Consider the form of a Bloch function in one dimension: \( \psi_k(x) = e^{ikx}u_k(x) \), where \( u_k(x) \) has the periodicity of the lattice \( u_k(x + a) = u_k(x) \). Hence, \( u_k(x) \) can be represented as a complex Fourier series:

\[
u_k(x) = \sum_{n=-\infty}^{\infty} U_n(k) e^{2\pi i n x / a},
\]

where the \( U_n(k) \) are the Fourier coefficients of \( u_k(x) \). We now introduce the notation used in the course notes: \( G_n = \frac{2\pi i}{a} \), in which case we can write

\[
u_k(x) = \psi_k(x) e^{-ikx} = \sum_{G} \tilde{U}_G(k) e^{iGx}.
\]

(1)

We now invoke the orthogonality property of the complex exponentials over the unit cell:

\[
\int_{0}^{a} e^{iGx} e^{-iG'x} dx = \int_{0}^{a} \exp \left[ \frac{2\pi i}{a} (n - n')x \right] dx = a \delta_{n,n'} = a \delta_{G,G'},
\]

to project individual Fourier components in (1):

\[
\int_{0}^{a} \psi_k(x) e^{-ikx} e^{-iG'x} dx = \int_{0}^{a} \psi_k(x) e^{i(k+G')x} dx
\]

\[
= \int_{0}^{a} \left[ \sum_{G} U_G(k) e^{iGx} \right] e^{-iG'x} dx
\]

\[
= \sum_{G} U_G(k) \int_{0}^{a} e^{i(G-G')x} dx
\]

\[
= a U_G^*(k)
\]

\[
\equiv \tilde{u}(k + G'),
\]

where in the last line we have redefined the Fourier coefficients to absorb the factor of \( a \). Therefore, the Fourier representation of a Bloch function can be written as

\[
\psi_k(x) = e^{ikx} \sum_{G} \tilde{u}(k + G) e^{iGx} = \sum_{G} \tilde{u}(k + G) e^{i(k+G)x}.
\]

(2)

The same procedure can be used to obtain analogous expressions in higher dimensions:

\[
\psi_k(x) = \sum_{G} \tilde{u}(k + G) e^{i(k+G)x}.
\]

The expansion in (2) has important consequences. Consider the operation of the momentum operator on a Bloch state:

\[
-i\hbar \frac{d\psi_k(x)}{dx} = -i\hbar \frac{d}{dx} \left[ \sum_{G} \tilde{u}(k + G) e^{i(k+G)x} \right] = -i\hbar \sum_{G} \tilde{u}(k + G) \frac{d e^{i(k+G)x}}{dx}
\]

\[
= k \sum_{G} \tilde{u}(k + G) e^{i(k+G)x} + \sum_{G} G \tilde{u}(k + G) e^{i(k+G)x}
\]

\[
= k \psi_k(x) + \sum_{G} G \tilde{u}(k + G) e^{i(k+G)x}.
\]
The first term on the right-hand side of this equation appears like a momentum eigenstate, but the second term includes contributions from other wave vectors, in effect, from the periodic lattice. In other words, the fact that an electron in a periodic lattice is not a momentum eigenstate is due to the fact that the lattice imposes a periodicity on the electron, so the momentum is characteristic of the electron-lattice system, rather than the electron alone. For this reason, $k$ is called a **crystal momentum**.

As a second example of the utility of (2), we can immediately write

$$
\psi_{k+G}(x) = \sum_{G'} \tilde{u}(k + G + G') e^{i(k+G+G')x}.
$$

The summation index $G'$ is a dummy variable whose range extends from minus to plus infinity. Thus, we can shift this variable by defining a new summation index $G'' = G + G'$, in which case we obtain

$$
\psi_{k+G}(x) = \sum_{G''} \tilde{u}(k + G'') e^{i(k+G'')x} = \sum_{G} \tilde{u}(k + G) e^{i(k+G)x} = \psi_k(x),
$$

where we have again used the fact that $G$ and $G''$ are both dummy variables. Hence,

$$
\psi_{k+G}(x) = \psi_k(x).
$$

An immediate consequence of this result is obtained from the Schrödinger equations solved by $\psi_k(x)$ and $\psi_{k+G}(x)$. We have

$$
\hat{H} \psi_k(x) = E(k) \psi_k(x),
$$

where $\hat{H}$ is the Hamiltonian operator and, by changing $k$ to $k + G$,

$$
\hat{H} \psi_{k+G}(x) = E(k + G) \psi_{k+G}(x),
$$

which is the same as

$$
\hat{H} \psi_k(x) = E(k + G) \psi_k(x).
$$

This implies that

$$
E(k + G) = E(k).
$$