{n} \times \hat{x} \sin \theta \]

You may assume that \( \mathbf{n} \cdot \hat{x} = 0 \).

6. Show that the orbital angular momentum operators, \( L_i = \epsilon_{ijk} \hat{x}_j \hat{p}_k \) obey the angular momentum commutation relations, Eq.(1).
Foundations of Quantum Mechanics – Problem Sheet 8

The following questions refer to the EPRB state, which is given by

\[ |\psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle) \]

1. Show that for any combination \( \Sigma \) of Pauli matrices,

\[ \langle \psi | \Sigma \otimes 1 |\psi\rangle = \frac{1}{2} \text{Tr} \Sigma \tag{1} \]

in the EPRB state, where \( \text{Tr} \) denotes the trace of the matrix (the sum of the diagonal elements) and may also be written

\[ \text{Tr} \Sigma = \langle \uparrow | \Sigma | \uparrow \rangle + \langle \downarrow | \Sigma | \downarrow \rangle \]

2. For a single spin half particle, the projection operators onto the two spin states in the \( a \) direction are

\[ P_\pm^a = \frac{1}{2} (1 \pm a \cdot \sigma) \tag{2} \]

where \( a \) is a unit vector.

(i) Confirm that these operators satisfy \( P^2 = P \) and write out their explicit matrix form for projections in the \( z \)-direction.

(ii) Two experimenters \( A \) and \( B \) make measurements of spin on an EPRB state \( |\psi\rangle \) in directions \( a \) and \( b \). The measured values of spin are denoted \( a, b \) which may each take values \( \pm 1 \). The probabilities \( p(a, b) \) for these values are obtained using the projectors Eq.(2), so, for example, we have

\[ p(+, -) = \langle \psi | P_+^a \otimes P_-^b |\psi\rangle \]

together with three more similar results. Compute the four probabilities \( p(a, b) \) in terms of the angle \( \theta \) between the unit vectors \( a, b \). You may use the results of Question 2
on Problem Sheet 5 and Eq.(1) in Question 1 above. (The results are simplest when expressed in terms of $\theta/2$).

(iii) Using the results of (ii), compute the probabilities $p(a)$ and $p(b)$ and the conditional probabilities $p(a|b)$. Confirm that your results are consistent with the expected anti-correlation of the spins when $a$ and $b$ are parallel.

3. Consider the correlation function

$$C(a, b) = \langle \psi | (a \cdot \sigma) \otimes (b \cdot \sigma) | \psi \rangle$$

we first computed in Question 2 on Problem Sheet 5, where $|\psi\rangle$ is the EPRB state. This question gives a different and somewhat quicker way to obtain the result. First use the property of the EPRB state

$$S^{AB}_{ij} |\psi\rangle = \frac{\hbar}{2} \left( \sigma^A_i \otimes 1 + 1 \otimes \sigma^B_i \right) |\psi\rangle = 0$$

to write $C(a, b)$ in the form of Eq.(1) in Question 1 and then use the properties of the Pauli matrices to show that

$$C(a, b) = -a \cdot b$$

Not Part of Rapid Feedback

4. In lectures we argued that

$$\mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2} = \mathcal{H}_0 \oplus \mathcal{H}_1$$

which means that the four-dimensional Hilbert space of two spin-half particles can be written in terms of a basis consisting of the three states of a spin-one system plus a single state with zero spin. In particular we defined the spin-one states as

$$|1, 1\rangle = |\uparrow\rangle \otimes |\uparrow\rangle$$

$$|1, 0\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle)$$

$$|1, -1\rangle = |\downarrow\rangle \otimes |\downarrow\rangle$$

and the spin zero states is

$$|0, 0\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle)$$
where we are using $|\uparrow\rangle$, $|\downarrow\rangle$ to denote the $S_z$ eigenstates in the spin-half Hilbert spaces (in lectures denoted $|\pm \frac{1}{2}\rangle$, or as $|\frac{1}{2}, \pm \frac{1}{2}\rangle$). The fact that they are spin one and zero follows from their construction described in lectures. However, the purpose of this question is to confirm this explicitly using the spin operator.

The spin operator on $\mathcal{H}_A \otimes \mathcal{H}_B$ is

$$S^{AB}_i = S^A_i \otimes 1 + 1 \otimes S^B_i$$

and the spin operator on the two spin-half Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ is expressed in terms of the Pauli spin matrices as $S_i = (\hbar/2) \sigma_i$. Show that

$$(S^{AB})^2 = \frac{\hbar^2}{2} \left(3 + \sigma_i \otimes \sigma_i\right)$$

where $\sigma_i \otimes \sigma_i$ means

$$\sigma_i \otimes \sigma_i = \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z$$

Determine the action of $\sigma_i \otimes \sigma_i$ on the four states $|s\rangle \otimes |s'\rangle$, where $s, s'$ denotes $\uparrow$ or $\downarrow$, and hence on the states $|1, m\rangle$ (where $m = 1, 0, -1$) and $|0, 0\rangle$. (You may use the properties of the Pauli matrices in Question 2 Problem Sheet 5). Hence show that these states are indeed eigenstates of $(S^{AB})^2$ with the appropriate eigenvalue.

Hint: note any simplifications that arise from the fact that the ladder operators $S^{AB}_\pm$ commute with $(S^{AB})^2$. 

3
Foundations of Quantum Mechanics – Problem Sheet 9

1. Consider the pure state

$$|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$$

and the mixed state

$$\rho = |a|^2|\uparrow\rangle \langle \uparrow| + |b|^2|\downarrow\rangle \langle \downarrow|$$

where $|a|^2 + |b|^2 = 1$ and $|\uparrow\rangle, |\downarrow\rangle$ denote spin up and down in the z-direction. Show that the two states gives the same probabilities for a measurement of spin up and down in the z-direction. Compute the probabilities for the spin to be in the $+$ and $-$ directions in the $x$-direction for each state. You may use the eigenstates $|\pm\rangle = (|\uparrow\rangle \pm |\downarrow\rangle)/\sqrt{2}$.

2. A device emits particles which are in state $|\psi_1\rangle$ with probability $p_1$ and in state $|\psi_2\rangle$ with probability $p_2$, where $p_1 + p_2 = 1$. Any predictions concerning measurements on the particle are made using the density operator

$$\rho = p_1|\psi_1\rangle \langle \psi_1| + p_2|\psi_2\rangle \langle \psi_2|$$

Confirm that this object is indeed a density operator even if the states $|\psi_1\rangle$ and $|\psi_2\rangle$ are not orthogonal. Suppose now that $|\psi_1\rangle = |\uparrow\rangle$ and $|\psi_2\rangle = |+\rangle$ (in the notation of Question 1). Compute the probabilities for spin up and down in the z-direction.

3. The EPRB state for a pair of particles $A$, $B$ is given by

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle_A \otimes |\downarrow\rangle_B + |\downarrow\rangle_A \otimes |\uparrow\rangle_B)$$

Show that the reduced density operator for particle $A$ only is

$$\rho_A = \frac{1}{2} 1$$

where $1$ denotes the identity operator on a single two-state system. (Note: this can either be done using the explicit form of the EPRB state above, or from making simple
observations using Eq.(1) in Question 1 on Problem Sheet 8.) Why would you expect this result, given what you know already about properties of the EPRB state?

4. The density matrix of a simple open quantum system evolves according to the equation

\[ \frac{d\rho}{dt} = -\frac{i}{\hbar} [H, \rho] - \left( L^2 \rho + \rho L^2 - 2L\rho L \right) \]

for some hermitian operator \( L \). Show that \( \text{Tr}\rho \) is preserved under this evolution. Show that \( \text{Tr}\rho^2 \) decreases with time. What is the significance of this result in terms of the evolution of pure and mixed states?

Hints: You will need to use the fact that \( \text{Tr}(AB) = \text{Tr}(BA) \) and \( \text{Tr}(ABC) = \text{Tr}(CAB) \). You will also need to prove and use the fact that an expression of the form \( \text{Tr}(A \hat{1} A) \) is non-negative and set \( A = [\rho, L] \).
The GHZ (Greenberger, Horne, Zeilinger) state is a three particle state which exhibits correlations similar to the EPRB state. It may be used to derive a very striking contradiction which rules out in a single stroke a hidden variables interpretation of the correlations.

The state is given by

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A \otimes |\uparrow\rangle_B \otimes |\uparrow\rangle_C + |\downarrow\rangle_A \otimes |\downarrow\rangle_B \otimes |\downarrow\rangle_C)$$

For notational convenience it may be written

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle)$$

The story below also works, with some minor changes, for a state with a $-$ sign between the two terms. Recall the following properties of the Pauli matrices

$$\sigma_x|\uparrow\rangle = |\downarrow\rangle, \quad \sigma_y|\uparrow\rangle = i|\downarrow\rangle, \quad \sigma_z|\uparrow\rangle = |\uparrow\rangle$$

$$\sigma_x|\downarrow\rangle = |\uparrow\rangle, \quad \sigma_y|\downarrow\rangle = -i|\uparrow\rangle, \quad \sigma_z|\downarrow\rangle = -|\downarrow\rangle$$

The correlations in the GHZ state are revealed using the three operators

$$T_1 = \sigma_x^A \otimes \sigma_y^B \otimes \sigma_y^C, \quad T_2 = \sigma_y^A \otimes \sigma_x^B \otimes \sigma_y^C, \quad T_3 = \sigma_y^A \otimes \sigma_y^B \otimes \sigma_x^C$$

1. Show that $T_1$, $T_2$ and $T_3$ all commute. Use the fact that

$$(A \otimes B \otimes C)(A' \otimes B' \otimes C') = AA' \otimes BB' \otimes CC'$$

Together with properties of the Pauli matrices (namely that they anticommute and square to the identity).

2. Show that the GHZ state is an eigenstate of $T_1$, $T_2$ and $T_3$ with eigenvalue $-1$.

3. Now suppose that observers at $A$, $B$ and $C$ measure $\sigma_x^A$, $\sigma_y^B$ and $\sigma_y^C$, and obtain values $m_x^A$, $m_y^B$, $m_y^C$, where each value $m$ may be $\pm 1$. The fact that $T_1|\psi\rangle = -|\psi\rangle$ implies that the values must satisfy

$$m_x^A m_y^B m_y^C = -1$$ (1)
(This is the analogue of the anticorrelation in the EPRB state). This means that \( m_x^A \) may be deduced from measurements of \( \sigma_y^B \) and \( \sigma_y^C \) and thus \( m_x^A \) is an element of reality.

Derive the two analogous relationships between values of spin for \( T_2 \) and \( T_3 \).

[The above three relationships imply that all the \( x \) spins may be deduced by measuring the \( y \) spins of the other two, so the \( x \) spins are elements of reality. Similarly, we can with certainty predict the value of \( y \) spin of one of the particles by measuring one \( x \) spin and one \( y \) spin of the other two particles. In this way, we deduce that all six values \( m_x^A, m_x^B, m_x^C, m_y^A, m_y^B, m_y^C \) are elements of reality since they may be deduced by making measurements on particles \( A, B, C \) and then using the three correlation relationships derived above to deduce the other three values. Of course we deduce in this way the values for non-commuting operators, but this is simply following the EPR definition of “elements of reality”, as we did in the EPRB case].

By taking the product of the three relationships of the form Eq.(1), show that

\[
m_x^A m_x^B m_x^C = -1
\]  

(2)

4. To check the deduction Eq.(2) in a different way, we examine operator corresponding to Eq.(2),

\[ R = \sigma_x^A \otimes \sigma_x^B \otimes \sigma_x^C \]

and repeat the above steps in terms of operators. Show that \( R \) commutes with \( T_1, T_2 \) and \( T_2 \). Show also that

\[ T_1 T_2 T_3 = -R \]

Hence show that

\[ R|\psi\rangle = +|\psi\rangle \]  

(3)

(This may also be shown directly using the properties of \( R \) acting on \( |\psi\rangle \)). Eq.(3) means that if the \( x \)-spins of all three particles were measured, their values must satisfy

\[
m_x^A m_x^B m_x^C = +1
\]  

(4)

Note the contradiction with Eq.(2)!

5. Identify the point at which the two “derivations” of Eq.(2) lead to a different sign.

1. **Definition.** A subspace $W$ of a vector $V$ is a nonempty subset of $V$ with the property that if $|a\rangle, |b\rangle \in W$, then $\alpha|a\rangle + \beta|b\rangle \in W$ for all $\alpha, \beta \in \mathbb{C}$.

(a) $U$ is nonempty since $0 = (0, 0, 0) \in U$.

Let $a = (a^1, a^2, a^3), b = (b^1, b^2, b^3) \in U$.

Then, $\sum_{i=1}^{3} a_i = \sum_{i=1}^{3} b_i = 0$.

Hence, $\sum_{i=1}^{3} (\alpha a_i + \beta b_i) = 0 \ \forall \alpha, \beta \in \mathbb{C}$,

i.e., $\alpha a + \beta b \in U$. So, $U$ is a subspace of $\mathbb{C}^3$.

(b) $V$ is not a subspace of $\mathbb{C}^3$. Take $a = b = (0, 1, 0) \in V$, $\alpha = \beta = 1 \in \mathbb{C}$; then, $\alpha a + \beta b = (0, 2, 0) \notin V$.

(c) $W$ is nonempty since $0 \in W$.

Let $a, b \in W$. Then, $\sum_{i=1}^{3} a_i = \sum_{i=1}^{3} b_i = 0$, $a_i = a_2, b_i = b_2$.

Then, $\sum_{i=1}^{3} (\alpha a_i + \beta b_i) = 0$, $\alpha a + \beta b = \alpha a_2 + \beta b_2 \ \forall \alpha, \beta \in \mathbb{C}$,

i.e., $\alpha a + \beta b \in W$. So, $W$ is a subspace of $\mathbb{C}^3$. 
1. (Continued)

(d) $X$ is not a subspace of $C^3$.
Take $a = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \in X$, $\alpha = \beta = 1 \in C$.
Then, $\alpha a + \beta b = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \notin X$.

(e) $Y$ is nonempty since $0 \in Y$.
Let $a, b \in Y$. Then, $\langle w, a \rangle = \langle w, b \rangle = 0$.
Hence, $\langle w, \alpha a + \beta b \rangle = \alpha \langle w, a \rangle + \beta \langle w, b \rangle = 0 \quad \forall \alpha, \beta \in C$,
i.e. $\alpha a + \beta b \in Y$.
So, $Y$ is a subspace of $C^3$.

**Note:** Geometrically, $Y$ is simply a complex plane, passing through the origin, with the normal vector $w$. 
2. \( \{ |e_i\rangle \} \) is an orthonormal basis
\[ \langle e_i | e_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases} \]

\[ = \delta_{ij} \quad \text{(Kronecker delta)} \]

(a) Given \( |\psi\rangle = 5|e_1\rangle + (2+i)|e_2\rangle \).
Then, \( \langle \psi | = 5 \langle e_1 | + (2-i) \langle e_2 | \).
Then, \( \langle \psi | \psi \rangle = 5 \cdot 5 \langle e_1 | e_1 \rangle + 5(2-i) \langle e_2 | e_1 \rangle + 5(2+i) \langle e_1 | e_2 \rangle + (2+i)(2-i) \langle e_2 | e_2 \rangle \)
\[ = 25 + 5 = 30. \]
Thus, \( ||\psi|| = \sqrt{\langle \psi | \psi \rangle} = \sqrt{30}. \)

(b) \( |\psi\rangle = \sum_{i=1}^{N} \psi_i |e_i\rangle \). Then, \( \langle \psi | = \sum_{j=1}^{N} \psi_j^* \langle e_j | \)
Hence, \( \langle \psi | \psi \rangle = \sum_{i,j=1}^{N} \psi_i^* \psi_j \langle e_j | e_i \rangle \)
\[ = \sum_{i,j=1}^{N} \psi_i^* \psi_j \delta_{ij} \]
\[ = \sum_{i=1}^{N} |\psi_i|^2 \quad \text{for all } i. \]
Thus, \( ||\psi|| = \sqrt{\sum_{i=1}^{N} |\psi_i|^2}. \)

Alternatively: If you're not good at summation sign, you just write things out explicitly!
This works! But if you want to be a GOOD
2.(b) Explicitly, we have

$$|\psi\rangle = \psi_1|e_1\rangle + \psi_2|e_2\rangle + \ldots + \psi_N|e_N\rangle.$$ 

$$<\varphi| = \psi_1^*<e_1| + \psi_2^*<e_2| + \ldots + \psi_N^*<e_N|.$$ 

But $$<e_1|e_1> = <e_2|e_2> = \ldots = 1,$$
and $$<e_1|e_2> = <e_1|e_3> = \ldots = 0.$$ 

Thus, $$<\varphi|\psi> = \psi_1\psi_1^* + \ldots + \psi_N\psi_N^*$$

$$\Rightarrow \|\psi\| = \sqrt{\psi_1\psi_1^* + \ldots + \psi_N\psi_N^*}.$$ 

(C) $$<\psi|\phi> = (-i\langle e_1|+(3-2i)\langle e_2|(6|e_1\rangle + |e_2\rangle + i|e_3\rangle)$$

$$= -6i + (3-2i) = 3 - 8i.$$
3. (i) $\{e_1, e_2, \ldots, e_N\}$ is a basis if

$$\forall \psi \in \mathcal{H}, \quad \psi = \sum_{i=1}^{N} \alpha_i \langle e_i | \psi \rangle$$

Suppose there is a different expansion

$$\psi = \sum_{i} \beta_i \langle e_i |$$

$$\Rightarrow \quad 0 = \sum_{i} (\alpha_i - \beta_i) \langle e_i |$$

$$\Rightarrow \quad | e_i \rangle \text{ lin. ind. } \implies \alpha_i - \beta_i = 0$$

So expansion is unique.

(ii) Consider set $\{f_1, f_2, \ldots, f_{N+1}\}$ and

$$\sum_{i=1}^{N+1} \langle f_i | = 0$$

In column vector notation:

$$\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_{N+1}
\end{pmatrix}
\begin{pmatrix}
f_{11} \\
f_{12} \\
\vdots \\
f_{1N}
\end{pmatrix}
\begin{pmatrix}
f_{21} \\
f_{22} \\
\vdots \\
f_{2N}
\end{pmatrix}
\begin{pmatrix}
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{pmatrix}
\begin{pmatrix}
f_{N+1,1} \\
f_{N+1,2} \\
\vdots \\
f_{N+1,N}
\end{pmatrix}
= 0$$

This N eqs for N+1 unknowns $\alpha_i$

$\Rightarrow \exists \text{ a non-trivial set of } \alpha_i \text{ so } \alpha_i \neq 0 \text{ and vectors } | f_i \rangle \text{ are linearly dependent.}$

If there are only $M < N$ vectors in the basis, can easily find a vector orthogonal to all vectors in the set, so it is not complete.
1. **Adjoint Operation:** \( A^\dagger = (A^*)^\dagger = (A^*)^* \).

   If we take a basis \( \{ |e_i\rangle \} \), then
   \[ \langle e_i^* | A^\dagger | e_j \rangle = \langle e_j | A | e_i^* \rangle^* \]

   Consider
   \[ \langle e_i^* | (B^\dagger)^\dagger | e_j \rangle = \langle e_j | B^\dagger | e_i^* \rangle^* \]
   \[ = (\langle e_i^* | B^\dagger | e_j \rangle^*)^* \]
   \[ = \langle e_i^* | B | e_j \rangle \quad \forall i, j \]

   Therefore, \( (B^\dagger)^\dagger = B \).

2. To prove that \( (AB)^\dagger = B^\dagger A^{-1} \), we want to check
   (1) \( (AB)(B^\dagger A^{-1}) = 1 \) and (2) \( (B^\dagger A^{-1})^\dagger X AB = 1 \).

   Check (1):
   \[ (AB)(B^\dagger A^{-1}) = A(BB^\dagger)A^{-1} = AA^{-1} = 1 \]
   Similarly for (2).

3. (i) Want to prove \( A = \sum \lambda_i |e_i^* \times e_i \rangle \langle e_i^* | \).

   **Proof:** Consider
   \[ (\sum \lambda_i |e_i^* \times e_i \rangle \langle e_i^* |) |e_j \rangle \]
   \[ = \sum \lambda_i |e_i^* \rangle \langle e_i^* | e_j \rangle \]
   \[ = \sum \lambda_i \delta_{ij} |e_i \rangle \]
   \[ = \sum \lambda_i \delta_{ij} |e_i \rangle = \lambda_j |e_j \rangle = A |e_j \rangle \quad \forall j \]
3. (i) (Cont.) Thus, $\sum \lambda_i e_i \langle e_i | e_i \rangle = A$. □

Observe that

$$\langle e_j | A e_k \rangle = \langle e_j | (\sum \lambda_i e_i \langle e_i | e_k \rangle) \rangle$$

$$= \sum \lambda_i \langle e_j | e_i \rangle \langle e_i | e_k \rangle$$

$$= \sum \lambda_i \delta_{j,i} \delta_{i,k}$$

$$= \begin{cases} \lambda_j & \text{if } j=k, \\
0 & \text{if } j \neq k. \end{cases}$$

Thus, the matrix rep. of $A$ wrt the basis $\{ | e_i \rangle \}$ is

$$\begin{pmatrix} \lambda_1 & 0 & \cdots \\
0 & \lambda_2 & \cdots \\
\vdots & \vdots & \ddots \end{pmatrix}.$$

(ii) Want to prove that $A^m | e_i \rangle = \lambda_i^m | e_i \rangle$ for $m \in \mathbb{N}$.

Induction:

STEP 1: $A | e_i \rangle = \lambda_i | e_i \rangle$. True for $m=1$.

STEP 2: Suppose $A^k | e_i \rangle = \lambda_i^k | e_i \rangle$. (*)

Then, $A^{k+1} | e_i \rangle = A^k (A | e_i \rangle)$

$$= A^k (\lambda_i | e_i \rangle)$$

$$= \lambda_i (A^k | e_i \rangle)$$

(* *) $\lambda_i (\lambda_i^k | e_i \rangle) = \lambda_i^{k+1} | e_i \rangle$. □
3. (ii) (Cont.) Suppose $\lambda_i \neq 0 \, \forall i$. WTP: $A^T l e_i > = \lambda_i^{-1} l e_i >$.

(It can be proven that $\det A = \prod \lambda_i \neq 0$, and so $A$ is invertible.)

Consider $A l e_i > = \lambda_i l e_i >$. Then, $l e_i > = A^{-1} l e_i >$. \(\star\)

Then, $A^{-1} l e_i > = A^{-1} (\lambda_i^{-1} l e_i >)$

$$= \lambda_i^{-1} l e_i >. \quad \square$$

(iii) WTP: $f(A) l e_i > = f(\lambda_i) l e_i >$.

**Proof**.

$f(A) l e_i > = \sum_{m=0}^{\infty} a_m A^m l e_i >$

$$= a_0 A^0 l e_i > + a_1 A l e_i > + a_2 A^2 l e_i > + \cdots$

$$= a_0 l e_i > + a_1 \lambda_i l e_i > + a_2 \lambda_i^2 l e_i > + \cdots$

$$= \sum_{m=0}^{\infty} a_m \lambda_i^m l e_i > = f(\lambda_i) l e_i >. \quad \square$

*********

**Projection Operators**

Example in Euclidean space $\mathbb{R}^2$.

Let $P_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $P_y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Observe: $P_x \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}$, $P_x \begin{pmatrix} \sqrt{3} \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ 0 \end{pmatrix}$

$P_y \begin{pmatrix} 7/8 \\ 3/2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3/2 \end{pmatrix}$.
Properties of $P_x$ and $P_y$

1. $P_x^2 = P_x$, $P_y^2 = P_y$.

2. $P_x$ acts on a vector $\mathbf{v} \in \mathbb{R}^2$ by picking out the part of $\mathbf{v}$ in the $x$-direction:

$$P_x \mathbf{v} = P_x \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} v_x \\ 0 \end{pmatrix}.$$ 

Similarly for $P_y$.

3. The eigenvalues of $P_x$ are 0 and 1, and the corresponding eigenvectors are $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

4. $P_x + P_y = \mathbb{I}$, $P_x P_y = P_y P_x = 0$.

5. Let $A$ be a diagonalisable $2 \times 2$ matrix with 2 distinct eigenvalues $\lambda_x, \lambda_y$ corresponding to eigenvectors $\hat{e}_x, \hat{e}_y$. Then, $A = \lambda_x P_x + \lambda_y P_y$.

[This just means that wrt. the basis $\{\hat{e}_x, \hat{e}_y\}$, $A$ has a matrix representation

$$\begin{pmatrix} \lambda_x & 0 \\ 0 & \lambda_y \end{pmatrix} = \lambda_x \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \lambda_y \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \lambda_x P_x + \lambda_y P_y.$$

Now try to generalise these to any vector spaces...]
3. (iV) Let \( P_i = |e_i\rangle \langle e_i| \).

WTP: \( P_i^2 = P_i \)

Proof: \( P_i^2 = (|e_i\rangle \langle e_i|)(|e_i\rangle \langle e_i|) = |e_i\rangle \langle \langle e_i| e_i\rangle | e_i\rangle = |e_i\rangle \langle e_i| = P_i \). □

Observe that if \( |\psi\rangle = \sum_j \psi_j |e_j\rangle \),

\[ P_i |\psi\rangle = (|e_i\rangle \langle e_i|) \sum_j \psi_j |e_j\rangle = |e_i\rangle \sum_j \psi_j \langle e_i| e_j\rangle = \psi_i |e_i\rangle, \]

i.e. \( P_i \) acts on a ket \( |\psi\rangle \in \mathcal{H} \) by picking out the part of \( |\psi\rangle \) in the direction \( |e_i\rangle \). We say \( P_i \) projects \( |\psi\rangle \) onto \( |e_i\rangle \).

Now consider eigenvectors & eigenvalues of \( P_i \). Since any vector in \( \mathcal{H} \) can be written as a linear combination of the basis \( \{|e_i\rangle\} \), it suffices to consider only the actions of \( P_i \) on each vector \( |e_j\rangle \).
3. (iv) (Cont.) 
\[ P_i |e_j\rangle = \begin{cases} 
|e_i\rangle & \text{if } j=i, \\
0 & \text{if } j \neq i.
\end{cases} \]

So, \( |e_i\rangle \) is an e.vector of \( P_i \) with e.value 1, and \( |e_j\rangle \) (for \( j \neq i \)) are e.vectors of \( P_i \) with e.value 0.

**Proposition.** 0 and 1 are the only eigenvalues of \( P_i \).

**Proof.** Let \( |\psi\rangle \) be an e.vector of \( P_i \) with e.value \( \lambda \).

Then, 
\[ P_i |\psi\rangle = \lambda |\psi\rangle. \]

Then, 
\[ P_i^2 |\psi\rangle = P_i |\psi\rangle = \lambda P_i |\psi\rangle. \]

Hence, 
\[ \lambda^2 |\psi\rangle = \lambda |\psi\rangle. \]

Applying \( \langle \psi | \) to both sides, we have 
\[ \lambda^2 \langle \psi | \psi \rangle = \lambda \langle \psi | \psi \rangle. \]

Since \( |\psi\rangle \) is an e.vector, \( \langle \psi | \psi \rangle \neq 0 \).

So, 
\[ \lambda^2 = \lambda \Rightarrow \lambda = 0 \text{ or } \lambda = 1. \]

\( \square \)

(V) Let \( Q = P_i + P_j \) with \( i \neq j \).

**Proposition.** \( Q^2 = Q \)

**Proof.** 
\[
Q^2 = (P_i + P_j)(P_i + P_j) = P_i^2 + P_j P_i + P_i P_j + P_j^2
\]

\[
= P_i + 0 + 0 + P_j \quad (P_i^2 = P_i \text{ & } P_j P_i = 0 \text{ i\#j})
\]

\[
= P_i + P_j = Q. \quad \square
\]
3. (v) (Cont.)
Observe that since $P_j |\psi\rangle = \psi_j |e_j\rangle$, $P_j |\psi\rangle = \psi_j |e_j\rangle$, it follows that
$Q |\psi\rangle = \psi_j |e_j\rangle + \psi_j |e_j\rangle \in \text{span} \{ |e_i\rangle, |e_j\rangle \}$.
So, $Q$ projects $|\psi\rangle$ into the subspace of $\mathcal{H}$ spanned by $|e_i\rangle$ and $|e_j\rangle$.

4. We see that
$$<e_i| A |e_j\rangle = \begin{cases} 0 & \text{if } i=j, \\ 1 & \text{if } i \neq j. \end{cases}$$
So, wrt. the basis $\{|e_i\rangle\}_{i=1}^3$, $A$ has a matrix rep.
$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$ 
We note that $\sum_{j=1}^{3} |e_j\rangle \times |e_j\rangle = |1\rangle$.
Then, $A = A |1\rangle$
$$= A \left( \sum_{j=1}^{3} |e_j\rangle \times |e_j\rangle \right)$$
$$= \sum_{j=1}^{3} (A |e_j\rangle) <e_j|$$
$$= (A |e_1\rangle) <e_1| + (A |e_2\rangle) <e_2| + (A |e_3\rangle) <e_3|$$
$$= (|e_2\rangle + |e_3\rangle) <e_1| + (|e_3\rangle + |e_1\rangle) <e_2|$$
$$+ (|e_1\rangle + |e_2\rangle) <e_3|$$
$$= \sum_{i=1}^{3} |e_i\rangle <e_i|.$$
4. (Cont.)

In order to find the values of \( A \), we solve the characteristic equation:

\[
\det (A - \lambda I) = 0
\]

\[
\begin{vmatrix}
-\lambda & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & -\lambda & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & -\lambda
\end{vmatrix} = 0
\]

\[
\begin{vmatrix}
-\lambda - 1 & 1 + \lambda & 0 \\
\frac{1}{2} & -\lambda & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & -\lambda
\end{vmatrix} = 0 \quad \text{(row } 1 \text{ - row } 2\text{)}
\]

\[
(-\lambda - 1)(\lambda^2 - 1) - (1 + \lambda)(\lambda - 1) = 0
\]

\[
(\lambda - 1)(\lambda^2 - \lambda - 2) = 0
\]

\[
(\lambda + 1)(\lambda - 2)(\lambda + 1) = 0
\]

\[
\lambda = -1 \text{ or } \lambda = 2.
\]

Consider \( \lambda = 2 \):

\[
A \mathbf{v} = 2 \mathbf{v}
\]

\[
\begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
\mathbf{v}_1 \\
\mathbf{v}_2 \\
\mathbf{v}_3
\end{pmatrix} =
\begin{pmatrix}
2\mathbf{v}_1 \\
2\mathbf{v}_2 \\
2\mathbf{v}_3
\end{pmatrix}
\]

\[
-2\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = 0 \quad \Rightarrow \quad \mathbf{v}_1 = \mathbf{v}_2, \quad \mathbf{v}_2 = \mathbf{v}_3
\]

\[
\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3 = 0
\]

\[
\mathbf{v}_1 + \mathbf{v}_2 - 2\mathbf{v}_3 = 0 \quad \Rightarrow \quad \text{A normalised e.vector is } \mathbf{x} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ i \\ 1 \end{pmatrix}.
\]

Dirac notation: \( \frac{1}{\sqrt{3}} (|1\rangle + |2\rangle + |3\rangle) \).
Consider $\lambda = -1$: $Av = -v$

$$\Rightarrow \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = -\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\Rightarrow v_1 + v_2 + v_3 = 0.$$

There will be 2 linear independent e.vectors with $\lambda = -1$. We can choose to make them orthogonal; e.g.,

$$\frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \text{ and } \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$ 

In Dirac notation,

$$\frac{1}{\sqrt{6}}(1e_1 + 1e_2 - 2e_3),$$

$$\frac{1}{\sqrt{2}}(1e_1 - 1e_2).$$
5. \[ \sum_{i} e_{i} e_{i} = 1 \]
   
   Act on \( e_{j} \) for any \( j \):
   
   \[ \sum_{i} e_{i} e_{i} e_{j} = e_{j} \]
   
   \[ e_{j} (e_{j} e_{j} - 1) + \sum_{i \neq j} e_{i} e_{i} e_{j} = 0 \]

   This has the form:
   
   \[ \sum_{i} \alpha_{i} e_{i} = 0 \Rightarrow \alpha_{i} = 0 \]

   so \( e_{j} e_{j} = 1 \) and \( e_{i} e_{j} = 0 \) \( i \neq j \)

   i.e. \( e_{i} e_{j} = \delta_{i,j} \).\[10\]
1. The matrix elements of $S$ wrt. the basis $\{|e_1\rangle, |e_2\rangle\}$ are

$$\langle e_1 | S | e_2 \rangle = 0 \quad \langle e_1 | S | e_1 \rangle = \frac{i}{2} \quad \langle e_2 | S | e_2 \rangle = 0 \quad \langle e_2 | S | e_1 \rangle = -\frac{i}{2}$$

Thus, the matrix representation of $S$ wrt. this basis is $S = \left( \begin{array}{cc} 0 & -i/2 \\ i/2 & 0 \end{array} \right)$.

Observe that $S^* = (S^*)^* = \left( \begin{array}{cc} 0 & i/2 \\ -i/2 & 0 \end{array} \right)^* = \left( \begin{array}{cc} 0 & -i/2 \\ i/2 & 0 \end{array} \right) = S$,

and so $S$ is a Hermitian matrix, and hence $S$ is a Hermitian operator.

**Postulate 1:** The only possible result of the measurement of a physical quantity $A$ is one of eigenvalues of the corresponding Hermitian operator $A$. 
**Postulate 2.1** (case of a discrete non-degenerate spectrum): When the physical quantity $A$ is measured on a system in the normalised state $|\psi\rangle$, the probability $P(a_n)$ of obtaining the non-degenerate eigenvalue $a_n$ of the corresponding operator $A$ is

$$P(a_n) = |\langle u_n | \psi \rangle|^2,$$

where $|u_n\rangle$ is the normalised eigenvector of $A$ with the eigenvalue $a_n$.

**Postulate 2.2** (case of a discrete spectrum): When the physical quantity $A$ is measured on a system in the normalised state $|\psi\rangle$, the probability $P(a_n)$ of obtaining the eigenvalue $a_n$ of the corresponding operator $A$ is

$$P(a_n) = \sum_{i=1}^{g_n} |\langle u_{n_i}^i | \psi \rangle|^2,$$

where $g_n$ is the degree of degeneracy of $a_n$ and $\{|u_{n_i}^i\rangle\}$ ($i=1,2,...,g_n$) is an orthonormal set of vectors which forms a basis in the eigensubspace $E_n$ associated with the eigenvalue $a_n$ of $A$. 
Postulate 3: If the measurement of the physical quantity $A$ on the system in the state $|\psi\rangle$ gives the result $a_n$, the state of the system immediately after the measurement is the normalised projection

$$\frac{|\psi_n\rangle}{\sqrt{\langle \psi | \psi_n \rangle}}$$

of $|\psi\rangle$ onto the eigensubspace associated with $a_n$.

[For more details, see chapter III of C. Cohen-Tannoudji, B. Diu, F. Laloe, Quantum Mechanics, Wiley-Interscience, 1977. Claude Cohen-Tannoudji was awarded a Nobel Prize in 1997 for development of methods to cool and trap atoms with laser light.]
Let us find the eigenvalues of $S$.

$$\text{det}(S - \lambda I) = 0 \iff \begin{vmatrix} -\lambda & -i/2 \\ i/2 & -\lambda \end{vmatrix} = 0$$

$$\iff \lambda^2 - \frac{1}{4} = 0 \iff \lambda = \pm \frac{1}{2}.\quad \text{(non-degenerate)}$$

For $\lambda = +\frac{1}{2}$, $Sv_+ = \frac{1}{2} v_+$

$$\Rightarrow -\frac{i}{2} v_+ y = \frac{1}{2} v_+ x, \quad \frac{i}{2} v_+ x = \frac{1}{2} v_+ y$$

Choose $v_x = 1$. Then, $v_y = i$

Then, the normalised e. vector with $\lambda = \frac{1}{2}$ is

$$v_+ = \frac{1}{\sqrt{2}} \left( \frac{1}{i} \right)$$

Similarly, the normalised e. vector with $\lambda = -\frac{1}{2}$ is

$$v_- = \frac{1}{\sqrt{2}} \left( \frac{1}{-i} \right)$$

In Dirac notation, we have

$$|v_+\rangle = \frac{1}{\sqrt{2}} (1\psi_1 + i\psi_2\rangle), \quad |v_-\rangle = \frac{1}{\sqrt{2}} (1\psi_1 - i\psi_2\rangle).$$

Now let us find probabilities:

$$P(+\frac{1}{2}) = |\langle v_+ | \psi \rangle|^2 \quad \text{(Postulate 2.1)}$$

$$= \frac{1}{2} |\langle e_1, -i\langle e_2| \times \psi_1 |e_1\rangle + \psi_2 |e_2\rangle|^2$$

$$= \frac{1}{2} |\psi_1 - i\psi_2|^2.$$  

Similarly, $P(-\frac{1}{2}) = |\langle v_- | \psi \rangle|^2 = \frac{1}{2} |\psi_1 + i\psi_2|^2.$
What Is The Physical Meaning of $S$?

1. $S$ is Hermitian, so it must correspond to some physical quantity.
2. The eigenvalues of $S$ are $+\frac{1}{2}$ and $-\frac{1}{2}$, and the corresponding eigenstates are $|\nu_+\rangle$, $|\nu_-\rangle$.
   - We know that $S$ is an operator corresponding to a spin component in some direction (say $y$) of the particle, and $|\nu_-\rangle$ and $|\nu_+\rangle$ are respectively the 'spin-left' and 'spin-right' eigenstates.
   - What about $|e_1\rangle$ & $|e_2\rangle$? In this convention, we know that if $S_z$ is a $z$-component of the spin operator, then $S_z |e_1\rangle = \frac{1}{2} |e_1\rangle$ and $S_z |e_2\rangle = -\frac{1}{2} |e_2\rangle$. Therefore, $|e_1\rangle$ & $|e_2\rangle$ are said to be the 'spin-up' and 'spin-down' eigenstates.
   - Observe that $|e_1\rangle$ & $|e_2\rangle$ are NOT eigenvectors of $S$, although they are eigenvectors of $S_z$. 
Therefore, if the particle is in an eigenstate of $S_z$ (i.e., it has a definite spin $z$-component), then the measurement of spin $y$-component in that state will give an indefinite result, and vice-versa. We say that the spin $y$-cpt and spin $z$-cpt are not simultaneously measurable.

2. The matrix rep. of $A$ wrt. the basis $\{|e_1\rangle, |e_2\rangle\}$

$|e_3\rangle^{1/2}$ is

$$A = \begin{pmatrix}
    \langle e_1|A|e_1 \rangle & \langle e_1|A|e_2 \rangle & \langle e_1|A|e_3 \rangle \\
    \langle e_2|A|e_1 \rangle & \langle e_2|A|e_2 \rangle & \langle e_2|A|e_3 \rangle \\
    \langle e_3|A|e_1 \rangle & \langle e_3|A|e_2 \rangle & \langle e_3|A|e_3 \rangle
\end{pmatrix} = \begin{pmatrix}
    0 & i & -i \\
    -i & 0 & i \\
    i & -i & 0
\end{pmatrix}.

Since $A^\dagger = (0 -i i 0 0 0 0 0 0)^\dagger = A$, it follows that $A$ is a Hermitian operator.

Now let us find eigenvalues of $A$:

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix}
    -\lambda & i & -i \\
    -i & -\lambda & i \\
    i & -i & -\lambda
\end{vmatrix} = 0$$
2. (Cont.)
\[ -\lambda (\lambda^2-1) - i(4\lambda + 1) - i(-1 + i\lambda) = 0 \]
\[ \iff -\lambda (\lambda^2-1) + \lambda + \lambda = 0 \]
\[ \iff \lambda (\lambda^2-3) = 0 \]
\[ \iff \lambda = 0, \lambda = \sqrt{3} \text{ or } \lambda = -\sqrt{3}. \]
(non-degenerate)

- For \(\lambda = 0\), choose a normalised e. vector to be
  \[ \mathbf{v}_0 = N \begin{pmatrix} a \\ b \\ 1 \end{pmatrix}. \]
  Then, \( A \mathbf{v}_0 = 0 \Rightarrow a = b = 1 \) and \( N = \frac{1}{\sqrt{3}} \).
  So, \( \mathbf{v}_0 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \)

- For \(\lambda = \sqrt{3}\), choose a normalised e. vector to be
  \[ \mathbf{v}_+ = N \begin{pmatrix} a \\ b \\ 1 \end{pmatrix}. \]
  Then, \( A \mathbf{v}_+ = \sqrt{3} \mathbf{v}_+ \Rightarrow -\sqrt{3}a + ib - i = 0 \)
  \[ \iff -ia - \sqrt{3}b + i = 0 \]
  \[ \Rightarrow -3a + a - \sqrt{3}i = 0 \]
  \[ \Rightarrow a = -\frac{1}{2}(1 + \sqrt{3}i), \]
  \[ b = 1 - \sqrt{3}a i = -\frac{1}{2}(1 - \sqrt{3}i). \]

Then, \( N^2 = \left[ \frac{1}{4}(1 + 3) + \frac{1}{4}(1 + 3) + 1 \right]^{-1} = \frac{1}{4} \).

So, \( \mathbf{v}_+ = \frac{1}{\sqrt{3}} \begin{pmatrix} -\frac{1}{2}(1 + \sqrt{3}i) \\ -\frac{1}{2}(1 - \sqrt{3}i) \\ 1 \end{pmatrix}. \)
For $\lambda = -\sqrt{3}$, choose a normalised eigenvector to be

$$\nu_- = N \left( \begin{array}{c} a \\ b \\ 1 \end{array} \right).$$

Then,

$$A \nu_- = -\sqrt{3} \nu_- \Rightarrow a\sqrt{3} + ib - i = 0 \Rightarrow a\sqrt{3} + ib + i = 0,$$

$$\Rightarrow a = \frac{1}{2} (-1 + \sqrt{3}i),$$

$$b = -\frac{1}{2} (1 + \sqrt{3}i),$$

and

$$N = \frac{1}{\sqrt{3}}.$$

So,

$$\nu_- = \frac{1}{\sqrt{3}} \left( \begin{array}{c} -\frac{1}{2} (1 - \sqrt{3}i) \\ -\frac{3}{2} (1 + \sqrt{3}i) \\ 1 \end{array} \right).$$

**Note:** We do not have to worry about the orthogonality of $\nu_0$, $\nu_+$ and $\nu_-$, because

**Theorem:** Two eigenvectors of a Hermitian operator corresponding to two different eigenvalues are orthogonal.

**Proof:** Consider two eigenvectors $|\Psi_1\rangle$ and $|\Psi_2\rangle$ of the Hermitian operator $A$:

$$A|\Psi_1\rangle = \lambda_1 |\Psi_1\rangle, \quad A|\Psi_2\rangle = \lambda_2 |\Psi_2\rangle.$$

Since $A$ is Hermitian, $<\Psi_2 | A | \Psi_1 > = \lambda_2 <\Psi_1 | \Psi_2 >$.  

Multiply $<\Psi_2 |$ on the left of (1) and $|\Psi_1\rangle$ on the right of (2):  

$$<\Psi_2 | A | \Psi_1 > = \lambda_1 <\Psi_2 | \Psi_1 >,$$

$$<\Psi_2 | | \Psi_1 > = \lambda_2 <\Psi_1 | \Psi_2 >.$$
Subtracting ③ from ④, we find 
\((\lambda_1 - \lambda_2) \langle \psi_1 | \psi_2 \rangle = 0.\)

Hence, if \(\lambda_1 \neq \lambda_2\), then \(\langle \psi_1 | \psi_2 \rangle = 0.\) □

By Postulate 2.1,
\[
\begin{align*}
\mathcal{P}(0) &= \left| \langle \psi_0 | \psi \rangle \right|^2 = \frac{1}{3} \left( \frac{1}{2} + \frac{1}{2} + \frac{\sqrt{2}}{2} \right)^2 = \frac{3 + 2\sqrt{2}}{6}, \\
\mathcal{P}(\sqrt{3}) &= \left| \langle \psi_4 | \psi \rangle \right|^2 \\
&= \frac{1}{3} \left( \frac{1}{4} \left( 1 - \sqrt{3}i \right) - \frac{1}{4} \left( 1 + \sqrt{3}i \right) + \frac{\sqrt{2}}{2} \right)^2 \\
&= \frac{3 - 2\sqrt{2}}{12} \\
\mathcal{P}(-\sqrt{3}) &= \left| \langle \psi_1 | \psi \rangle \right|^2 \\
&= \frac{1}{3} \left( \frac{1}{4} \left( 1 + \sqrt{3}i \right) - \frac{1}{4} \left( 1 - \sqrt{3}i \right) + \frac{\sqrt{2}}{2} \right)^2 \\
&= \frac{3 - 2\sqrt{2}}{12}
\end{align*}
\]

Check: \(\mathcal{P}(0) + \mathcal{P}(\sqrt{3}) + \mathcal{P}(-\sqrt{3}) = 1, \text{ OK!}\)

3. **Proposition 1:** Let \(A\) be a self-adjoint operator. Then, for any orthonormal basis \(\{ |e_i\rangle \}\),
\[
\langle e_i | A | e_j \rangle = \langle e_j | A^* | e_i \rangle^* \]

**Proof:** By definition of a self-adjoint operator,
\[
\langle \psi_1 | A | \psi_2 \rangle = \langle \psi_2 | A^* | \psi_1 \rangle^* \]
for any \(|\psi_1\rangle, |\psi_2\rangle \in \mathcal{H}\). We simply take \(|\psi_i\rangle = |e_i\rangle\), \(|\psi_i\rangle = |e_j\rangle\) and we are done. □
Since \( \langle e_i | A | e_j \rangle \) is an \( ij \)-entry of the matrix representation \( A \), proposition 1 tells us that indeed \( A^* = (A^*)^* = A \).

**Proposition 2:** Let \( B \) be a unitary operator. Then, for any orthonormal basis \( \{ |e_i \rangle \} \),

\[
\sum_k \langle e_i | B | e_k \rangle \langle e_k | B^t | e_j \rangle = \sum_k \langle e_i | B^t | e_k \rangle \langle e_k | B | e_j \rangle = \delta_{ij}
\]

**Proof:** By definition of a unitary op., \( BB^t = B^t B = I \).

Then, \( \langle e_i | BB^t | e_j \rangle = \langle e_i | B^t B | e_j \rangle = (I)_{ij} = \delta_{ij} \). (\#)

Since \( \sum_k |e_k \rangle \langle e_k | = I \), from (\#),

\[
\langle e_i | B | B^t | e_j \rangle = \langle e_i | B^t | e_j \rangle = \delta_{ij}
\]

\( \Rightarrow \sum_k \langle e_i | B | e_k \rangle \langle e_k | B^t | e_j \rangle = \sum_k \langle e_i | B^t | e_k \rangle \langle e_k | B | e_j \rangle = \delta_{ij} \). \( \Box \)

Proposition 2 tells us that \( BB^t = B^t B = I \), where \( B \) is a matrix rep. of \( B \) wrt. \( \{ |e_i \rangle \} \).
4. **Theorem**: The eigenvalues of a unitary operator are pure phases.

**Proof**: Let \( |\psi_u\rangle \) be a normalised e.vector of the unitary operator \( U \) with e.value \( u \): \( U|\psi_u\rangle = u|\psi_u\rangle \).

The square of the norm of \( U|\psi_u\rangle \) is

\[
\langle \psi_u | U^\dagger U | \psi_u \rangle = u^*u \quad \langle \psi_u | \psi_u \rangle = u^*u = |u|^2.
\]

Since \( U^\dagger U = I \),

\[
\langle \psi_u | U^\dagger U | \psi_u \rangle = \langle \psi_u | \psi_u \rangle = 1.
\]

From (4) and (5), \( |u|^2 = 1 \Rightarrow u = e^{i\theta}, \theta \in \mathbb{R} \). \( \square \)

Let \( B = \sum_4 e_i^* X e_j \). Then, \( B^\dagger = \sum_4 e_i^* X e_j^\dagger \).

So,

\[
BB^\dagger = \left( \sum_4 e_i^* \langle e_x \mid e_j \rangle \right) \left( \sum_3 e_j^\dagger \langle e_d \mid e_j \rangle \right)
\]

\[
= \sum_4 \sum_3 \delta_{ij} = I_4 \quad \text{since } \{ e_i \} \text{ is o.n.}
\]

Thus, \( B \) is unitary.

Also,

\[
B|e_k\rangle = \sum_4 e_i^* \langle e_x \mid e_k \rangle |e_k\rangle = |e_k\rangle.
\]

So, \( B: |e_k\rangle \rightarrow |e_k\rangle \).
\[ G = \sum_i |f_i\rangle \langle f_i| + 1 \]

\[ \langle f_i | f_j \rangle = \delta_{ij} \]

\[ \langle \psi | G | \psi \rangle = \sum_i |\langle \psi | f_i \rangle|^2 > 0 \]

and cannot = 0 since \( \{ |f_i\rangle \} \) complete

\[ G \text{ has +ve e-values} \]

\[ G = G^\dagger \Rightarrow G = \sum_i \lambda_i |g_i\rangle \langle g_i| \quad G |g_i\rangle = \lambda_i |g_i\rangle \]

\[ G^{-\frac{1}{2}} = \sum_i \lambda_i^{-\frac{1}{2}} |g_i\rangle \langle g_i| \]

so \( G^{-\frac{1}{2}} \) exists.

Define \( |e_i\rangle = G^{-\frac{1}{2}} |f_i\rangle \)

\[ \sum_i |e_i\rangle \langle e_i| = G^{-\frac{1}{2}} \sum_i |f_i\rangle \langle f_i| G^{-\frac{1}{2}} \]

\[ = G^{-\frac{1}{2}} G G^{-\frac{1}{2}} \]

\[ = \frac{1}{G} \]

\[ \Rightarrow \{ |e_i\rangle \} \text{ are orthonormal.} \]
1. \( A|e_1\rangle = \frac{1}{2}|e_1\rangle \), \( A|e_2\rangle = -\frac{1}{2}|e_2\rangle \)

\[ \Rightarrow |e_1\rangle \text{ and } |e_2\rangle \text{ are eigenvectors of } A \]

with eigenvalues \( +\frac{1}{2} \) and \( -\frac{1}{2} \) respectively.

- \( B \) is the same operator as \( S \) in Question 1 of sheet 3: \( |v_+\rangle = \frac{1}{\sqrt{2}}(|e_1\rangle + i|e_2\rangle) \) and

\[ |v_-\rangle = \frac{1}{\sqrt{2}}(|e_1\rangle - i|e_2\rangle) \]

are eigenvectors of \( B \) with eigenvalues \( +\frac{1}{2} \) and \( -\frac{1}{2} \) respectively.

- To prove that \( AB \neq BA \), it suffices to show that, for example, \( AB|e_1\rangle \neq BA|e_1\rangle \).

\[ AB|e_1\rangle = A\left(\frac{1}{2}|e_2\rangle\right) = \frac{1}{2}(A|e_2\rangle) = \frac{1}{2}\left(\frac{1}{2}|e_2\rangle\right) = -\frac{1}{4}|e_2\rangle \]

\[ BA|e_1\rangle = B\left(\frac{1}{2}|e_1\rangle\right) = \frac{1}{2}(B|e_1\rangle) = \frac{1}{2}\left(\frac{1}{2}|e_2\rangle\right) = \frac{1}{4}|e_2\rangle \]

\[ \Rightarrow AB|e_1\rangle \neq BA|e_1\rangle \Rightarrow AB \neq BA \]

- Suppose that the system is in the state \( |e_1\rangle \).

Then, \( \text{Tr}(\text{measuring value of } A = \frac{1}{2}) = 1 \).

Immediately after the measurement the system is still in the state \( |e_1\rangle \).
Then, \( P(\text{measuring value of } B = \frac{1}{2}) \)
\[ = 1 \langle v_+ | e_1 \rangle^2 = |\frac{1}{\sqrt{2}}|^2 = \frac{1}{2} \]

Thus, \( P(\text{measuring } \frac{1}{2} \text{ for } A \text{ and then } \frac{1}{2} \text{ for } B) \)
\[ = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \]  

(*)

- Suppose again that the system is in the state \( |e_1\rangle \).
  \( P(\text{measuring value of } B = \frac{1}{2}) = 1 \langle v_+ | e_1 \rangle^2 = \frac{1}{2} \).

Suppose that the outcome of the measurement of \( B \) is \( \frac{1}{2} \). Then, the state collapses to the eigenstate corresponding to the value obtained immediately after the measurement:

\[ |\psi\rangle = |e_1\rangle \quad \rightarrow \quad |v_+\rangle \]

Then, \( P(\text{measuring value of } A = \frac{1}{2}) \)
\[ = 1 \langle e_1 | v_+ \rangle^2 = \frac{1}{2} \]

Thus, \( P(\text{measuring } \frac{1}{2} \text{ for } B \text{ and then } \frac{1}{2} \text{ for } A) \)
\[ = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \]  

(***)

- \( P(\text{measuring } \frac{1}{2} \text{ for } A \text{ and then } \frac{1}{2} \text{ for } B) \) \( \neq \) \( P(\text{measuring } \frac{1}{2} \text{ for } B \text{ and then } \frac{1}{2} \text{ for } A) \).
Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

Given $$|e_i\rangle = \frac{1}{2} |e_2\rangle, \quad |e_2\rangle = \frac{1}{2} |e_1\rangle$$, we find $$\langle e_1 | e_i \rangle = 0, \quad \langle e_1 | e_2 \rangle = \frac{1}{2}$$, so $$C = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2} \sigma_x.$$  

Similarly, $$B = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{1}{2} \sigma_y,$$  

and $$A = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2} \sigma_z.$$  

Note that $$\det (C - \lambda I) = 0 \Rightarrow \left| \begin{array}{cc} \frac{1}{2} & -\lambda \\ -\lambda & \frac{1}{2} \end{array} \right| = 0 \Rightarrow \lambda = \pm \frac{1}{2}$$.  

i.e. the eigenvalues of $$C$$ are $$\frac{1}{2}$$ and $$-\frac{1}{2}$$.

For $$\lambda = +\frac{1}{2}$$, let us choose a normalised eigenvector to be $$w_+ = N \begin{pmatrix} 1 \\ x \end{pmatrix}$$. Then,  

$$C w_+ = +\frac{1}{2} w_+ \Rightarrow x = 1, \quad N = \frac{1}{\sqrt{2}}$$  

Thus, $$w_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$  

For $$\lambda = -\frac{1}{2}$$, let us choose a normalised eigenvector to be $$w_- = N \begin{pmatrix} 1 \\ y \end{pmatrix}$$. Then,  

$$C w_- = -\frac{1}{2} w_- \Rightarrow y = -1, \quad N = \frac{1}{\sqrt{2}} \Rightarrow w_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$
In the Dirac notation, 
\[ |W_+\rangle = \frac{1}{\sqrt{2}} (|e_1\rangle + |e_2\rangle) \quad (e.\ value \ \frac{1}{2}) \]
\[ |W_-\rangle = \frac{1}{\sqrt{2}} (|e_1\rangle - |e_2\rangle) \quad (e.\ value \ -\frac{1}{2}) \]

Let us summarise what we have got so far and write them in better notation.

"I think that most physicists would say that the 19th-century greats missed these two crucial symmetries [Lorentz & gauge invariance] because of their lousy notation ... It is said, and I agree, that one of Einstein's great contributions is the repeated indices summation convention. Try to read Maxwell's treatises and you will appreciate the importance of good notation."

Anthony Zee
GFT in a nutshell, p. 457
OLD notation ↔ BETTER notation

\[ |e_1\rangle, |e_2\rangle \leftrightarrow |\pm\rangle_2 \text{ or } |\pm\rangle_3 \]
\[ |w_\pm\rangle \leftrightarrow |\pm\rangle_y \text{ or } |\pm\rangle_2 \]
\[ |w_\pm\rangle \leftrightarrow |\pm\rangle_x \text{ or } |\pm\rangle_1 \]

A ↔ \[ S_2 \text{ or } S_3 \]
B ↔ \[ S_y \text{ or } S_2 \]
C ↔ \[ S_x \text{ or } S_1 \]

Then, \[ S = \frac{1}{2} \sigma, \]
\[ |\pm\rangle_x = \frac{1}{\sqrt{2}} (|+\rangle_2 \pm |-\rangle_2), \]
\[ |\pm\rangle_y = \frac{1}{\sqrt{2}} (|+\rangle_2 \pm i|-\rangle_2). \]

Now suppose that the system is in the state \[ |+\rangle_2 \] (i.e. it has spin \[ \uparrow \] in \[ z \]-direction).

Then, \[ P(\text{measuring } t_y) = |\langle + | + \rangle_2|^2 \]
\[ = \left| \frac{1}{\sqrt{2}} (2 + i) - \frac{1}{\sqrt{2}} (-1) \right|^2 \]
\[ = \frac{1}{2}, \]

\[ P(\text{measuring } t_y) = \left| \frac{1}{\sqrt{2}} (2 + i) - \frac{1}{\sqrt{2}} (-1) \right|^2 = \frac{1}{2}, \]

\[ P(\text{measuring } t_x) = \left| \frac{1}{\sqrt{2}} (2 + i) - \frac{1}{\sqrt{2}} (-1) \right|^2 = \frac{1}{2}, \]

\[ P(\text{measuring } t_x) = \left| \frac{1}{\sqrt{2}} (2 + i) - \frac{1}{\sqrt{2}} (-1) \right|^2 = \frac{1}{2}. \]
Let us label $x, y, z$ directions by $1, 2, 3$. Then:

$$1 \pm \|_1 = \frac{1}{\sqrt{2}} \left( 1 + \|_3 \pm 1 - \|_3 \right),$$
$$1 \pm \|_2 = \frac{1}{\sqrt{2}} \left( 1 + \|_3 \pm i\|_3 \right).$$

Observe that

$$| \langle \pm | \pm \rangle_i |^2 = | \langle \pm | \pm \rangle_j |^2 = \frac{1}{2} \quad \text{for} \quad i \neq j.$$ 

Thus, if the state is an eigenstate of spin in any one of the three directions $x, y, z$, then the prob. of measuring spin up (or down) in either of the other two directions is $\frac{1}{2}$. 
2. \[ \langle \phi | \hat{\rho} \hat{\psi} | \psi \rangle = \int dx \, \langle \phi | x \rangle \langle x | \hat{\rho} | \psi \rangle \]

\[ = \int dx \, x \, \phi^*(x) \frac{i}{\hbar} \nabla \psi(x) \]

\[ < \phi | \hat{\rho} | \psi > = \int dx \, < \phi | x > \langle x | \hat{\rho} | \psi > \]

\[ < \phi | \hat{\rho} | \psi > = < x | \hat{\rho}^\dagger | \phi >^* \]

\[ = < x | \hat{\rho} | \phi >^* \]

\[ = (-i \hbar \frac{\partial}{\partial x})^* = i \hbar \frac{\partial}{\partial x}^* \]

\[ < \phi | \hat{\rho} | \psi > = \int dx \, x \, i \hbar \frac{\partial}{\partial x}^* \psi(x) \]

\[ < \phi | (\hat{\rho} \hat{\psi}) | \psi > = \int dx \, x \, -i \hbar (\phi^* \frac{\partial \psi}{\partial x} + 2 \phi \frac{\partial \psi}{\partial x}) \]

\[ = -i \hbar \int dx \, x \, \frac{\partial}{\partial x} (\phi^* \psi) \]

integrate by parts.

\[ = i \hbar \int dx \, \phi^* \psi - i \hbar \int x \phi^* \psi \]

\[ = i \hbar \langle \phi | \psi > \]

\[ = 0 \]

compatible with \[ [\hat{\rho}, \hat{\psi}] = i \hbar \]
Given that \( [\hat{x}, \hat{p}] = i\hbar \hat{1} \) (3)

Then, \( \hat{x} \hat{p} - \hat{p} \hat{x} = +i\hbar \hat{1} \).

Taking a Hermitian conjugate, we find
\[ \hat{p}^+ \hat{x} - \hat{x} \hat{p}^+ = -i\hbar \hat{1} \]
\[ \Rightarrow [\hat{x}, \hat{p}^+] = i\hbar \hat{1}. \] (4)

From (3), (4) \[ [\hat{x}, \hat{p} - \hat{p}^+] = 0. \]

Let \( \hat{A} = \hat{p} - \hat{p}^+ \). Then \([\hat{x}, \hat{A}] = 0.\)

Consider a matrix representation of \( \hat{A} \) in \( \{|x\rangle\} \) basis as follows:
\[ \langle x' | [\hat{x}, \hat{A}] | x \rangle = 0 \]
\[ \Rightarrow \langle x' | \hat{x} \hat{A} - \hat{A} \hat{x} | x \rangle = 0 \]
\[ \Rightarrow x' \langle x' | \hat{A} | x \rangle - \langle x' | \hat{A} | x \rangle x = 0 \]
\[ \Rightarrow (x' - x) \langle x' | \hat{A} | x \rangle = 0 \]
\[ \Rightarrow x' = x \text{ or } \langle x' | \hat{A} | x \rangle = 0. \]

Take \( x' \neq x \), then this means that all of the off-diagonal elements of the matrix rep. \( \hat{A} \) are zero. Hence, \( \hat{A} \) is a diag. matrix wrt. the \( \{|x\rangle\} \) basis. Thus, \( |x\rangle \) is an eigenvector of \( \hat{A} \).
Therefore, we say that
\[ \hat{A}|x\rangle = y(x) |x\rangle \quad \text{for} \quad y(x) \in \mathbb{C}. \]
\[ = y(\hat{x}) |x\rangle. \]
Since this is true for all \(|x\rangle\),
\[ \hat{A} = y(\hat{x}). \quad (\dagger) \]
Observe that
\[ \hat{A}^\dagger = (p - \hat{p})^\dagger = \hat{p}^\dagger - \hat{p} = -\hat{A}, \]
i.e. \( \hat{A} \) is anti-Hermitian.

**Proposition:** The expectation value of an anti-Hermitian operator is purely imaginary.

**Proof:** Let \( C \) be an anti-Hermitian op.
Then,
\[ \langle \psi | C | \psi \rangle = \langle \psi | C^\dagger | \psi \rangle \quad (C^\dagger = -C) \]
\[ = -\langle \psi | C | \psi \rangle \]
\[ = -\langle \psi | C | \psi \rangle^*, \]
for any state \(|\psi\rangle \in \mathcal{H} \).
Thus, \( \langle \psi | C | \psi \rangle \) is purely imaginary. \( \square \)
Hence,
\[ \hat{A} = \hat{p} - \hat{p}^\dagger = i f(\hat{x}), \]
for a real function \( f \).
4. Let $|\psi\rangle = M(a) V(b) |\psi\rangle$

For some $a, b$. $U(a) = e^{\frac{i}{\hbar} a \hat{X}}$, $V(b) = e^{\frac{i}{\hbar} b \hat{X}}$

$M(a) U(a) \hat{X} U(a) = \hat{X} + a$, $V^\dagger(b) \hat{P} V(b) = \hat{P} + b$.

$\langle \phi | \hat{X} | \psi \rangle = \langle \phi | V^\dagger(b) U^\dagger(a) \hat{X} U(a) V(b) | \psi \rangle$

$= \langle \phi | V^\dagger(b) (\hat{X} + a) V(b) | \psi \rangle$

$= \langle \phi | (\hat{X} + a) | \psi \rangle$ since $[V, \hat{X}] = 0$

$= \langle \phi | \hat{X} | \psi \rangle + a$

so choose $a = -\langle \hat{X} | \psi \rangle$.

$\langle \phi | \hat{P} | \psi \rangle = \langle \phi | V^\dagger(b) U^\dagger(a) \hat{P} U(a) V(b) | \psi \rangle$

$= \langle \phi | V^\dagger(b) \hat{P} V(b) | \psi \rangle$ since $[\hat{P}, U] = 0$

$= \langle \phi | (\hat{P} + b) | \psi \rangle$

$= \langle \phi | \hat{P} | \psi \rangle + b$

so choose $b = -\langle \hat{P} | \psi \rangle$. 
Keeping $a, b$ general,

above argument also extended to show

$$\langle \phi | \hat{x}^2 | \phi \rangle = \langle \psi | (\hat{x} + a)^2 | \psi \rangle$$

$$\langle \phi | \hat{p}^2 | \phi \rangle = \langle \psi | (\hat{p} + b)^2 | \psi \rangle.$$ 

so

$$\begin{align*}
(\Delta x)^2_{\phi} &= \langle \phi | \hat{x}^2 | \phi \rangle - \langle \phi | \hat{x} | \phi \rangle^2 \\
&= \langle \psi | (\hat{x} + a)^2 | \psi \rangle - \left( \langle \psi | \hat{x} | \psi \rangle + a \right)^2 \\
&= \langle \psi | \hat{x}^2 | \psi \rangle + 2a \langle \psi | \hat{x} | \psi \rangle + a^2
- \langle \psi | \hat{x} | \psi \rangle^2 - 2a \langle \psi | \hat{x} | \psi \rangle - a^2 \\
&= \frac{(\Delta x)^2_{\psi}}.
\end{align*}$$

Similarly in $(\Delta p)^2$. 
5. \[ \hat{x} + a = U^+(a) \hat{x} M(a) \]

\[ \hat{p} + b = V^+(b) \hat{p} V(b) \]

\[ H_{ab} = \frac{(\hat{p} + b)^2}{2m} + \frac{1}{2} m \omega^2 (\hat{x} + a)^2 \]

\[ = V^+(b) \frac{\hat{p}^2}{2m} V(b) + \frac{1}{2} m \omega^2 U^+(a) \hat{x}^2 U(a) \]

\[ = V^+(b) \frac{\hat{p}^2}{2m} V(b) + \frac{1}{2} m \omega^2 U^+(a) \hat{x}^2 U(a) \]

\[ = V^+(b) \frac{\hat{p}^2}{2m} V(b) + \frac{1}{2} m \omega^2 U^+(a) \hat{x}^2 U(a) \]

\[ = \sum_{n=0}^{\infty} \text{En} \langle U_n \rangle \langle U_n \rangle V(b) \]

\[ = \sum_{n=0}^{\infty} \text{En} |U_n^{ab}\rangle \langle U_n | \]

where \[ |U_n^{ab}\rangle = V^+(b) U^+(a) |U_n\rangle \]

and clearly \[ H_{ab} |U_n^{ab}\rangle = \text{En} |U_n^{ab}\rangle. \]

(*) Note: we make use of the fact that

\[ \hat{x} + a = V^+(b) (\hat{x} + a) V(b) \quad ([V, \hat{x}] = 0) \]

\[ = V^+(b) U^+(a) \hat{x} U(a) V(b) \]

and similarly for \[ \hat{p} + b. \]
6. \( R = \int \text{d}x \left< -x \right| \left< x \right> \)

\[ R \left< x \right| = \int \text{d}y \left< y \right| \left< -y \right| \left< y + x \right| \] = 1 - x

\[ R^2 \left< x \right| = R \left< -x \right| = x \] \Rightarrow R^2 = 1.

\[ R \hat{x} R = \int \text{d}x x \left< x \right| \left< x \right> = -x \]

\[ = - \int \text{d}x x \left< x \right| \left< x \right> = -x \]

\[ R = \int \text{d}x \int \text{d}p \int \text{d}p' \left| p \right> \left< p \mid x \right> \left< x \mid -xp' \right> \left< p' \right| \]

\[ = \int \text{d}x \int \text{d}p \int \text{d}p' \left| p \right> e^{-\frac{ipx - ip'x}{\hbar}} \frac{1}{2\pi \hbar} \left< p' \right| \]

\[ = \int \text{d}p \int \text{d}p' \left| p \right> S(p + p') \left< p' \right| \]

\[ = \int \text{d}p \left| p \right> \left< -p \right|. \]
7. \[ F(\lambda) = e^{\lambda(A+B)} e^{-\lambda B} \]
\[ F'(\lambda) = e^{\lambda(A+B)} (A+B-\lambda B) e^{-\lambda B} \]
\[ = e^{\lambda(A+B)} e^{-\lambda B} e^{\lambda B} A e^{-\lambda B} \]
\[ = F(\lambda) A(\lambda) \]

where \[ A(\lambda) = e^{\lambda B} A e^{-\lambda B} \]
\[ A'(\lambda) = e^{\lambda B} (BA-AB) e^{-\lambda B} \]
\[ = -e^{\lambda B} C e^{\lambda B} \]
\[ = -c \quad \text{([}\ c, \ B]\ =\ 0\ ] \]

\[ \Rightarrow A(\lambda) = A - \lambda C \] since \[ A(0) = A. \]

Now \[ F'(\lambda) = F(\lambda) (A-\lambda C) \]
may check that
\[ F(\lambda) = e^{A\lambda - \lambda^2/2 C} \]
since \[ F(0) = 1 \] and \[ F'(\lambda) = F(\lambda) (A-\lambda C). \]

so \[ F(\lambda), \quad e^{\lambda(A+B)-\lambda B} e = e^{A\lambda - \lambda^2/2 C} \]
\[ \lambda=1 \Rightarrow e^{A+B} = e^A e^B e^{-1/2 C} \]
1. \((A \otimes B)^+ = A^+ \otimes B^+\)

For any \(A\), \(A^+\) is defined by
\[
\langle \psi | A^+ | \phi \rangle = \langle \phi | A | \psi \rangle^* \quad \text{for all} \quad |\psi\rangle, |\phi\rangle.
\]

Similarly,
\[
\begin{align*}
&\langle \alpha_1 | \alpha_2 \rangle (A \otimes B)^+ (|\alpha_1\rangle \otimes |\alpha_2\rangle) \\
= &\left( \langle \phi_1 | (A \otimes B) | \alpha_1 \rangle \otimes |\alpha_2\rangle \right)^* \\
= &\left( \langle \phi_1 | A | \phi_1 \rangle \langle \phi_2 | B | \phi_2 \rangle \right)^* \\
= &\langle \phi_1 | A^+ | \phi_1 \rangle \langle \phi_2 | B^+ | \phi_2 \rangle \\
= &\langle \alpha_1 \rangle \otimes \langle \alpha_2 \rangle A^+ \otimes B^+ (|\alpha_1\rangle \otimes |\alpha_2\rangle) \\
\Rightarrow &\quad (A \otimes B)^+ = A^+ \otimes B^+.
\end{align*}
\]
2. \( \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \)

\( |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \)

(i) \( \sigma_z |\uparrow\rangle = |\uparrow\rangle, \quad \sigma_z |\downarrow\rangle = -|\downarrow\rangle \)

\( \sigma_x |\uparrow\rangle = |\downarrow\rangle, \quad \sigma_x |\downarrow\rangle = |\uparrow\rangle \)

\( \sigma_y |\uparrow\rangle = i |\downarrow\rangle, \quad \sigma_y |\downarrow\rangle = -i |\uparrow\rangle \)

\( q_2 |\uparrow\rangle = (a_1 + i a_2) |\uparrow\rangle + q_3 |\downarrow\rangle \)

\( q_2 |\downarrow\rangle = (a_1 - i a_2) |\uparrow\rangle - q_3 |\downarrow\rangle \)

(ii) \( \langle \uparrow | q_2 | \uparrow \rangle = a_3 \)

\( \langle \downarrow | q_2 | \downarrow \rangle = -a_3 \)

\( \langle \uparrow | q_2 | \downarrow \rangle = a_1 - i a_2 \)

\( \langle \downarrow | q_2 | \uparrow \rangle = a_1 + i a_2 \)
(iii)  $\langle 4\vert A \otimes B \vert 14 \rangle$

$$= \frac{1}{2} \left[ A^{\uparrow \downarrow} B^{\downarrow \uparrow} + A^{\downarrow \uparrow} B^{\uparrow \downarrow} - A^{\uparrow \downarrow} B^{\downarrow \uparrow} - A^{\downarrow \uparrow} B^{\uparrow \downarrow} \right]$$

where $A^{\uparrow \downarrow} = \langle \uparrow \vert A \downarrow \rangle = \langle 4\vert a \pi \vert 1 \rangle$.

(iv)  $\langle 4\vert a \sigma \otimes b \sigma \vert 14 \rangle$

$$= \frac{1}{2} \left[ -a_3 b_3 - a_3 b_3 

- (a_1 - i a_2) (b_1 + i b_2) - (a_1 + i a_2) (b_1 - i b_2) \right]$$

$$= \frac{1}{2} \left[ -2 a_3 b_3 - (a_1 b_1 + a_2 b_2 + i a_1 b_2 - i a_2 b_1) 

- (a_1 b_1 + a_2 b_2 - i a_1 b_2 + i a_2 b_1) \right]$$

$$= -a \cdot b$$

as required.
3. \( \mathbf{n} = \omega \mathbf{n} \cdot \mathbf{\sigma} \)

\[
(n \cdot \sigma)^2 = n \cdot n = 1
\]

\[
(n \cdot \sigma)^3 = (n \cdot \sigma)(n \cdot \sigma)^2 = n \cdot \sigma
\]

\[
(n \cdot \sigma)^k = \begin{cases} 
(n \cdot \sigma)^{k/2} & \text{if } k \text{ even} \\
(n \cdot \sigma)^{k/2} \sigma \cdot \sigma & \text{if } k \text{ odd}
\end{cases}
\]

\[
e^{ \frac{-i \mathbf{H} t}{\hbar} } = \sum_{k=0}^{\infty} (-i \frac{t}{\hbar})^k \mathbf{w}^k (n \cdot \sigma)^k
\]

\[
= \sum_{k=0}^{\infty} (-i \frac{t}{\hbar})^k \mathbf{w}^k + \sum_{u=1}^{\infty} (-i \frac{t}{\hbar})^k \mathbf{w}^k \sigma \cdot \sigma
\]

\[
= \cos \left( \frac{\omega t}{\hbar} \right) \mathbf{1} + i \sin \left( \frac{\omega t}{\hbar} \right) n \cdot \sigma
\]

\[
e^{ \frac{-i \mathbf{H} t}{\hbar} } | \uparrow \rangle = \cos \left( \frac{\omega t}{\hbar} \right) | \uparrow \rangle - i \sin \left( \frac{\omega t}{\hbar} \right) \frac{n_3}{\hbar} | \uparrow \rangle
\]

\[
= \left( \cos \left( \frac{\omega t}{\hbar} \right) - i n_3 \sin \left( \frac{\omega t}{\hbar} \right) \right) | \uparrow \rangle
\]

\[- i (n_1 + i n_2) \sin \left( \frac{\omega t}{\hbar} \right) | \downarrow \rangle
\]
4. \( \Psi(x) = \frac{1}{(2\pi \sigma^2)^{1/4}} e^{-x^2/4\sigma^2} \)

Minimum uncertainty state is:

\( \widetilde{\Psi}(p) = \frac{1}{(2\pi \sigma_p^2)^{1/4}} e^{-p^2/4\sigma_p^2} \)

\[ \sigma_p^2 \sigma_x^2 = \frac{k^2}{4} \]

\( \sigma_x \), by explicit computation:

\[ \widetilde{\Psi}(p) = \int dx \ e^{i px} \frac{1}{2\pi h} \cdot \frac{1}{(2\pi \sigma^2)^{1/4}} \int dx \ e^{-x^2/4\sigma^2 - \frac{i}{\hbar} px} \]

\[ = \frac{1}{(2\pi h)^{1/2}} \frac{1}{(2\pi \sigma^2)^{1/4}} \int dx \ e^{-x^2/4\sigma^2 - \frac{i}{\hbar} px} (4\pi \sigma^2)^{1/2} e^{-p^2/2\sigma^2} \]

\[ = \frac{1}{(2\pi h)^{1/2}} \frac{1}{(2\pi \sigma^2)^{1/4}} \int dx \ e^{-x^2/4\sigma^2 - \frac{i}{\hbar} px} (4\pi \sigma^2)^{1/2} e^{-p^2/2\sigma^2} \]

\[ = \left( \frac{2\pi h \sigma^2}{2\hbar} \right)^{1/4} \frac{1}{(2\pi \sigma^2)^{1/4}} \frac{1}{(2\pi \sigma^2)^{1/4}} e^{-p^2/4\sigma_p^2} \]

\[ = \frac{1}{(2\pi \sigma_p^2)^{1/4}} e^{-p^2/4\sigma_p^2} \]

Using: \( \int dx \ e^{-ax^2+bx} = e^{b^2/4a} \sqrt{\frac{\pi}{a}} \)
\[
\psi(x, t) = \int dp \left< x \bigg| e^{-\frac{i}{\hbar} \mathbf{p} \cdot \mathbf{x} + i \frac{p^2}{2m}} \right| \psi(p) \right>
\]

\[
= \int dp \left< x \bigg| p \right> e^{-\frac{i}{\hbar} \frac{p^2}{2m}} \frac{1}{(2\pi\hbar)^{1/2}} e^{-\frac{p^2}{4\sigma_p^2}}
\]

\[
= \frac{1}{(2\pi\hbar)^{1/2}} \frac{1}{(2\pi\sigma_p^2)^{1/4}} \int dp \ e^{\frac{i}{\hbar} p x - \frac{p^2}{2m} \left( \frac{\sigma^2 + \frac{i}{2} \frac{\hbar t}{2m}}{\sigma^2 + \frac{i}{2} \frac{\hbar t}{2m}} \right)}
\]

\[
= \frac{1}{(2\pi\sigma_p^2)^{1/4}} \left( \frac{\sigma^2}{\sigma^2 + \frac{i}{2} \frac{\hbar t}{2m}} \right)^{1/4} \exp \left( -\frac{x^2}{4 \left( \sigma^2 + \frac{i}{2} \frac{\hbar t}{2m} \right)} \right)
\]

\[
|\psi(x, t)|^2 = \frac{1}{(2\pi\sigma_p^2)^{1/4}} \left| \sigma^2 \right| \exp \left( -\frac{x^2}{4 \left( \sigma^2 + \frac{i}{2} \frac{\hbar t}{2m} \right)} \right)
\]

\[
= \exp \left( -\frac{x^2}{2(\Delta x)^2} \right)
\]

\[
(\Delta x)^2 = \frac{\sigma^2 + \frac{\hbar^2 t^2}{4m^2}}{2m \sigma^2}
\]
5. \[
\langle \psi_t | \psi \rangle = \langle \psi | e^{\frac{itH}{\hbar}} | \psi \rangle \\
= \langle \psi | 1 - \frac{itH}{\hbar} - \frac{t^2H^2}{2\hbar^2} + O(t^3) | \psi \rangle + O(t^3) \\
= 1 - \frac{it}{\hbar} \langle H \rangle - \frac{t^2}{2\hbar^2} \langle H^2 \rangle + \ldots \\
\]
\[
|\langle \psi_t | \psi \rangle|^2 = \left(1 - \frac{t^2}{2\hbar^2} \langle H^2 \rangle \right)^2 + \frac{t^2}{\hbar^2} \langle H^2 \rangle \\
= 1 - \frac{t^2}{\hbar^2} \langle H^2 \rangle + O(t^4) + \frac{t^2}{\hbar^2} \langle H^2 \rangle \\
= 1 - \frac{t^2}{\hbar^2} \langle H^2 \rangle - \langle H^2 \rangle \langle H^2 \rangle + \ldots \\
= 1 - \frac{t^2}{\hbar^2} \langle H \rangle^2 + \ldots \\
= 1 - \frac{t^2}{\hbar^2} + \ldots \\
\]
where \( t \) = \frac{\hbar}{\Delta H} = \text{Zero Time}.

Can also be written
\[
|\langle \psi_t | \psi \rangle|^2 \approx e^{-\frac{t^2}{2t^2}} \text{ for small } t.
\]
\[
\begin{align*}
\alpha |n\rangle &= \sqrt{n} |n-1\rangle \\
\alpha^\dagger |n\rangle &= \sqrt{n+1} |n+1\rangle \\
\alpha |0\rangle &= 0 \\
\hat{x} &= \frac{\hbar}{\sqrt{2 \hbar m \omega}} (\alpha^\dagger + \alpha) \\
\hat{p} &= i \sqrt{\frac{\hbar m \omega}{2}} (\alpha^\dagger - \alpha)
\end{align*}
\]

\[
\langle n | \hat{x} | n \rangle = \sqrt{\frac{\hbar}{2 \hbar m \omega}} \langle n | (\alpha^\dagger + \alpha) | n \rangle
\]

\[
= 0
\]

\text{since } \langle n | m \rangle = 0 \text{ for } n \neq m.

\text{Similarly, } \langle n | \hat{p} | n \rangle = 0.

\[
\begin{align*}
\langle n | \hat{x}^2 | n \rangle &= \frac{\hbar}{2 \hbar m \omega} \langle n | (\alpha^\dagger + \alpha)^2 | n \rangle \\
&= \frac{\hbar}{2 \hbar m \omega} \langle n | (\alpha^\dagger)^2 + \alpha^2 + \alpha \alpha^\dagger + \alpha^\dagger \alpha | n \rangle
\end{align*}
\]

Note \( \langle n | (\alpha^\dagger)^2 | n \rangle \propto \langle n | n+2 \rangle = 0 \)

\( \langle n | \alpha^2 | n \rangle \propto \langle n | n-2 \rangle = 0 \)

\[
\begin{align*}
\langle n | \alpha^\dagger \alpha | n \rangle &= n \langle n | n \rangle = n \\
\langle n | \alpha \alpha^\dagger | n \rangle &= \langle n | (\alpha^\dagger \alpha + 1) | n \rangle \\
&= n+1
\end{align*}
\]
\[ \langle n | \hat{x}^2 | n \rangle = \frac{\hbar}{2m_w} (n + n + 1) \]

\[ = \frac{\hbar}{m_w} (n + \frac{1}{2}) \]

\[ \langle n | \hat{\rho}^2 | n \rangle = -\frac{m_w \hbar}{2} \langle n | (a^+)^2 + a^2 - a^+a - aa^+ | n \rangle \]

\[ = \frac{m_w \hbar}{2} \langle n | (a^+a + aa^+) | n \rangle \]

\[ = 2n + 1 \quad \text{as in } \hat{x} \text{ case} \]

\[ = \frac{m_w \hbar}{2} (n + \frac{1}{2}) \]

\[ \langle \Delta x \rangle^2 = \langle \hat{x}^2 \rangle - \langle x \rangle^2 = \langle \hat{x}^2 \rangle \]

\[ \langle \Delta \rho \rangle^2 = \langle \hat{\rho}^2 \rangle - \langle \rho \rangle^2 = \langle \hat{\rho}^2 \rangle \]

\[ \langle \Delta x \rangle^2 \langle \Delta \rho \rangle^2 = \langle \hat{x}^2 \rangle \langle \hat{\rho}^2 \rangle = \hbar^2 (n + \frac{1}{2})^2 \]

\[ \Delta x \cdot \Delta \rho = \hbar (n + \frac{1}{2}) \geq \hbar/2 \]

\[ \langle PE \rangle = \frac{1}{2} m_w \hbar^2 \langle \hat{x}^2 \rangle = \frac{1}{2} m_w (n + \frac{1}{2}) \]

\[ \langle KE \rangle = \frac{\langle \hat{\rho}^2 \rangle}{2m} = \frac{1}{2} m_w (n + \frac{1}{2}) \]

\[ \langle PE \rangle = \langle KE \rangle = \frac{1}{2} \text{ energy of state } | n \rangle. \]
2. Compute \[ [\hat{x}(t), \hat{x}(t')] \]

\[ = \begin{bmatrix} \hat{x} \cos \omega t + \frac{\hat{p}}{m} \sin \omega t, \hat{x} \cos \omega t + \frac{\hat{p}}{m} \sin \omega t \end{bmatrix} \]

\[ = \begin{bmatrix} \hat{x}, \hat{p} \end{bmatrix} \begin{bmatrix} \cos \omega t \sin \omega t - \sin \omega t \cos \omega t \end{bmatrix} \frac{1}{m} \]

\[ = \frac{i \hbar}{m} \sin \omega (t'-t) \rightarrow 0 \text{ as } t' \rightarrow t. \]

\[ [\hat{x}(t), \hat{p}(t')] = \begin{bmatrix} m \partial_t \hat{x}(t), \hat{p}(t') \end{bmatrix} \]

\[ = \frac{i \hbar}{m} \cos \omega (t'-t) \]

\[ \rightarrow i \hbar \cos t' \rightarrow t. \]

\[ [\hat{p}(t), \hat{p}(t')] = \begin{bmatrix} m \partial_t \hat{p}(t), \hat{p}(t') \end{bmatrix} \]

\[ = \frac{i \hbar}{m} \sin \omega (t'-t) \]

\[ \rightarrow 0 \text{ as } t' \rightarrow t. \]
\begin{align*}
\langle 0 | \hat{x}(t) \hat{x}(t') | 0 \rangle &= \frac{\hbar}{\sqrt{2m\omega}} \left( e^{-i\omega t} a + e^{i\omega t} a^\dagger \right) \\
\hat{x}(t') | 0 \rangle &= \frac{\hbar}{\sqrt{2m\omega}} e^{i\omega t'} | 11 \rangle \\
\langle 0 | \hat{x}(t) \hat{x}(t') | 11 \rangle &= \frac{\hbar}{2m\omega} e^{-i\omega(t-t')} <11| \langle 11 | \\
&= \frac{\hbar}{2m\omega} e^{-i\omega(t-t')} \\
\Rightarrow \langle 0 | \hat{x}(t') \hat{x}(t) | 0 \rangle &= \frac{\hbar}{2m\omega} e^{i\omega(t-t')} \\
\Rightarrow \langle 0 | [\hat{x}(t), \hat{x}(t')] | 0 \rangle &= \frac{\hbar}{2m\omega} \left( e^{-i\omega(t-t')} - e^{i\omega(t-t')} \right) \\
&= \frac{\hbar}{2m\omega} + 2i \sin \omega (t-t') \\
&= i \frac{\hbar}{m\omega} \sin \omega (t-t') \quad \text{consistent with above.}
\end{align*}
\[ \langle x | \psi \rangle = \frac{1}{(2\pi \hbar)^{1/2}} \exp \left( -\frac{(x-\xi)^2}{4\delta^2} + \frac{i}{\hbar} \psi(x) \right) \]

We have that
\[ \int \frac{dp dq}{2\pi \hbar} \langle \psi | \psi \rangle = 1 \]

\[ \langle x | \text{LHS} | y \rangle = \int \frac{dp dq}{2\pi \hbar} \langle x | \psi \rangle \langle \psi | y \rangle \]

\[ = \int \frac{dp dq}{2\pi \hbar} \frac{1}{(2\pi \hbar)^{1/2}} \exp \left( -\frac{(x-\xi)^2}{4\delta^2} + \frac{i}{\hbar} \psi(x) \right) \exp \left( -\frac{(y-\xi)^2}{4\delta^2} + \frac{i}{\hbar} \psi(y) \right) \]

\[ = \int \frac{dq}{(2\pi \hbar)^{1/2}} \exp \left( -\frac{(x-\xi)^2 - (y-\xi)^2}{4\delta^2} \right) \times \int \frac{dp}{2\pi \hbar} \exp \left( -\frac{i}{\hbar} p (x-y) \right) \delta(x-y) \]

\[ = \delta(x-y) \int_{-\infty}^{\infty} dq \frac{1}{(2\pi \hbar)^{1/2}} \exp \left( -\frac{(x-\xi)^2}{2\delta^2} \right) \]

\[ = 1 \]

\[ = \delta(x-y) \text{ as required.} \]
\[ 4. \quad \psi(x) = \frac{1}{(2\pi \sigma^2)^{\frac{1}{4}}} e^{-\frac{(1+i\beta)x^2}{4\sigma^2}} = \sqrt{\frac{\hbar}{2\pi \sigma^2}} e^{-x^2/2\sigma^2} = \frac{\hbar}{2\pi \sigma^2} \]

\[ \langle \hat{x}^2 \rangle = \int dx |\psi(x)|^2 = \int \frac{dx}{(2\pi \sigma^2)^{\frac{1}{2}}} e^{-x^2/2\sigma^2} = \frac{\hbar}{2\pi \sigma^2} \]

\[ \Sigma = \frac{1}{2} \langle \hat{\chi} \hat{\rho} + \hat{\rho} \hat{\chi} \rangle - \frac{1}{2} \langle \hat{\chi} \hat{\rho} + \hat{\rho} \hat{\chi} - i\hbar \rangle = i\hbar \langle \hat{\rho} \hat{\chi} \rangle - i\hbar \frac{1}{2} \]

\[ \langle \hat{\rho} \hat{\chi} \rangle = \int dx \psi^*(x) \chi \cdot -i\hbar \frac{\hbar}{2\pi \sigma^2} \]

\[ = \int dx \psi^*(x) \chi \cdot (-i\hbar) - \frac{1}{2} (1+i\beta) \chi \cdot \psi \]

\[ = \frac{i\hbar (1+i\beta)}{2\sigma^2} \langle \hat{x}^2 \rangle - \frac{i\hbar (1+i\beta)}{2} \]

\[ = \frac{i\hbar}{2} - \frac{\hbar \beta}{2} \]

\[ \Rightarrow \Sigma = -\frac{\hbar \beta}{2} \]

\[ \langle \hat{y}^2 \rangle = \int dx \left( -i\hbar \frac{\hbar}{5\pi} \right) \left( i\hbar \frac{\hbar}{5\pi} \right) \]

\[ = \int dx \chi^2 |\psi|^2 \cdot \frac{\hbar^2}{4\pi} \frac{(1+i\beta)(1-i\beta)}{4\sigma^4} \]

\[ = \langle \hat{x}^2 \rangle \frac{\hbar^2 (1+\beta^2)}{4\sigma^4} = \frac{\hbar^2 (1+\beta^2)}{4\sigma^2} \]
\[\sum = \frac{1}{2} \left< \hat{X}\hat{p} + \hat{p}\hat{X} \right> .\]

\[
\frac{d}{dt} \left< \hat{X}^2 \right>_t = \frac{d}{dt} \left< \hat{X}(t)^2 \right> = \frac{d}{dt} \left< \hat{X} \hat{p}(t) + \hat{p} \hat{X}(t) \right> = \frac{1}{m} \left< \hat{X}(t) \hat{p}(t) + \hat{p}(t) \hat{X}(t) \right> = \frac{2}{m} \sum(t)
\]

\[
\frac{d}{dt} \sum(t) = \frac{1}{2} \left< \frac{d\hat{X}}{dt} \hat{p} + \frac{d\hat{p}}{dt} \hat{X} + \frac{dp}{dt} \hat{X} + \frac{d\hat{X}}{dt} \hat{p} \right> = \frac{1}{2} \left< \frac{\hat{p}^2}{m} - m\omega^2 \hat{X}^2 - m\omega^2 \hat{X}^2 + \frac{\hat{p}^2}{m} \right> = \frac{\left< \hat{p}^2 \right>}{m} - m\omega^2 \left< \hat{X}^2 \right>
\]

\[
\frac{d}{dt} \left< \hat{p}^2 \right> = \left< \hat{p} \frac{d\hat{p}}{dt} + \frac{d\hat{p}}{dt} \hat{p} \right> = -m\omega^2 \left< \hat{X} \hat{p} + \hat{p} \hat{X} \right> = -2m\omega^2 \sum(t)
\]

\[
\left< \hat{X}^2 \right>_t, \left< \hat{p}^2 \right>_t \text{ constant } \Rightarrow \sum(t) = 0
\]

\[
\Rightarrow \beta = 0
\]

\[
\frac{d\sum}{dt} = 0 \Rightarrow \left< \hat{p}^2 \right> = m^2 \omega^2 \left< \hat{X}^2 \right>
\]

\[
\Rightarrow \frac{h^2}{4m^2} = m^2 \omega^2 \sigma^2
\]

\[
\Rightarrow \sigma^2 = \frac{h^2}{4m^2} \text{ expected to stay constant}
\]

\[
\Rightarrow \sigma^2 = \frac{h^2}{2m\omega^2}
\]
In free particle $W = 0$

and $d\Sigma = \frac{\langle p^2 \rangle}{m}$

so cannot have $\Sigma_t = 0$ ideally, so there is wave packet spreading.

The SHO has the key property that $d\Sigma$ may be zero and this is why coherent states are possible.

[Note echo for the free particle we can solve for $\langle x^2 \rangle_t$]

\[
\frac{d^2}{dt^2} \langle x^2 \rangle_t = \frac{2}{m} \frac{d\Sigma}{dt} = \frac{2}{m^2} \langle p^2 \rangle = \text{constant}
\]

Assume $\Sigma = \infty$ initially, we have

\[
\langle x^2 \rangle_t = \langle x^2 \rangle_0 + \langle p^2 \rangle \frac{t^2}{m^2}
\]

\[
= \frac{\Delta^2}{4} + \frac{t^2 \Delta^2}{4 m^2}
\]

the expected expression for wave packet spreading.
4. \[ [\mathbf{J}^2, \mathbf{J}_c] = [\mathbf{J}_u \mathbf{J}_u, \mathbf{J}_c] \]
\[ = \mathbf{J}_u [\mathbf{J}_u, \mathbf{J}_c] + [\mathbf{J}_u, \mathbf{J}_c] \mathbf{J}_u \]
\[ = \mathbf{i} \hbar \left( \mathbf{J}_u \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{z} \mathbf{j}} \mathbf{J}_d + \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} \mathbf{J}_d \mathbf{J}_u \right) \]
\[ = \mathbf{i} \hbar \left( \mathbf{J}_k \mathbf{J}_d + \mathbf{J}_d \mathbf{J}_u \right) \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} \]
\[ = \mathbf{i} \hbar \left( \mathbf{J}_k \mathbf{J}_d \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} + \mathbf{J}_d \mathbf{J}_u \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} \right) \]
\[ = \mathbf{i} \hbar \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} \left( \mathbf{J}_k \mathbf{J}_d + \mathbf{J}_d \mathbf{J}_u \right) \]
\[ = \mathbf{i} \hbar \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} \left( \mathbf{J}_k \mathbf{J}_d \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} + \mathbf{J}_d \mathbf{J}_u \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} \right) \]
\[ = \mathbf{i} \hbar \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} \left( \mathbf{J}_k \mathbf{J}_d \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} + \mathbf{J}_d \mathbf{J}_u \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} \right) \]
\[ = \mathbf{i} \hbar \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} \left( \mathbf{J}_k \mathbf{J}_d \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} + \mathbf{J}_d \mathbf{J}_u \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} \right) \]
\[ = \mathbf{i} \hbar \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} \left( \mathbf{J}_k \mathbf{J}_d \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} + \mathbf{J}_d \mathbf{J}_u \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} \right) \]
\[ = \mathbf{i} \hbar \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} \left( \mathbf{J}_k \mathbf{J}_d \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} + \mathbf{J}_d \mathbf{J}_u \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} \right) \]
\[ = \mathbf{i} \hbar \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} \left( \mathbf{J}_k \mathbf{J}_d \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} + \mathbf{J}_d \mathbf{J}_u \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} \right) \]
\[ = \mathbf{i} \hbar \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} \left( \mathbf{J}_k \mathbf{J}_d \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} + \mathbf{J}_d \mathbf{J}_u \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} \right) \]
\[ = \mathbf{i} \hbar \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} \left( \mathbf{J}_k \mathbf{J}_d \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} + \mathbf{J}_d \mathbf{J}_u \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} \right) \]
\[ = \mathbf{i} \hbar \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} \left( \mathbf{J}_k \mathbf{J}_d \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} + \mathbf{J}_d \mathbf{J}_u \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} \right) \]
\[ = \mathbf{i} \hbar \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} \left( \mathbf{J}_k \mathbf{J}_d \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} + \mathbf{J}_d \mathbf{J}_u \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} \right) \]
\[ = \mathbf{i} \hbar \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} \left( \mathbf{J}_k \mathbf{J}_d \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} + \mathbf{J}_d \mathbf{J}_u \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} \right) \]
\[ = \mathbf{i} \hbar \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} \left( \mathbf{J}_k \mathbf{J}_d \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} + \mathbf{J}_d \mathbf{J}_u \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} \right) \]
\[ = \mathbf{i} \hbar \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} \left( \mathbf{J}_k \mathbf{J}_d \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} + \mathbf{J}_d \mathbf{J}_u \mathbf{E}_{\mathbf{k} \mathbf{i} \mathbf{i} \mathbf{z} \mathbf{j}} \right) \]
\[ = \pm \hbar \left( \mathbf{J}_1 \pm \mathbf{J}_2 \right) \]
\[ = \pm \hbar \mathbf{J}_+ \]
\[ J_+ J_- = (J_1 + iJ_2)(J_1 \pm iJ_2) \]
\[ = J_1^2 \pm i J_1 J_2 \mp J_2 J_1 + J_2^2 \]
\[ = (J_1^2 + J_2^2 + J_3^2) - J_3^2 \pm i [J_1, J_2] \]
\[ = J^2 - J_3^2 \pm i \hbar J_3 \]
\[ = \frac{J^2}{\hbar} \pm \hbar J_3 \]
\( J = \frac{1}{2} \) case

\[ J = \frac{1}{2}, \pm \frac{1}{2} > = \pm \frac{h}{2} \frac{1}{2}, \pm \frac{1}{2} > \]

\[ J^+ \left( \frac{1}{2}, \frac{1}{2} > = 0 \right) \quad J^- \left( \frac{1}{2}, -\frac{1}{2} > = \frac{\hbar}{2} \left( \frac{1}{2}, \frac{1}{2} > \right) \]

\[ J^+ \left( \frac{1}{2}, -\frac{1}{2} > \right. = \frac{\hbar}{2} \left( \frac{1}{2}, \frac{1}{2} > \right) \quad J^- \left( \frac{1}{2}, -\frac{1}{2} > \right. = 0 \]

Let \( \frac{1}{2}, \frac{1}{2} > = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \frac{1}{2}, -\frac{1}{2} > = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \)

Clearly \( J_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar}{2} \sigma_z \)

\[ J^+ = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad J^- = \frac{\hbar}{2} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \]

\[ J_\pm = J_1 \pm i J_2 \]

\( \Rightarrow \) \[ J_1 = \frac{1}{2} (J^+ + J^-) = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2} \sigma_x \]

\[ J_2 = -\frac{i}{2} (J^+ - J^-) = -\frac{i\hbar}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]

\[ = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar}{2} \sigma_y \]
2(b) \( J^z = 1 \) case \( \mathcal{H} = \{ |1,1\rangle, |1,0\rangle, |1,-1\rangle \} \)

\[
J^z |1, m\rangle = m |1, m\rangle \quad m = 1, 0, -1
\]

\[
J^+ |1, 1\rangle = 0 \quad J^- |1, 1\rangle = \sqrt{2} J^+ |1, 0\rangle
\]

\[
J^+ |1, 0\rangle = \sqrt{2} J^+ |1, 1\rangle \quad J^- |1, 0\rangle = \sqrt{2} J^- |1, -1\rangle
\]

\[
J^+ |1, -1\rangle = \sqrt{2} J^+ |1, 0\rangle \quad J^- |1, -1\rangle = 0
\]

Let \( |1, 1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \) \( |1, 0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \) \( |1, -1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \)

Then \( J^z = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \)

\[
J^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \sqrt{2} 
\]

\[
J^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \frac{1}{\sqrt{2}}
\]

\[
J^x = \frac{1}{2} (J^+ + J^-) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
\]

\[
J^y = \frac{-i}{2} (J^+ - J^-) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}
\]
\[ R(n, \omega) = \exp \left( -\frac{i \omega n}{\hbar} \right) \]

\[ S = \frac{k}{2} \]

\[ R = \exp \left( -\frac{i \omega n}{2} \right) \]

\[ \text{Sheet 5, Q3} \Rightarrow \]

\[ R = \sum_{n=0}^{\infty} \frac{1}{k^n} \left( \frac{-i \omega}{2} \right)^n \left( \frac{n \sigma}{2} \right)^k \]

\[ = \cos \left( \frac{\omega}{2} \right) 1 - i \sin \left( \frac{\omega}{2} \right) \frac{n \sigma}{2} \]

\[ \text{unity} \quad (n \sigma) = 1 \quad k \text{ even} \]

\[ = n \frac{\sigma}{2} \quad k \text{ odd} \]

If \( \omega = 2 \pi \), \( \cos \left( \frac{\omega}{2} \right) = -1 \)

\[ \sin \left( \frac{\omega}{2} \right) = 0 \]

\[ R = -1 \]
4. \[ R = \cos \left( \frac{\omega}{2} \right) I - i \sin \left( \frac{\omega}{2} \right) \]

\[ R \cdot \sigma R^\dagger = (a \cos \omega + n \times a \sin \omega) \cdot \sigma \]

\[ \sigma_x = a \cdot \sigma \text{ where } a = (0, 0, 1) \]

(i) To rotate \( \sigma_x \rightarrow \sigma_x \), choose axes \( n = (0, 1, 0) \)
and \( \omega = \frac{\pi}{12} \)

\[ R \sigma_x R^\dagger = n \times a \cdot \sigma = \varepsilon_{ijk} n_j a_k \sigma_i \]

\[ = \varepsilon_{123} \sigma_j = 0 \text{ as required} \]

\[ R = \cos \left( \frac{\pi}{4} \right) I - i \sigma_y \sin \left( \frac{\pi}{4} \right) \]

\[ = \frac{1}{\sqrt{2}} \left( 1 - i \sigma_y \right) \]

\[ \sigma_y \uparrow\downarrow = i \downarrow\uparrow, \quad \sigma_y \downarrow\uparrow = -i \uparrow\downarrow. \]

\[ \uparrow\downarrow = R \uparrow\downarrow = \frac{1}{\sqrt{2}} (\uparrow\downarrow + \downarrow\uparrow) \]

\[ \downarrow\uparrow = R \downarrow\uparrow = \frac{1}{\sqrt{2}} (\downarrow\uparrow - \uparrow\downarrow) \]

\[ \text{apply Euler's rotation of } \sigma_x. \]
(ii) Denote & state \( |\sigma \rangle \), \( |\alpha \rangle \) (arrow pointing away / towards us)

To realize \( g_2 \rightarrow \sigma y \), choose \( n = (-1, 0, 0) \) and \( \omega = \frac{\pi}{2} \).

\[
\begin{align*}
R g_2 R^+ = n \times a \cdot \sigma &= \varepsilon_{ijk} n_j a_k \sigma_i \\
&= \varepsilon_{213} n_2 a_3 \sigma_2 = \sigma_y \text{ as required}
\end{align*}
\]

\[
R = \frac{1}{\sqrt{2}} \left( 1 - i n \cdot \sigma \right) = \frac{1}{\sqrt{2}} \left( 1 + i \sigma_3 \right)
\]

\[
\sigma_x \left| \uparrow \right\rangle = \left| \downarrow \right\rangle \quad \sigma_x \left| \downarrow \right\rangle = \left| \uparrow \right\rangle
\]

\[
|\sigma\rangle = R |\alpha\rangle = \frac{1}{\sqrt{2}} \left( 1 + i \sigma_3 \right) |\uparrow\rangle = \frac{1}{\sqrt{2}} \left( |\uparrow\rangle + i |\downarrow\rangle \right)
\]

\[
|\sigma\rangle = R |\alpha\rangle = \frac{1}{\sqrt{2}} \left( 1 + i \sigma_3 \right) |\downarrow\rangle = \frac{1}{\sqrt{2}} \left( |\downarrow\rangle + i |\uparrow\rangle \right)
\]
\[ \hat{x}_i(\theta) = R^t(n, \theta) \hat{x}_i R(n, \theta) \]
\[ = e^{i \theta n \cdot \hat{x}_i} e^{-i \theta n \cdot \hat{x}_i} \]
\[ \frac{d}{d\theta} \hat{x}_i(\theta) = i \frac{R^t}{\hbar} [n \cdot \hat{x}_i, \hat{x}_c] R \]
\[ = \frac{i}{\hbar} R^t n_k \varepsilon_{jkm} [\hat{x}_j \hat{p}_m, \hat{x}_c] R \]
\[ = R^t n_k \hat{x}_j \varepsilon_{kji} R \]
\[ = \varepsilon_{kji} n_k \hat{x}_j(\theta) \]
\[ = (n \times \hat{x}(\theta)) \iota \]
\[ \frac{d}{d\theta} \hat{x}(\theta) = n \times \hat{x}(\theta) \]

for which we know the solution is

\[ \hat{x}(\theta) = \hat{x} \cos \theta + n \times \hat{x} \sin \theta \]

since it is identical to classical case considered in lectures.
\[ L_i = E_{jk} \hat{\chi}_j \hat{\rho}_n \]

\[
[\hat{L}_i, \hat{L}_j] = E_{ik} E_{jm} \left[ \hat{\chi}_k \hat{\rho}_l, \hat{\chi}_m \hat{\rho}_n \right]
\]

\[
= E_{ik} E_{jm} \left( \hat{\chi}_k \left[ \hat{\rho}_l, \hat{\chi}_m \hat{\rho}_n \right] + \left[ \hat{\chi}_k, \hat{\chi}_m \hat{\rho}_n \right] \hat{\rho}_l \right)
\]

\[
= E_{ik} E_{jm} \left( \hat{\chi}_k \left[ \hat{\rho}_l, \hat{\chi}_m \hat{\rho}_n \right] + \hat{\chi}_m \left[ \hat{\chi}_k \hat{\rho}_n, \hat{\rho}_l \right] \right)
\]

\[
= i \hbar \delta_{km} \hat{\chi}_j \hat{\rho}_n
\]

\[
= i \hbar \left( -E_{ik} E_{jm} \hat{\chi}_k \hat{\rho}_n + E_{ik} E_{jm} \hat{\chi}_m \hat{\rho}_l \right)
\]

\[
= i \hbar \left( (S_{ij} S_{kn} - S_{in} S_{jk}) \hat{\chi}_k \hat{\rho}_n 
\right.
\]

\[
+ S_{ij} \left( S_{im} S_{kj} - S_{jm} S_{ik} \right) \hat{\chi}_m \hat{\rho}_l \right)
\]

\[
= i \hbar \left( S_{ij} \hat{\chi}_i \hat{\rho}_j - \hat{\chi}_j \hat{\rho}_i - S_{ij} \hat{\chi}_i \hat{\rho}_j 
\right.
\]

\[
+ \hat{\chi}_i \hat{\rho}_j \right)
\]

\[
= i \hbar \left( \hat{\chi}_i \hat{\rho}_j - \hat{\chi}_j \hat{\rho}_i \right)
\]

\[
= i \hbar \left( S_{ik} S_{jl} - S_{il} S_{jk} \right) \hat{\chi}_k \hat{\rho}_l
\]

\[
= i \hbar E_{ikm} E_{jm} \hat{\chi}_k \hat{\rho}_l
\]

\[
= i \hbar E_{ijm} L_m
\]
\[ \langle 4 | \Sigma \otimes 1 | 4 \rangle \]

\[ = \frac{1}{\sqrt{2}} \langle 4 | (\Sigma \uparrow \downarrow \otimes \downarrow \downarrow) + (\Sigma \downarrow \uparrow \otimes \uparrow \uparrow) \rangle \]

\[ = \frac{1}{2} \left( \langle \uparrow \Sigma \uparrow \rangle - \langle \downarrow \Sigma \downarrow \rangle + \langle \downarrow \Sigma \uparrow \rangle + \langle \uparrow \Sigma \downarrow \rangle \right) \]

\[ = \frac{1}{2} \left( \langle \uparrow \Sigma \uparrow \rangle + \langle \downarrow \Sigma \downarrow \rangle \right) \]

\[ = \frac{1}{2} \text{Tr}(\Sigma) \]
\[ p_\pm = \frac{1}{2} \left( 1 \pm a \cdot \sigma \right) \]

\[ p_\pm^2 = \frac{1}{4} \left( 1 \pm 2 a \cdot \sigma + (a \cdot \sigma)^2 \right) \]

\[ a \cdot q = 1 \]

\[ = \frac{1}{2} \left( 1 \pm a \cdot \sigma \right) \]

\[ = \frac{p_\pm}{p_\pm} \]

\[ p_\pm^2 = \frac{1}{2} \left( 1 \pm 6 \tau \right) = \frac{1}{2} \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \]

\[ p_+^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad p_-^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \]

(i)

\[ p(+) = \langle + | p_+^a \otimes p_-^b | + \rangle \]

\[ = \frac{1}{4} \langle + | (1 + a \cdot \sigma) \otimes (1 - b \cdot \sigma) | + \rangle \]

\[ = \frac{1}{4} \left[ 1 + \langle + | a \cdot \sigma \otimes 1 | + \rangle \right. \]

\[ + \langle + | 1 \otimes b \cdot \sigma | + \rangle - \langle + | a \cdot \sigma \otimes b \cdot \sigma | + \rangle \]

\[ \langle + | a \cdot \sigma \otimes 1 | + \rangle = \frac{1}{2} \text{Tr} (a \cdot \sigma) = 0 \]

\[ \Rightarrow pS5 \Rightarrow \langle + | a \cdot \sigma \otimes b \cdot \sigma | + \rangle = -a \cdot b \]
\[ p(\pm, -) = \frac{1}{4} (1 + a \cdot b) = \frac{1}{4} (1 + \cos \theta) = \frac{1}{2} \cos^2 \frac{\theta}{2} \]

Similarly, \[ p(-, +) = p(+, -) = \frac{1}{2} \cos^2 \frac{\theta}{2} \]

\[ p(+, +) = p(-, -) = \frac{1}{4} (1 - a \cdot b) = \frac{1}{2} \sin^2 \frac{\theta}{2} \]

(iii) \[ p(a) = \sum_b p(a, b) = p(a, +) + p(a, -) \]

\[ p(a = +1) = p(+, +) + p(+, -) = \frac{1}{2} \]

\[ p(a = -1) = p(-, +) + p(-, -) = \frac{1}{2} \]

Similarly, \[ p(b = \pm 1) = \frac{1}{2} \]

\[ p(a | b) = \frac{p(a, b)}{p(b)} = \frac{1}{2} \]

\[ p(a = +1 | b = +1) = 2p(+, +) = \sin^2 \frac{\theta}{2} = 0 \text{ at } \theta = 0 \]

\[ p(a = +1 | b = -1) = 2p(+, -) = \cos^2 \frac{\theta}{2} = 1 \text{ at } \theta = 0 \]

\[ p(a = -1 | b = +1) = \cos^2 \frac{\theta}{2} \]

\[ p(a = -1 | b = -1) = \sin^2 \frac{\theta}{2} \]

At \( \theta = 0 \) (\( a, b \) parallel), \( p(+1-) = p(-1+) = 1 \), indicating perfect anticorrelation.
3 \quad C(a, b) = <4| a \otimes b | 1 > \\

For EPRB state

\[ S_{14}^{AB} = 0 \]

\[ S_{14}^{AB} = \frac{1}{2} \left( \sigma_i^A \otimes 1 + 1 \otimes \sigma_i^B \right) | 1 > = 0 \]

\[ \Rightarrow (\sigma_i^A \otimes 1 + 1 \otimes \sigma_i^B) | 1 > = 0 \]

\[ 1 \otimes b \sigma_i^B | 1 > = -a \sigma_i^A \otimes 1 | 1 > \]

Now we have

\[ C(a, b) = -<4| (a \sigma_i^A)(b \sigma_i^B) | 1 > \]

\[ = -\frac{1}{2} \text{Tr} (a \sigma_i^A b \sigma_i^B) \]

\[ = -\frac{1}{2} a_k b_j \text{Tr}(\sigma_k \sigma_j) \frac{1}{2} \delta_{k,j} \]

\[ = -a \cdot b \quad \text{as required.} \]
\[ S_{A^B} = S_A \otimes 1 + 1 \otimes S_B \]

i.e.
\[ S_{A^B} = S_A^A \otimes 1 + 1 \otimes S_A^B \]
\[ S_i^A = \frac{\hbar}{2} \sigma_i = S_i^B \]

\[ 0 \cdot 0 = \sigma_i \cdot 0 = \sigma_i^x + \sigma_i^y + \sigma_i^z = 3.1 \]

\[ (S_{A^B})^2 = (S_A^A \otimes 1 + 1 \otimes S_A^B)^2 \]
\[ = (S_A^A)^2 \otimes 1 + 1 \otimes (S_A^B)^2 \]
\[ + 2 S_A^A \otimes S_A^B \]
\[ = \frac{\hbar^2}{4} \left( 61 + 26 \sigma_i^A \otimes \sigma_i^B \right) \]
\[ = \frac{\hbar^2}{2} \left( 31 + 6 \sigma_i^A \otimes \sigma_i^B \right) \]

Now compute \( \sigma_i^A \otimes \sigma_i^B \) on the states:
\[ |\uparrow\rangle |\uparrow\rangle, |\uparrow\rangle |\downarrow\rangle, |\downarrow\rangle |\uparrow\rangle, |\downarrow\rangle |\downarrow\rangle \]
\( \sigma_2 \uparrow \uparrow = \uparrow \uparrow \) \( \sigma_2 \downarrow \downarrow = - \downarrow \downarrow \)
\( \sigma_x \uparrow \downarrow = \downarrow \uparrow \) \( \sigma_x \downarrow \uparrow = \uparrow \downarrow \)
\( \sigma_y \uparrow \uparrow = i \downarrow \uparrow \) \( \sigma_y \downarrow \downarrow = - i \uparrow \downarrow \)

\( \left( \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y \right) \uparrow \uparrow \uparrow \uparrow = \uparrow \uparrow \downarrow \downarrow + (i)^2 \downarrow \downarrow \uparrow \uparrow \downarrow \downarrow \)
\( \left( \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y \right) \downarrow \downarrow \downarrow \downarrow = \uparrow \uparrow \downarrow \downarrow + (-i)^2 \downarrow \downarrow \uparrow \uparrow \downarrow \downarrow \)
\( \left( \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y \right) \uparrow \downarrow \uparrow \downarrow = \uparrow \downarrow \uparrow \downarrow + \uparrow \downarrow \downarrow \uparrow \downarrow \)
\( \left( \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y \right) \downarrow \uparrow \uparrow \downarrow = \downarrow \uparrow \uparrow \downarrow + \downarrow \uparrow \downarrow \uparrow \downarrow \)

\( \sigma_2 \otimes \sigma_z \uparrow \downarrow = - \downarrow \uparrow \downarrow \)

\( \left( \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y \right) \downarrow \uparrow \uparrow \downarrow = \downarrow \uparrow \uparrow \downarrow + \downarrow \uparrow \downarrow \uparrow \downarrow \)
\( \left( \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y \right) \downarrow \downarrow \uparrow \uparrow = \uparrow \downarrow \uparrow \uparrow + \downarrow \downarrow \downarrow \downarrow \)

\( \sigma_x \otimes \sigma_x \uparrow \uparrow \uparrow \uparrow = \sigma_x \otimes \sigma_x \uparrow \uparrow \downarrow \downarrow = \uparrow \uparrow \uparrow \uparrow \)

\( \sigma_i \otimes \sigma_i \downarrow \downarrow \downarrow = \sigma_i \otimes \sigma_i \downarrow \downarrow \uparrow \uparrow = \downarrow \downarrow \downarrow \downarrow \)

\( \sigma_i \otimes \sigma_i \downarrow \downarrow \downarrow = \downarrow \downarrow \downarrow \downarrow \)

\( \sigma_i \otimes \sigma_i \uparrow \uparrow \downarrow \downarrow = 2 \uparrow \uparrow \downarrow \downarrow - \uparrow \uparrow \downarrow \downarrow \)

\( \sigma_i \otimes \sigma_i \downarrow \uparrow \uparrow \downarrow = 2 \uparrow \uparrow \downarrow \downarrow - \downarrow \uparrow \uparrow \downarrow \)
\[ \begin{align*}
|c \otimes \bar{c} : (\uparrow \uparrow \downarrow \downarrow + \downarrow \downarrow \uparrow \uparrow) &= \uparrow \downarrow \downarrow \downarrow + \downarrow \downarrow \uparrow \uparrow \\
|c \otimes \bar{c} : (\uparrow \uparrow \downarrow \downarrow - \downarrow \downarrow \uparrow \uparrow) &= \frac{-2}{\sqrt{3}} (\uparrow \downarrow \downarrow \downarrow - \downarrow \downarrow \uparrow \uparrow) \end{align*} \]

\[
(S^{AB})^2 = \frac{t^2}{2} (3 \mathbb{1} + \sigma_i \otimes \sigma_i)
\]

\[
(S^{AB})^2 |\uparrow \uparrow \rangle = 2 \frac{t^2}{2} |\uparrow \uparrow \rangle = s (s+1) \frac{t^2}{2} |\uparrow \uparrow \rangle
\]

\[
(S^{AB})^2 |\downarrow \downarrow \rangle = 2 \frac{t^2}{2} |\downarrow \downarrow \rangle
\]

\[
(S^{AB})^2 (|\uparrow \downarrow \rangle + |\downarrow \uparrow \rangle) = 2 \frac{t^2}{2} (|\uparrow \downarrow \rangle + |\downarrow \uparrow \rangle)
\]

\[
(S^{AB})^2 (|\uparrow \downarrow \rangle - |\downarrow \uparrow \rangle) = 0
\]

States \(|\uparrow \uparrow \rangle\) and \(|\downarrow \downarrow \rangle\) are not \(S\)-states.

A simplification is to note that \(|11, 1 \rangle, |11, 0 \rangle\) and \(|11, -1 \rangle\) are related by ladder operators \(S^+\), which commute with \((S^{AB})^2\), so it is necessary only to show that, for example, \((S^{AB})^2 |11, 1 \rangle = 0\), and it follows that \((S^{AB})^2 |11, 0 \rangle = 0 = (S^{AB})^2 |11, -1 \rangle\).
1. \( |\uparrow\rangle = a |\uparrow\rangle + b |\downarrow\rangle \quad |a|^2 + |b|^2 = 1 \)

\( \rho = |a|^2 |\uparrow\rangle\langle \uparrow| + |b|^2 |\downarrow\rangle\langle \downarrow| \)

\[ \rho (\uparrow) = \begin{cases} |\langle \uparrow | \uparrow \rangle|^2 = |a|^2 \\ \langle \uparrow | e \uparrow \rangle = |a|^2 \end{cases} \]

\[ \rho (\downarrow) = \begin{cases} |\langle \downarrow | \uparrow \rangle|^2 = |b|^2 \\ \langle \downarrow | e \downarrow \rangle = |b|^2 \end{cases} \]

\( 1 \pm > = \frac{1}{\sqrt{2}} (|\uparrow\rangle \pm |\downarrow\rangle) \)

\[ \langle \uparrow | \uparrow \rangle = \frac{1}{\sqrt{2}} \quad \langle \uparrow | \downarrow \rangle = \frac{1}{\sqrt{2}} \]

\[ \langle \downarrow | \uparrow \rangle = \frac{1}{\sqrt{2}} \quad \langle \downarrow | \downarrow \rangle = -\frac{1}{\sqrt{2}} \]

\[ \langle +|\uparrow \rangle = a \langle +|\uparrow \rangle + b \langle +|\uparrow \rangle = \frac{a+b}{\sqrt{2}} \]

\[ \rho (\uparrow) = |\langle +|\uparrow \rangle|^2 = \frac{(a+b)^2}{2} \]

\[ \rho_c (\uparrow) = \langle +| e + \rangle = |a|^2 |\langle +|\uparrow \rangle|^2 + |b|^2 |\langle +|\downarrow \rangle|^2 \]

\[ = \frac{|a|^2 + |b|^2 + |a+b|^2}{2} \]

Similarly for \( |\langle -|\downarrow \rangle|^2 \) and \( \langle -| e \downarrow \rangle \).
2. \[ E = \rho_1 |\psi_1\rangle\langle\psi_1| + \rho_2 |\psi_2\rangle\langle\psi_2| \]

\[ Tr(E) = \rho_1 Tr(|\psi_1\rangle\langle\psi_1|) + \rho_2 Tr(|\psi_2\rangle\langle\psi_2|) \]

\[ = \rho_1 + \rho_2 = 1 \]

Clearly, \[ E = \mathbb{1} \]

Also, \[ \langle \phi | E | \phi \rangle = \rho_1 |\langle \phi | \psi_1 \rangle|^2 + \rho_2 |\langle \phi | \psi_2 \rangle|^2 \]

\[ \geq 0 \quad \text{for any } |\phi\rangle \]

So, 3 density operator properties are satisfied and \[ |\psi_1\rangle \]

does not need to be orthogonal to \[ |\psi_2\rangle \].

\[ |\psi_1\rangle = |\uparrow\rangle, \quad |\psi_2\rangle = |\downarrow\rangle \]

\[ p(\uparrow) = \langle \uparrow | E | \uparrow \rangle = \rho_1 |\langle \uparrow | \psi_1 \rangle|^2 + \rho_2 |\langle \uparrow | \psi_2 \rangle|^2 \]

\[ = \rho_1 + \frac{1}{2} \rho_2 \quad \text{since } \langle \uparrow | \psi_2 \rangle = \frac{1}{\sqrt{2}} \]

\[ p(\downarrow) = \langle \downarrow | E | \downarrow \rangle = 0 + \rho_2 |\langle \downarrow | \psi_2 \rangle|^2 \]

\[ = \frac{1}{2} \rho_2 \]

Check: \[ p(\uparrow) + p(\downarrow) = \rho_1 + \frac{1}{2} \rho_2 + \frac{1}{2} \rho_2 = 1 \]
3. \[ |\psi\rangle \langle \psi| = \frac{1}{2} \left( |\uparrow\rangle_A |\uparrow\rangle_B + |\downarrow\rangle_A |\downarrow\rangle_B - |\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B \right) \]

\[ \rho_A = \text{Tr}_B |\psi\rangle \langle \psi| \]

Recall \( \text{Tr}_B |\phi\rangle \langle \phi| = \langle \phi | B \rangle \)

so \( \text{Tr}_B |\uparrow\rangle_B |\downarrow\rangle_B = \langle \downarrow | B \rangle = 1 \)

\( \text{Tr}_B |\uparrow\rangle_B |\uparrow\rangle_B = \langle \uparrow | B \rangle = 0 \).

\[ \Rightarrow \text{3rd and 4th terms} \to 0 \text{ in trace} \]

\[ \rho_A = \frac{\text{Tr}_B |\psi\rangle \langle \psi|}{2} = \frac{1}{2} \left( |\uparrow\rangle_A |\uparrow\rangle_B + |\downarrow\rangle_A |\downarrow\rangle_B \right) \]

\[ = \frac{1}{2} \mathbf{1}_A. \]

\[ \implies \text{PS8, 4.1} \Rightarrow \text{in EPRB state} \]

\[ \langle \psi | \Sigma_A \otimes \frac{1}{2} |\psi\rangle = \frac{1}{2} \text{Tr}_A (\Sigma) \text{ for any } \Sigma \]

but \[ \langle \psi | \Sigma_A \otimes \frac{1}{2} |\psi\rangle = \text{Tr}_A (\Sigma \rho_A) \]

by def. of reduced \( \rho_A \), so \( \rho_A = \frac{1}{2} \mathbf{1}_A. \)

We know that measurement of spin on one particle in any direction has probability \( \frac{1}{2} \)
for \( \uparrow \) or \( \downarrow \) which means that \( \rho_A = \frac{1}{2} \mathbf{1}_A. \)
4. \[ \dot{\vec{e}} = -i [H, \vec{e}] - L^2 (\vec{e}) \]

\[ L^2 (\vec{e}) = L^2 \vec{e} + \vec{e} L^2 - 2 \vec{e} L \vec{e} \]

\[ \frac{d}{dt} \text{Tr} \vec{e}^2 = 2 \text{Tr} \vec{e} \dot{\vec{e}} \]

\[ = 2 \text{Tr} (\vec{e} \dot{\vec{e}} - i [H, \vec{e}]) - 2 \text{Tr} (L^2 (\vec{e})) \]

\[ = -2i \text{Tr} (\vec{e} H \vec{e} - \vec{e}^2 H) - 2 \text{Tr} (L^2 (\vec{e})) \]

\[ \text{(Tr part at end)} \]

\[ \text{Tr (ABC)} = \text{Tr (CAB)} \]

\[ \frac{d}{dt} \text{Tr} \vec{r}^2 = -2 \text{Tr} (L^2 (\vec{r})) \]

\[ = -2 \text{Tr} (L^2 \vec{e} + \vec{e} L^2 - 2 \vec{e} L \vec{e}) \]

\[ = -2 \text{Tr} (L^2 - \vec{e} L \vec{e}) \]

Now use for any \( A \)

\[ \text{Tr} (A^* A) = \sum \langle n | A^* A | n \rangle = \sum \| A | n \rangle \|^2 > 0 \]

Choose \( A = [L, \vec{e}] = \vec{e} L - \vec{L} \vec{e} \)

\[ \implies \text{Tr} (\vec{e} L - \vec{L} \vec{e})^* (\vec{e} L - \vec{L} \vec{e}) > 0 \]

\[ \implies \text{Tr} (\vec{e} L - \vec{e} L)(\vec{e} L - \vec{L} \vec{e}) > 0 \]
\[ \text{Tr} \left( L e^2 L - L e L e L - e L e L + e^2 L e L \right) \geq 0 \]

\[ 2 \text{Tr} \left( e^2 L^2 - L e L e L \right) \geq 0. \]

\[ \Rightarrow \quad \frac{d}{dt} \text{Tr} e^2 \leq 0. \]

Hence \( \text{Tr} e^2 \) may decrease so pure states can go to a mixed state.

\[ \frac{d}{dt} \text{Tr} e = \text{Tr} \frac{de}{dt} \]

\[ = -\frac{i}{\hbar} \text{Tr} \left( [H, e] \right) - \text{Tr} \left( L^2 e + e L^2 - 2 L e L \right) \]

\[ = -\frac{i}{\hbar} \text{Tr} \left( H e - e H \right) - \text{Tr} \left( L^2 e + L^2 e - 2 L^2 e \right) \]

\[ = 0 \quad \text{using} \quad \text{Tr} (AB) = \text{Tr} (BA) \]

\[ \text{Tr} (ABC) = \text{Tr} (CAB). \]
1. \[
T_1 = \mathcal{O}_{XX}^{A} \otimes \mathcal{O}_{YY}^{B} \otimes \mathcal{O}_{YY}^{C}
\]
\[
T_2 = \mathcal{O}_{YY}^{A} \otimes \mathcal{O}_{XX}^{B} \otimes \mathcal{O}_{YY}^{C}
\]
\[
T_3 = \mathcal{O}_{YY}^{A} \otimes \mathcal{O}_{YY}^{B} \otimes \mathcal{O}_{XX}^{C}
\]

Note: \(\mathcal{O}_{XX} \otimes \mathcal{O}_{YY} = - \mathcal{O}_{YY} \otimes \mathcal{O}_{XX}\)

All commutators \([T_1, T_2]\) etc. involve two anti-commutations.

\[
T_1 T_2 = \mathcal{O}_{XX}^{A} \otimes \mathcal{O}_{YY}^{B} \otimes \mathcal{O}_{YY}^{6x} \otimes (G_y^2)^c
\]
\[
= \mathcal{O}_{YY}^{A} \otimes \mathcal{O}_{XX}^{B} \otimes \mathcal{O}_{YY}^{6x} \otimes (G_y^2)^c
\]
\[
= T_2 T_1
\]

Similarly for other commutators.
2. \[ T_1 |\uparrow\uparrow\uparrow\uparrow\rangle = (i)^2 |\downarrow\downarrow\downarrow\rangle = - |\downarrow\downarrow\downarrow\rangle \]
\[ T_1 |\downarrow\downarrow\downarrow\rangle = (i)^2 |\uparrow\uparrow\uparrow\rangle = - |\uparrow\uparrow\uparrow\rangle \]
\[ T_1 (4) = - |4\rangle \]

Similarly for other two.

3. \[ T_2 \text{ case } \Rightarrow M_y^A M_x^B M_y^C = -1 \]
\[ T_3 \text{ case } \Rightarrow M_y^A M_y^B M_x^C = -1 \]
\[ (T_1 \text{ case } \Rightarrow M_x^A M_y^B M_y^C = -1) \]

Product \[\Rightarrow \frac{(M_y^A)^2 (M_y^B)^2 (M_y^C)^2 M_x^A M_x^B M_x^C}{(-1)^3} = 1 \]
\[\Rightarrow M_x^A M_x^B M_x^C = -1 \]
\[ R = \sigma_x^A \otimes \sigma_y^B \otimes \sigma_x^C \]

\[ [R, T_i] = 0 \text{ because as before} \]

get two anti-commutators.

\[ T_1 T_2 T_3 = \]

\[ \sigma_x^A (6y)^2 \otimes \sigma_y^B \sigma_x^B \sigma_y^C \otimes (6y)^2 \sigma_x^C \]

\[ = - \sigma_x^A \otimes \sigma_y^B (6y)^2 \]

\[ = - R. \]

\[ R |1+\rangle = - T_1 T_2 T_3 |1+\rangle = (-1)(-1)^3 |1+\rangle \]

\[ = + |1+\rangle. \]

5. Difference between 3 and \( T_1 T_2 T_3 \) anti-commutator involved is \( T_1 T_2 T_3 \).