1. Taking a solar luminosity of $L_\odot = 3.8 \times 10^{26} \text{W}$, a solar radius of $R_\odot = 7.0 \times 10^8 \text{m}$ and a Sun-Earth distance of $1\text{AU} = 1.5 \times 10^{11} \text{m}$, calculate the following quantities:
   (i) The solar flux at the top of the Earth’s atmosphere, $S$;
   (ii) The solar flux at the distance of Pluto (~39 AU), Mars (~1.5 AU) and Mercury (~0.4 AU) as a fraction of $S$.
   (iii) The solar flux at the Sun’s surface and the effective temperature of the Sun.
   (iv) The brightest star noticeable in the night sky is Sirius which lies at a distance of 8.48 light years and has a luminosity ~23$L_\odot$. Calculate the flux of the light from Sirius at the top of the Earth’s atmosphere. If the effective temperature of Sirius is 9600K what is its radius?

2. The dynamical or ‘free-fall’ timescale, $t_{\text{ff}}$, can be approximated by the ratio of the stellar radius to the escape velocity. Calculate the escape velocity of a test particle at radius $R$ and derive an approximate expression for $t_{\text{ff}}$. Express $t_{\text{ff}}$ in terms of the mean solar density and compare your expression to the one given in lecture 2. Evaluate the mean solar density and hence $t_{\text{ff}}$ for the Sun. You may use $M_\odot = 2.0 \times 10^{30} \text{kg}$ and $R_\odot = 7.0 \times 10^8 \text{m}$.

3. A region of the interior of the Sun contains fully ionised hydrogen, helium and heavier elements in the fractions $X, Y, Z$ (by mass). A typical atom of the heavier elements can be assumed to have approximately $A/2 >> 1$ protons, where $A$ is the element’s atomic mass. Show that the number density of electrons in that region is:
   
   \[ n_e = \frac{\rho}{m_H} \left( X + \frac{Y}{2} + \frac{Z}{2} \right) \]

   and that the mean molecular weight $\mu$ is:
   
   \[ \mu \approx \frac{1}{\left( 2X + 3Y/4 + Z/2 \right)} = \frac{4}{(5X + 3 - Z)} \]

   In the centre of the Sun, it is estimated from a theoretical model that $X = 0.40$. Neglecting elements heavier than helium, estimate the value of $\mu$ at the centre of the Sun.

4. Write down expressions for the gas pressure $P_{\text{GAS}}$ and the radiation pressure $P_{\text{RAD}}$. Assuming the mean molecular weight $\mu = 1$, evaluate the ratio $P_{\text{RAD}} / P_{\text{GAS}}$ at the centre of the Sun and near its surface.

   You may assume for the solar centre $T_C = 1.5 \times 10^7 \text{K}$, $\rho_C = 1.5 \times 10^5 \text{kg m}^{-3}$, and near the surface $T_{\text{ns}} = 10^4 \text{K}$, $\rho_{\text{ns}} = 10^{-3} \text{kg m}^{-3}$. Use radiation constant $a = 7.55 \times 10^{-16} \text{J m}^{-2} \text{K}^{-4}$, and $\mathcal{R} = 8.26 \times 10^3 \text{J K}^{-1} \text{kg}^{-1}$.

   Compare the central pressure you calculate here to the estimates from lecture 2 in your notes.

5. (a) Calculate the energy, in MeV, produced when 4 protons fuse to produce one helium atom.
   (b) Show that all three branches of the proton-proton reaction chain produce one $^4\text{He}$ ion, two positrons and two electron neutrinos from the fusion of four protons.
   (c) Why do the three branches of the proton-proton reaction chain produce slightly different amounts of heat from one another in fusing four protons?
6. (a) For two protons to get close enough to fuse, they must approach one another in spite of the Coulomb repulsion of their two positive charges. Assuming that the two protons have the average kinetic energy \( (= \frac{3}{2} k T) \) and have a head-on collision, show that the temperature would have to be about \( 10^{10} \) K for the two protons to approach to within a nuclear radius \( (= 10^{-15} \) m). [You may use \( \varepsilon_0 = 8.26 \times 10^{-12} \) farad \( m^{-1} \).]

(b) Even taking into account the tails of the Boltzmann distribution, there is no way to get a reasonably large number of proton-proton collisions above the Coulomb barrier. The resolution of this problem is quantum tunnelling. A simple estimate of tunnelling can be obtained by assuming that the proton must be within one de Broglie wavelength \( (= h/p, \) where \( h \) is the Planck constant and \( p \) the momentum of the proton) of its target for tunnelling to occur. Estimate the temperature at which two average protons in head-on collision can get to within this distance of one another.

5. Derive the Schwarzschild criterion:

\[
\frac{1}{\gamma} \frac{d \ln p}{dr} > \frac{d \ln p}{dr}
\]

for the stratification of the Sun to be stable. In the Sun’s convection zone, the stratification is marginally unstable (so replace the “>” sign with an “=” sign in the above inequality). Assuming the adiabatic exponent \( \gamma \) is a constant \( 5/3 \), show that the pressure and density are related by \( p \propto \rho^{5/3} \).

6. In the Sun’s convective envelope (the outer 30% of the Sun), the pressure \( P \) and density \( \rho \) are related to a good approximation by

\[
P = K \rho^{5/3}
\]

where \( K \) is a constant. The convection zone contains only 2% of the Sun’s mass, so the gravitational acceleration \( g = G M_\odot / r^2 \). Starting from the equation of hydrostatic support that can also be written as \( \frac{dp}{dr} = - \rho g \) show that:

\[
\rho(r)^{2/3} - \rho_s^{2/3} = \frac{2 G M_\odot}{5 K} \left( \frac{1}{r} - \frac{1}{R_\odot} \right)
\]

where \( \rho_s \) is the density at the surface \( r = R_\odot \). Given that the surface density is very much smaller than the typical density in the rest of the convection zone, deduce that:

\[
\rho \approx \left( \frac{2 G M_\odot}{5 K R_\odot^2} \right)^{3/2} z^{3/2}
\]

at depths \( z \ll R_\odot \) below the surface \( r = R_\odot \), and derive a similar expression for the pressure. Deduce that the temperature is roughly proportional to depth throughout the convection zone.

7. A star has observed angular diameter = 0.044 arcseconds, and lies at a distance of 427 light years from the Earth, what is its radius? [answer, radius = \( 4.3 \times 10^{11} \) m ] – part of 2005 Q B2 (v)
1. The monochromatic intensity $B_\nu(T)$ of a black body of temperature $T$ is the energy per unit time at frequency $\nu$ radiated per unit frequency per unit solid angle into a small cone from a piece of the surface, the piece having unit area in projection perpendicular to the axis of the cone. The Planck function is given by:

$$B_\nu(T) = \frac{2\hbar \nu^3}{c^2} \frac{1}{e^{(\hbar \nu/kT)} - 1}$$

(a) Show that the energy per unit time from a piece of surface of area $\delta A$ is:

$$\int_0^\pi d\theta \int_0^{2\pi} \sin\theta d\phi \int_0^\infty dv \cos\theta B_\nu \times \delta A$$

where $\theta=0$ is the direction normal to the surface.

(b) Deduce that the flux $F$ (energy per unit time per unit area) from the surface is $F = \sigma T^4$ where $\sigma$ is a constant.

Using the standard identity $\int_0^\infty x^3 (e^{x^2} - 1) dx = \pi^4/15$ and the definition $a = \frac{8\pi^5 k^4}{15(hc)^3}$ of the radiation constant $a$, show further that $\sigma = ac/4$.

2. The energy levels of atomic hydrogen are:

$$E_n = -\frac{m_e}{2\hbar^2} \left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 \frac{1}{n^2} \quad (n = 1, 2, 3 \ldots)$$

where $m_e$ and $e$ are the mass and charge of an electron, and $\hbar = h/2\pi$, and $\varepsilon_0$ is the electrical permittivity.

(a) Show that the wavelength of a photon emitted when an electron undergoes a transition from level $n'$ to a lower energy level $n$ is

$$\lambda = 4\pi \left(\frac{\hbar c}{(e^2/4\pi\varepsilon_0)}\right) \left(\frac{\hbar^2}{m_e(e^2/4\pi\varepsilon_0)}\right) \left(\frac{n'^2n^2}{(n'^2 - n^2)}\right)$$

The first fraction, $\left(\frac{\hbar c}{(e^2/4\pi\varepsilon_0)}\right) \approx 137$, is the inverse of the “fine-structure constant”.

(b) Calculate the numerical value of the wavelength of the transition from upper level $n'=2$ to lower level $n=1$ (this is the Lyman alpha emission line). Are the $n'>1$ to $n=1$ transitions of the Lyman series in the infrared, optical or ultraviolet?

(c) Calculate the wavelength of the Balmer $\alpha$ ($n'=3$ to $n=2$) and Balmer $\beta$ ($n'=4$ to $n=2$) transitions. In what part of the spectrum are these? In which part of the spectrum would you expect subsequently higher series (eg the Paschen series down to level $n=3$)?
3. (a) Explain briefly why sunspots appear dark.
(b) The measured speeds of convection in granular cells at the surface of the sun are about 2 km/s and the photospheric density is of order $10^{-3}$ kg m$^{-3}$. A magnetic field of strength $B$ will be able to alter the motion of a plasma in which it is embedded when the magnetic energy density $B^2/2\mu_0$ is greater than the kinetic energy density associated with the plasma motions. Estimate this critical density for magnetic field in the Sun's surface layers. Comment on your answer to part (a) in the light of this calculation: maximum field strengths in sunspots are about 0.3 T (3000 Gauss). [The value of the magnetic permeability $\mu_0$ in SI units is $4 \times 10^{-7}$ henry m$^{-1}$.]

4. Let the apparent stellar magnitudes $m_U$, $m_B$, and $m_V$ be defined by:

$$m_U = -2.5 \log F_{\lambda_U} + c_1$$
$$m_B = -2.5 \log F_{\lambda_B} + c_2$$
$$m_V = -2.5 \log F_{\lambda_V} + c_3$$

where $F_\lambda$ is the flux received at Earth in units of J m$^{-2}$ s$^{-1}$ nm$^{-1}$ at wavelengths $
\lambda_U = 360$ nm, $\lambda_B = 440$ nm, $\lambda_V = 550$ nm.

Show that the colour indices $U - B \equiv m_U - m_B$, and $B - V \equiv m_B - m_V$ are independent of stellar distance, assuming that the flux varies purely as given by the inverse square law.

5. The bolometric flux received from the Sun at the Earth's distance is 1370 W m$^{-2}$. If a planet orbits a star which is ten times as luminous as the Sun, what is the radius of its orbit if the bolometric flux it receives from its star is 2740 W m$^{-2}$?

6. The apparent magnitude of the Sun as seen from the Earth is -26.7.
(a) What is the apparent magnitude of the Sun as seen from Jupiter (orbital radius 5.2 AU)?
(b) What is the Sun's absolute magnitude?

7. (a) Reproduce the homology argument from lectures to deduce how the mean density and central pressure and temperature scale with stellar mass and radius amongst an homologous group of stars. Assume that the pressure is solely that of an ideal gas.
(b) Assuming that the stars are wholly radiative and that the opacity to radiation is Kramer's opacity, derive also how the stellar luminosity $L$ scales with stellar mass and radius.
(c) Assuming further that energy generation is by the p-p reaction chain, so that $\epsilon = \rho T^4$, obtain a second scaling relation between luminosity and mass and radius; and so find how stellar radius scales with mass. Use this scaling relation between radius and mass to find scaling relations for mean density, central pressure and temperature, and luminosity solely in terms of mass.

For what stars are the assumptions in parts (b) and (c) appropriate?
(d) By finding also how effective temperature $T_{\text{eff}}$ scales with mass for these stars, deduce a relation between log $L$ and log $T_{\text{eff}}$.

8. A main sequence star with the same radius as the Sun has a maximum energy output in visible light at a wavelength of 500 nm. If this star goes on to become a red giant with a radius 100 times larger and maximum energy output at wavelength of 750 nm, assuming black-body radiation laws, estimate by what factor the star's luminosity increases.

9. The Sun has a radius of $7 \times 10^8$ m and a bolometric luminosity of $3.8 \times 10^{26}$ W.
(a) Calculate the effective surface temperature of the Sun.
(b) At what wavelength is the solar spectrum expected to have the greatest intensity?
(c) The Sun's continuous spectrum actually peaks at 470 nm. What could be concluded from this?
(d) Why can the effective surface temperature of the Moon not be calculated from its B-V colour index?
1. Given that the luminosity of the Sun is about $4 \times 10^{26}$ W and that its absolute bolometric magnitude is $M_{\text{bol}} = 4.72$, estimate the distance at which the Sun could just be seen by the naked eye. [The naked eye can detect a star of apparent magnitude 6.]

2. Barnard’s star has a measured parallax of 0.55 arcsec and a proper motion of 10.3 arc-sec per year. [How are the parallax and proper motion distinguished from one another observationally?] Calculate the distance to Barnard’s star. What component of its velocity relative to the Sun can be deduced from the proper motion? Calculate this component, in km s$^{-1}$. 

3. In an HR diagram the main sequence of the Pleiades cluster of stars consists of stars with mass less than 6 $M_\odot$: the more massive stars have already evolved off the main sequence. Estimate the age of the Pleiades cluster. [The Sun is about half-way through its main-sequence lifetime with age just under $5 \times 10^9$ years.]

4. (a) A binary star system consists of stars of masses $m_1$ and $m_2$ moving in circular orbits of radii $a_1$ and $a_2$ about their common centre of mass. The orbital period is $P$. Derive expressions for the mass ratio, $m_1 / m_2$, and the sum of the masses, $m_1 + m_2$, in terms of $a_1$, $a_2$ and $P$ (or equivalently the angular velocity $\Omega = 2 \pi / P$).

   If the mass ratio is $r$, say, and the sum of the masses is $s$, write down expressions for the individual masses $m_1$ and $m_2$.

   (b) A binary system, which has period of 50 years, consists of two stars of masses 1 $M_\odot$ and 2.5 $M_\odot$ moving in circular orbit about their centre of mass. Find the separation between the two stars. [Express your answer in AUs or parsecs as you think most appropriate.]

5. (a) The Doppler line-of-sight velocities of two components in a spectroscopic binary system are measured to be 75 km s$^{-1}$ and 100 km s$^{-1}$; the measured period is 11 days. Find the mass ratio between the stars.

   (b) Suppose now that the system is also an eclipsing binary. Assuming therefore that the system is seen edge-on (ie with inclination angle $i=90^\circ$), what are the masses of the two components? By how much would your answers change if the inclination angle $i = 80^\circ$ ?

6. (a) As the Sun evolved towards the main sequence, it contracted under gravity whilst remaining close to hydrostatic equilibrium, and its internal temperature in the core changed from about 30 000 K to about $6 \times 10^6$ K. Estimate the total energy radiated during this contraction.

   (b) Assuming that the luminosity during this contraction was comparable to the present luminosity of the Sun, estimate the time taken to reach the main sequence.
7. The colour indices of some bright stars are given below:

<table>
<thead>
<tr>
<th>Star</th>
<th>B - V</th>
<th>Star</th>
<th>B - V</th>
</tr>
</thead>
<tbody>
<tr>
<td>the Sun</td>
<td>0.65</td>
<td>Sirius</td>
<td>0.00</td>
</tr>
<tr>
<td>Canopus</td>
<td>0.15</td>
<td>Alpha Centauri A</td>
<td>0.71</td>
</tr>
<tr>
<td>Arcturus</td>
<td>1.23</td>
<td>Rigel</td>
<td>-0.03</td>
</tr>
<tr>
<td>Altair</td>
<td>0.22</td>
<td>Aldebaran</td>
<td>1.54</td>
</tr>
<tr>
<td>Antares</td>
<td>1.83</td>
<td>Spica</td>
<td>-0.23</td>
</tr>
</tbody>
</table>

Arrange these stars in decreasing order of temperature.

What colour would you expect each star to appear if photographed on colour film?

What are the spectral types?
1. A file named sheet4_data.csv is available on Blackboard. This contains basic data (radius, orbital semi-major axis and mass) for a range of Solar System bodies. The file can be read into your favourite plotting programme - Origin, Excel, Numbers etc. Use this to calculate the densities of these objects, and then see how density varies with object radius, and how density changes with orbital semi-major axis. You will need to plot the log of the radius of the objects so that you can easily see the full dynamic range.

What do these plots tell you about the nature of the objects and about their formation?

2. The first part of this question involves some rather awkward and lengthy algebra with probably not much gain in physical insight. I include it here for completeness. You should read it, and be aware that equation (2) is a solution to equation (1), but you may chose to skip the algebra if you are pressed for time.

The equation for an ellipse in Cartesian coordinates \((x, y)\) is:
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{1}
\]

Here \(a\) is the length of the semi-major axis, \(b\) the length of the semi-minor axis, and the eccentricity \(e\) of the ellipse is defined by \(e^2 = 1 - b^2/a^2\). Show that the solution
\[
\frac{1}{r} = \frac{1}{r_0} \left(1 + e \cos \theta \right) \tag{2}
\]

of the orbital equations (with \(0 < e < 1\)) satisfies Eq. 1 with the following definitions for \(x, y, a\) and \(b\)
\[
x = r \cos \theta + \frac{e r_0}{1 - e^2} \\
y = r \sin \theta \\
a = \frac{r_0}{1 - e^2} \\
b = \frac{r_0}{\sqrt{1 - e^2}}.
\]

Confirm also that \(1 - b^2/a^2\) is equal to \(e^2\), so the quantity \(e\) in Eq. 2 is indeed the eccentricity of the ellipse.

3. The area of an ellipse with semi-major axis \(a\) and semi-minor axis \(b\) is \(\pi ab\). First, by convincing yourself that the area element swept out by a radius vector of length \(r\) moving through a small angle \(d\theta\) is \(\frac{1}{2} r^2 d\theta\), show therefore that if the ellipse is described in polar coordinates (as in Eq. 2 above) then
\[
\pi ab = \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^P \frac{1}{2} r^2 \frac{d\theta}{dt} dt,
\]
where \(P\) is the period of the orbit.

You can now prove Kepler’s third law, \(P^2 = \frac{(2\pi)^2 a^3}{(GM)}\), where \(M\) is the mass of the star. To do this, evaluate the ellipse area above, using the relation \(r_0 = h^2/\left(G M\right)\), the expressions for \(a\) and \(b\) from the previous question and the fact that the specific angular momentum \(h = r^2 d\theta / dt\) of a planet’s orbital motion is constant.

4. Saturn’s mean distance from the Sun is 9.5 AU. By Kepler’s third law, how many years does it take for Saturn to orbit once around the Sun?

5. This question forms the first half of an exam question from 2013
(a) Find a general equation for the temperature $T$ of a planet's surface given the luminosity of the star it is orbiting is $L$ and the radius of the orbit is $d$, ignoring the effect of any planetary atmosphere that might be present, and assuming the planet's orbit is circular.

(b) A planetary system is discovered including a normal main sequence star, whose luminosity is $2 \times 10^{26}$ W, i.e. half that of the Sun, and a planet orbiting with a radius of $85 \times 10^6$ km. Use the equation derived above, and assuming the planet's albedo is 0.25, the calculate the surface temperature of the planet.

Is an albedo of 0.25 a reasonable assumption?

The presence of what material is thought to be a pre-requisite for the possible presence of life on a planet? Given the expected surface temperature you have calculated, what conclusion can you draw about the possible habitability of this planet?

If the planet has an atmosphere, what difference might this make to the habitability of the planet?

6. Mars has a mass of $6.4 \times 10^{23}$ kg, a radius of 3400 km and an albedo of 0.25. Its mean distance from the Sun is 1.5 AU.

(a) Calculate the ‘no-atmosphere’ temperature of Mars. Do you expect the actual mean surface temperature to be higher or lower than the no-atmosphere temperature and why?

(b) Calculate the escape velocity from Mars. Using the temperature from above and assuming that gas will be lost completely from a planet over the age of the solar system if its thermal velocity exceeds 20% of the escape velocity, can Mars retain (a) molecular hydrogen, or (b) water vapour, or (c) carbon dioxide?
Problem Sheet 5

1. **Computational** Write a Python program that explores the development of the Kirkwood Gaps in the asteroid belt. You should set up the Sun and Jupiter as gravitating bodies and then have a roughly uniform distribution of bodies between the orbits of Mars and Jupiter that act under the effects of the gravity of the Sun and Jupiter. The effects of gravity from Mars and from the asteroid bodies themselves can be neglected.

   A hints sheet on how to approach this project is available on Blackboard, called kirkwood-hints.pdf. Example code that solves this problem will be posted with the solutions to the problem sheet.

2. Jupiter emits roughly twice as much energy into space as it receives from the Sun. The difference is thought to be produced by continuing slow gravitational contraction of the planet. Assuming that the gravitational potential energy of the planet $\simeq GM^2/R$ (where the missing constant of proportionality is of order unity) estimate by how much the radius of the planet must change each year.

3. The following is an old exam question (2004, B2).
   (a) Explain what is meant by:
      i. the solar constant,
      ii. the greenhouse effect, and
      iii. the habitable zone of a planetary system.
   (b) The time-averaged value of the solar constant is 1368 W m$^{-2}$. Name two kinds of surface feature on the Sun that cause measurable changes in the solar constant on timescales of the solar rotation and the 11-year solar cycle, and state whether they cause the solar constant to increase or decrease.
   (c) The incoming solar radiation flux incident on the Earth, averaged over time, is 342 W m$^{-2}$. Show how this can be derived from the above value of the solar constant. It is estimated that on average 77 W m$^{-2}$ is reflected back by clouds and other atmospheric constituents, and 30 W m$^{-2}$ is reflected back by the surface. What is the effective albedo of the Earth?
   (d) Because of the greenhouse effect, the Earth’s surface must radiate upwards an average of 390 W m$^{-2}$. Assuming a black-body emission, what average surface temperature is required to produce this amount of surface radiation?
      Additional question: At what wavelengths is most of this radiation emitted and in what part of the spectrum is that?
   (e) Assuming that the surface radiation scales proportionally with the incoming radiation, what (in Astronomical Units) are the inner and outer limits of the Sun’s habitable zone? How would your answers change for a planetary system around a star of mass 1.5 $M_\odot$ and luminosity 5.0$L_\odot$?

4. and a short one from 2009, A(iv)
   It was suggested by Roche that there is a limit to the distance $d$ that a body with mass $m$ and radius $r$ held together by self-gravity can approach a more massive body with mass $M$ before it will be disrupted by tidal forces. Assuming that $r \ll d$, show that this limit is given by
   $$d_R = \left(\frac{2M}{m}\right)^{1/3} r.$$

   Comment on how this can help explain the presence of rings around gas giants.

5. These next two questions are from a past paper (2004, Section A).
(a) Calculate the escape velocity from the surface of Mars. By what factor is it smaller than the escape velocity from Earth? Comment on the relevance to finding Martian meteorites on Earth.

Earth's mass and radius: $6 \times 10^{24}$ kg and $6 \times 10^6$ m, respectively. Mars' mass and radius: $6 \times 10^{23}$ kg and $3 \times 10^6$ m, respectively.

(b) The Kirkwood gaps in the distribution of asteroids correspond to simple resonances such as $3 : 1$ with the period of Jupiter. Find all such resonances of the form $n_1 : n_2$ for which both $n_1$ and $n_2$ are positive integers smaller than 8 between the $3 : 1$ and $2 : 1$ resonances. At what distance (semi-major axis length) from the Sun is the $3 : 1$-resonance Kirkwood gap situated?

Jupiter is 5.2 AU distant from the Sun.
Problem Sheet 6

1. A planet with mass $M \sin i = 3.9 \, M_J$ (where $M_J$ is the mass of Jupiter) has been found to orbit the star $\tau$ Boo (in the constellation Bootes). The planet has been detected from studying the variation in Doppler velocity of the star; the star's mass may be taken to be 1.2 $M_\odot$.

(a) The system's orbital period is 3.313 days. What is the semi-major axis length of the planet's orbit in AU?

(b) The planet is an example of what are called 'hot Jupiters'. Calculate the no-atmosphere temperature for this planet. If the planet were a gas giant with composition similar to gas giants in our solar system, could it retain its atmosphere?

(c) Assuming circular orbits, how big is the Doppler velocity signal (in m s$^{-1}$) of $\tau$ Boo due to the presence of this planet?

(d) If the inclination of the planetary orbit is 90°, what fraction of the star's light do you expect to be blocked during transit? Would this be detectable (a) from the ground and (b) from space? Why are detections from space more sensitive?

2. It has been suggested that the tendency for planet-bearing stars to have higher metallicity is caused by infalling planets. If all four terrestrial planets were to fall into the Sun, calculate by how much the metal abundance $Z$ (i.e., the fractional abundance by mass of all elements heavier than helium) would be increased if the material were

(a) uniformly mixed throughout the Sun,

(b) mixed only throughout the convective envelope (the outer 30% by radius, which contains only 2% of the Sun’s mass).

3. Jeremiah Horrocks, curate at Hoole, Lancashire, made an observation of a transit of Venus, i.e., Venus passing in front of the Sun, on Sunday 24 November 1639. Apparently, Horrocks observed the transit between church services; how long did the transit last?

[Note: you are expected to be able to calculate the duration of a transit, but the detailed calculation of the more exact transit duration time of Venus across the Sun as observed from Earth and as given in the answer sheet, is included only for interest.]

4. The Hipparcos satellite is capable of astrometric measurements accurate to 500 $\mu$ arcsec. If a star with measured parallax $p = 0.02''$ and a planetary system identical to our solar system were viewed using Hipparcos, would one be capable of detecting the presence of any planets?

The specifications of the GAIA space observatory are such that measurements should be accurate to about 2 $\mu$ arcsec. Would this be able to detect a planet the equivalent of Jupiter or Earth orbiting a solar-mass star?
1. Answer:

(i) Flux, \( F \), is given by \( L/A \), where \( L \) is the luminosity and \( A \) is the surface area through which the radiation passes. Considering the solar flux at Earth, \( S \), the surface area is given by \( A = 4\pi d^2 \) where \( d = 1 \text{ AU} \) is the distance between the Sun and the Earth. Thus

\[
S = \frac{3.8 \times 10^{26}}{4\pi (1.5 \times 10^{11})^2} = 1340 \text{Wm}^{-2}
\]

(ii) For Pluto and Mars their respective distances from the Sun need to be inserted for \( d \).
For Pluto, \( d = 39 \text{ AU} \)

\[
F_\oplus = \frac{L_\odot}{4\pi (39\text{AU})^2} = \frac{1}{39^2} \frac{S}{1521} = 0.88 \text{ Wm}^{-2}
\]

For Mars, \( d = 1.5 \text{ AU} \)

\[
F_\oplus = \frac{L_\odot}{4\pi (1.5\text{AU})^2} = \frac{1}{1.5^2} \frac{S}{2.25} = 595.6 \text{ Wm}^{-2}
\]

For Mercury, \( d = 0.4 \text{ AU} \)

\[
F_\oplus = \frac{L_\odot}{4\pi (0.4\text{AU})^2} = \frac{1}{0.4^2} \frac{S}{0.16} = 8375 \text{ Wm}^{-2}
\]

(iii) At the Sun’s surface, \( A = 4\pi R^2_\odot \), and so the flux at the solar surface is:

\[
F_\odot = \frac{L_\odot}{4\pi R^2_\odot} = \frac{3.8 \times 10^{26}}{4\pi (7.0 \times 10^8)^2} = 6.2 \times 10^7 \text{ Wm}^{-2}
\]

(iv) To calculate flux at Earth from Sirius, we need to convert light years to SI units and use

\[
F_{\text{Sirius}} = \frac{L_{\text{Sirius}}}{4\pi (d)^2}
\]

\[
d = 8.48 \text{ lyrs} = 8.48 \times (3 \times 10^8) \times (365 \times 24 \times 60 \times 60) = 8.48 \times 9.46 \times 10^{15} \text{ m}
\]

\[
F_{\text{Sirius}} = \frac{L_{\text{Sirius}}}{4\pi d^2} = \frac{23 \times 3.8 \times 10^{26}}{4\pi (8.48 \times 9.46 \times 10^{15})^2} = 1.1 \times 10^{-7} \text{ Wm}^{-2}
\]
2. Answer:

Use approximation that $t_{ff} = R/v_{esc}$ for the "free-fall" or dynamical timescale. Equate gravitational and kinetic energy of a test particle of mass $m$ to find an expression for escape velocity $v_{esc}$:

$$\left[ -\frac{GMm}{r} \right]_{\infty}^{R} = \frac{1}{2} m v_{esc}^2$$

where $M$ is the stellar mass and $R$ is the stellar radius. Therefore

$$v_{esc} = \sqrt{\frac{2GM}{R}}$$

$$t_{ff} = \sqrt{\frac{R^3}{2GM}} = \sqrt{\frac{3}{8\pi G\bar{\rho}}}$$

where the expression for mean solar density $\bar{\rho} = 3M/(4\pi R^3)$ has been used.

We find the mean solar density to be:

$$\bar{\rho} = \frac{3 \times 2 \times 10^{30}}{4\pi (7 \times 10^8)^3} = \frac{6 \times 10^6}{4\pi \times 7^3} = 1.4 \times 10^3 \text{ kg m}^{-3}$$

and so the free-fall timescale for the Sun is:

$$t_{ff} = \sqrt{\frac{3}{8\pi G\bar{\rho}}} = \frac{3}{8\pi \times 6.7 \times 10^{-11} \times 1.4 \times 10^3} \approx 1100 \text{ s}$$

or approximately 20 minutes.

The expression from lecture 2 only differs slightly.
3. Mass of:

- hydrogen = $\rho X$ number density of ions $n_H = \frac{\rho X}{m_H}$
- helium = $\rho Y$ number density of ions $n_{He} = \frac{\rho Y}{4m_H}$
- metals = $\rho Z$ number density of ions $n_{metals} = \frac{\rho Z}{Am_H}$

since fully ionised, the number density of electrons is:

$$n_e = 1 \cdot n_H + 2n_{He} + \frac{A}{2} n_{metals}$$

$$= \frac{\rho X}{m_H} + 2 \frac{\rho Y}{4m_H} + \frac{A}{2} \frac{\rho Z}{Am_H}$$

$$= \frac{\rho}{m_H} (X + \frac{1}{2} Y + \frac{1}{2} Z)$$

and

$$\mu = \frac{\rho}{m_H n} = \frac{\rho}{m_H (n_H + n_{He} + n_{metals} + n_e)}$$

$$= \frac{1}{(X + \frac{Y}{4} + \frac{Z}{A} + X + \frac{Y}{2} + \frac{Z}{2})}$$

$$= \frac{1}{2X + \frac{3}{4} Y + \frac{1}{2} + \frac{1}{A} Z}$$

using $X + Y + Z = 1$, and that $A/2 \gg 1$, and $Y = 1 - X - Z$

$$\mu \approx \frac{1}{2X + \frac{3}{4} Y + \frac{Z}{2}}$$

$$\mu = \frac{4}{5X + 3 - Z}$$

Estimate $\mu$ in solar centre, $X = 0.4$, so $Y = 0.6$. Substituting in gives:

$$\mu = \frac{4}{5 \times 0.4 + 3} = 0.8$$
4. Gas pressure:

\[ P_g = \frac{\gamma \rho T}{\mu} \]

and radiation pressure:

\[ P_r = \frac{1}{3} a T^4 \]

\[ \frac{P_g}{P_r} = \frac{3\gamma}{\mu a} \rho T^{-3} \approx 3.7 \times 10^{19} \rho T^{-3} \text{ m}^3 \text{ K}^3 \text{ kg}^{-1} \]

where we have assumed mean molecular weight, \( \mu \approx 1 \).

<table>
<thead>
<tr>
<th></th>
<th>T</th>
<th>( \rho )</th>
<th>( P_{\text{rad}} )</th>
<th>( P_{\text{gas}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>centre</td>
<td>1.5 \times 10^7</td>
<td>1.5 \times 10^5</td>
<td>1.3 \times 10^{13}</td>
<td>1.9 \times 10^{16}</td>
</tr>
<tr>
<td>surface</td>
<td>10^4</td>
<td>10^{-3}</td>
<td>2.5</td>
<td>8 \times 10^4</td>
</tr>
<tr>
<td></td>
<td>K</td>
<td>kg m^{-3}</td>
<td>N m^{-2}</td>
<td>N m^{-2}</td>
</tr>
</tbody>
</table>

Therefore at the solar centre:

\[ \frac{P_r}{P_g} = 10^{-3} \]

and at the solar surface:

\[ \frac{P_r}{P_g} = 10^{-5} \]

The value of the gas pressure obtained is about 500 times larger than the minimum pressure we calculated during the lectures and this is mainly due to \( 1/R^4 \) being significantly smaller than \( 1/r^4 \).

Note: there now follow handwritten answers to the remaining questions. The numbering of the answers is incorrect, but they follow in the correct order.
2 a) \[ M_{\text{He}} = 3.97m_H \quad \text{and} \quad m_H = 1.67 \times 10^{-27} \text{kg} \]

\[ \therefore \text{energy produced} = (4m_H - 3.97m_H) \cdot c^2 \]
\[ = 0.03 \cdot m_H \cdot c^2 \]
\[ = 0.03 \times (1.67 \times 10^{-27}) \times (3.00 \times 10^8)^2 \text{ J} \]
\[ = 4.51 \times 10^{-12} \text{ J} \]

\[ 1 \text{ eV} = 1.6 \times 10^{-19} \text{ J} \quad \therefore \quad 1 \text{ MeV} = 1.6 \times 10^{-13} \text{ J} \]

so energy produced = \[ \frac{4.51 \times 10^{-12}}{1.6 \times 10^{-13}} \text{ J} = 28.2 \text{ MeV} \]

b) \text{Branch I}
\[ p + p \rightarrow ^2H + e^+ + \nu_e \] \[ \text{twice} \]
\[ p + ^2H \rightarrow ^3\text{He} + \gamma \]
\[ ^3\text{He} + ^3\text{He} \rightarrow ^4\text{He} + 2p \]

\[ \therefore \text{in total} \]
\[ \begin{array}{c}
\text{on LHS} \\
6p + 2 ^2H + 2 ^3\text{He} \\
\text{Net} \\
4p \rightarrow 4\text{He} + 2e^+ + 2\nu_e ( + 2\gamma) \\
\end{array} \]
Branch II

\[ \begin{align*}
\text{LHS} & \quad \text{RHS} \\
4p + 2H & \quad 2H + e^+ + \nu_e \\
3\text{He} + 4\text{He} & \quad + 3\text{He} + 4\text{He} \\
7\text{Be} + e^- & \quad + 7\text{Be} + e^- \\
7\text{Li} & \quad + 7\text{Li} \\
2\text{He} & \quad + 2\text{He}
\end{align*} \]

\[ \text{Net} \quad 4p + e^- \rightarrow 4\text{He} + e^+ + 2\nu_e (\pm 2\gamma) \]

( you can think of $e^-$ on LHS as equivalent to another $e^+$ on RHS )

Branch III

\[ \begin{align*}
\text{LHS} & \quad \text{RHS} \\
4p + 2H + 3\text{He} & \quad 2H + e^+ + \nu_e \\
4p + 7\text{Be} + 8\text{Be} & \quad + 3\text{He} + 3\gamma + 7\text{Be} \\
8\text{B} + \nu_e & \quad + 8\text{B} + 8\text{Be} \\
8\text{Be} & \quad + 8\text{Be} + e^+ + \nu_e \\
2\text{He} & \quad + 2\text{He}
\end{align*} \]

\[ \text{Net} \quad 4p \rightarrow 4\text{He} + 2e^+ + 2\nu_e (\pm 3\gamma) \]

3) The difference in production of energy vs heat is the different neutrino energies. The neutrinos in step 1 are fairly low energy, but the neutrino in Branch III ($^8\text{B} \rightarrow ^8\text{Be}$) is a high energy neutrino which carries away 30% of the energy produced.

[The effective energy released by the branches — including energy from the annihilation of positrons, but not energy carried away by neutrinos—is: Branch I: 26.2 MeV, Branch II: 25.7 MeV, Branch III: 19.1 MeV.]
4a) \[ \langle \frac{1}{2} m v^2 \rangle = \frac{3}{2} kT = \frac{e^2}{4\pi\varepsilon_0 r} \] at closest approach distance \( r \),

for \( r = r_{\text{nucl}} \ (10^{-15} \text{ m}) \)

need \[ T = \frac{2/3 e^2}{4\pi\varepsilon_0 k r_{\text{nucl}}} \]

\[ = \frac{2/3 (1.6 \times 10^{-19})^2}{(4\pi \times 8.85 \times 10^{-12}) \times 1.38 \times 10^{-23} \times 10^{-15}} \]

\[ = 1 \times 10^{10} \text{ K} \]

b) \[ \frac{p^2}{2m} = \frac{1}{2} m v^2 = \frac{3}{2} kT \]

\[ \therefore \quad p = (3m kT)^{1/2} \]

so de Broglie wavelength

\[ \lambda = \frac{\hbar}{p} = (3m kT)^{-1/2} \hbar \]

for \( r = \lambda \)

need \[ T = \frac{2/3 e^2}{4\pi\varepsilon_0 k (\hbar/p)} = \frac{2/3 e^2}{4\pi\varepsilon_0 k \hbar} \]

\[ \therefore \quad T^{1/2} = \frac{2/3 e^2 (3m_H/k)^{1/2}}{4\pi\varepsilon_0 \hbar} \]

\[ = \frac{2/3 (1.6 \times 10^{-19})^2 (3 \times 1.67 \times 10^{-24} / 1.38 \times 10^{-23})^{1/2}}{(4\pi \times 8.85 \times 10^{-12}) \times 6.62 \times 10^{-34}} \]

\[ = 4.4 \times 10^3 \]

\[ \therefore \quad T = 10^7 \text{ K} \]

(Much closer to actual central temperature of sun; reactions only occur at the "sun's low" temperature of 15 million K because of quantum tunnelling through the Coulomb barrier.)
5. Blob ascends \( \delta r \) in pressure equilibrium and adiabatically.

\[
P_1 = P_0 + \delta r \frac{dp}{dr}
\]

Adiabatic means that blob's new pressure \( P_1 \) and density \( \rho_1 \) are related to old pressure and density by

\[
\frac{P_1}{P_0} = \left( \frac{\rho_1}{\rho_0} \right)^\gamma
\]

\[\gamma = \Gamma, \text{ adiabatic exponent}\]

\[
\therefore \rho_1 = \rho_0 \left( \frac{P_1}{P_0} \right)^{\frac{1}{\gamma}} = \rho_0 \left( 1 + \delta r \frac{dp}{P} \right)^{\frac{1}{\gamma}}
\]

\[
\therefore \rho_0 \left( 1 + \delta r \frac{dp}{P} \right) > \rho_0 + \delta r \frac{dp}{dr}
\]

Blob sinks back (i.e., stratification stable) if \( \rho_1' > \rho_1 \),

\[
\rho_0 \left( 1 + \frac{\delta r \frac{dp}{P}}{\delta P \frac{dp}{dr}} \right) > \rho_0 + \delta r \frac{dp}{dr}
\]

\[
\frac{\delta r}{\delta P} \frac{dp}{dr} > \frac{\delta r \frac{dp}{P}}{\delta P \frac{dp}{dr}}
\]

\[
\therefore \frac{1}{\delta P} \frac{dp}{dr} > \frac{1}{P} \frac{dp}{dr}
\]

\[
\frac{1}{\delta \ln P} > \frac{d \ln P}{dr}
\]

Note: don't worry about "\( \delta \)" subscript in \( \delta r \) terms, as these are already first order in small quantity \( \delta r \).
in convection zone $Y = 5/3$ and

$$\frac{d \ln \tau}{dr} = \frac{5}{3} \frac{d \ln \rho}{dr}$$

integrating \quad \ln \tau = \frac{5}{3} \ln \rho + \text{const}

\therefore \quad \tau = K \rho^{5/3}, \quad \text{a constant}
6. \[ P = k \rho^{5/3} \quad (1) \]

\[ \text{and} \quad \frac{dP}{dr} = -\rho \frac{dP}{dr} = -\frac{GM_0 \rho}{r^2} \quad (2) \]

\[ \therefore \text{using (1) in LHS of (2)} \quad \Rightarrow \quad \frac{5}{3} k \rho^{2/3} \frac{dP}{dr} = -\frac{GM_0 \rho}{r^2} \]

\[ \Rightarrow \quad \frac{5}{3} k \rho^{2/3} \frac{dP}{dr} = -\frac{GM_0}{r^2} \]

Integrating:

\[ \frac{5}{3} k \left[ \frac{3}{2} \rho^{2/3} \right]_{r = R_0}^{r} = \left[ \frac{GM_0}{r} \right]_R \]

\[ \Rightarrow \quad \frac{5}{2} k \left( \rho^{2/3} - \rho_s^{2/3} \right) = GM_0 \left( \frac{1}{r} - \frac{1}{R_0} \right) \]

\[ \Rightarrow \quad \rho^{2/3} - \rho_s^{2/3} = \frac{2GM_0}{5k} \left( \frac{1}{r} - \frac{1}{R_0} \right) \]

where \( r = R_0 - \frac{z}{R_0} \) (\( z \ll R_0 \))

\[ r = R_0 \left( 1 - \frac{z}{R_0} \right) \]

\[ \Rightarrow \quad \frac{1}{r} = \frac{1}{R_0} \left( 1 - \frac{z}{R_0} \right)^{-1} \approx \frac{1}{R_0} \left( 1 + \frac{z}{R_0} \right) \]

provided \( \frac{z}{R_0} \ll 1 \)

Also \( \rho^{2/3} \gg \rho_s^{2/3} \) almost everywhere

\[ \Rightarrow \quad \rho^{2/3} \approx \frac{2GM_0}{5k} \cdot \frac{z}{R_0} \]

\[ \rho = \left( \frac{2GM_0}{5kR_0^2} \right)^{3/2} \cdot \frac{z^{3/2}}{R_0} \]

\[ P = k \rho^{5/3} \quad \Rightarrow \quad P = k \left( \frac{2GM_0}{5kR_0^2} \right)^{5/2} \cdot \frac{z^{5/2}}{R_0} \]
finally \[ T \propto \frac{P}{\rho} \propto \frac{z^{5/2}}{z^{3/2}} \propto z \]

so \( T \) is proportional to depth, in the convection zone \( \left( \frac{z}{R_0} \lesssim 0.3 \right) \)
1a)  
\[ \text{In direction at angle } \theta \text{ to the surface normal, the projected area of element } \delta A \text{ is } \delta A \cos \theta \]

So radiated energy in that direction per unit time per unit solid angle and per unit frequency from \( \delta A \) in \( B_{\nu}(T) \delta A \cos \theta \)

Element of which angle is \( d\theta \sin \theta d\phi \) (NB solid angle = area on unit sphere)

\[ \text{total radiated energy into the half space } \theta < \frac{\pi}{2} \]

(integrated over all frequencies)

\[ \int_{\nu = 0}^{\infty} \left( \int_{\theta = 0}^{\pi} \int_{\phi = 0}^{2\pi} d\theta \sin \theta d\phi \right) B_{\nu}(T) \delta A \cos \theta \, d\nu \]

b) setting \( \delta A = 1 \), get

\[ F = \int_{\nu = 0}^{\infty} \int_{\theta = 0}^{\pi} \int_{\phi = 0}^{2\pi} d\nu \sin \theta \cos \theta B_{\nu}(T) \]

\[ = \int_{0}^{\infty} d\nu \cdot \int_{0}^{\pi} \sin \theta \cos \theta \cdot \frac{2\pi T_{0}^{4}}{c^{2}} \cdot \frac{1}{e^{\frac{h\nu}{kT}} - 1} \]

\[ = \frac{2\pi T_{0}^{4}}{c^{2}} \cdot \int_{0}^{\infty} \frac{\nu^{3} d\nu}{e^{\frac{h\nu}{kT}} - 1} \]

let \( x = \frac{h\nu}{kT} \)

\[ = \frac{2\pi T_{0}^{4}}{c^{2}} \cdot \left( \frac{kT}{h} \right)^{4} \cdot \int_{0}^{\infty} \frac{x^{3} dx}{e^{x} - 1} \]

so

\[ F = \sigma T^{4} \]

and so

\[ \sigma = \frac{2\pi k^{4}}{h^{3} c^{2}} \cdot \int_{0}^{\infty} \frac{x^{3} dx}{e^{x} - 1} \]

where \( \sigma \) is a constant just a number
For transitions down to higher values of $n$ we would expect the wavelength to increase. In the case of the Paschen series ($n = 3$), the final term converges to 9, thus all Paschen transitions are at wavelengths between 820 nm (Paschen limit) and 1876 nm (Pa$\alpha$, the $4 \rightarrow 3$ transition) in the (near)-IR.
In a strong magnetic field, ionized plasma is constrained so it cannot move across the field lines. Sunspots have strong (up to $0.3 \, T \approx 3000 \, \text{Gauss}$) magnetic fields, so the solar granular convection is inhibited or even suppressed in the spot. This means that less heat energy escapes from here, and the temperature is lower: the spot appears dark (compared to the surroundings). Spot temperature $\approx 4000 \, \text{K}$; rest of surface $\approx 5800 \, \text{K}$.

b) Magnetic field will have a dominant role in the plasma dynamics if $\frac{B^2}{2 \mu_0} \gg \frac{1}{2} \rho v^2$

Define $B_{\text{crit}}$ when these energy densities are equal.

$\therefore B_{\text{crit}} = (\mu_0 \rho)^{\frac{1}{2}} v$

$= (4 \times 10^{-7} \times 10^{-3})^{\frac{1}{2}} \times (2 \times 10^3) \, T$

$= 0.04 \, T \quad (= 400 \, \text{Gauss})$

Sunspot has much stronger field than this (in umbra region) so expect it to dominate the plasma motion and suppress the overturning convection – as discussed in (a).
4a. This question originally had two parts, a and b, but I covered part 4a in an earlier lecture on the blackboard, but I will leave the answer here as it is of use. It relates to writing the Planck distribution in wavelength and frequency domain, converting from one to the other.

\[ B_\nu (\nu) = \frac{2\pi}{c^2} \frac{\nu^3}{e^{\frac{h\nu}{kT}} - 1} \]

\[ \int_0^\infty B_\nu d\nu = \int_0^\infty B_\lambda d\lambda \]

so \[ B_\lambda = B_\nu \left| \frac{d\nu}{d\lambda} \right| \]

\[ \nu = \frac{c}{\lambda} \quad \Rightarrow \quad \frac{d\nu}{d\lambda} = -\frac{c^2}{\lambda^2} \]

\[ \therefore B_\lambda (\lambda) = 2\pi \left( \frac{c}{\lambda} \right)^3 \frac{1}{c^2} \frac{1}{e^{\frac{hc}{\lambda kT}} - 1} \cdot \left| -\frac{c}{\lambda^2} \right| \]

\[ = \frac{2\pi c^2}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda kT}} - 1} \]

6) \[ m_B = -2.5 \log \frac{L_{\lambda_B}}{4\pi d^2} + c_2 \]

\[ = -2.5 \log \left( \frac{L_{\lambda_B}}{4\pi d^2} \right) + c_2 \] \[ \text{where } L_{\lambda_B} \text{ is luminosity at wavelength } \lambda_B \]

and likewise for \( m_B \) and \( m_V \)

\[ m_B - m_V = -2.5 \left( \log L_{\lambda_B} - \log 4\pi d^2 \right) + c_2 \]

\[ \text{(i.e. } B-V) \]

\[ \{ -2.5 (\log L_{\lambda_B} - \log 4\pi d^2) + c_3 \} \]

\[ = -2.5 \log L_{\lambda_B} + 2.5 \log L_{\lambda_V} + c_2 - c_3 \]

\[ \log 4\pi d^2 \text{ terms cancel, so } B-V \text{ is independent of distance } d, \]

Likewise \( m_K - m_B \) \((\equiv U-B)\) is independent of distance.

This is assuming flux falls off exactly as \( 1/d^2 \) i.e., no scattering or absorption of radiation as it travels from star to observer.
5. The variables in the question are luminosity, flux and distance. The expression relating these is

\[ F = \frac{L}{4\pi d^2} \]

Quantities relating to Earth are denoted by \( L, f, d \) and those for the other planet by \( L', f', d' \), then \( L' = 10L, F' = 2F \), so

\[ \frac{F'}{F} = \frac{L'/L}{d'^2/d^2} \]

Substituting for \( L'/L \) and \( F'/F \) gives \( d'^2/d^2 = 5 \), so \( d'/d = 2.24 \). So orbital radius is 2.24 AU.

6. (a) Use equation that relates magnitude and distance: \( m - M = 5 \log(d/10 \text{ pc}) \)

For Earth and Jupiter:

\[ m_E - M = 5 \log \left( \frac{d_E}{10 \text{ pc}} \right) \quad m_J - M = 5 \log \left( \frac{d_J}{10 \text{ pc}} \right) \]

Subtracting these two equations from each other eliminates \( M \):

\[ m_E - m_J = 5 \left[ \log \left( \frac{d_E}{10 \text{ pc}} \right) - \log \left( \frac{d_J}{10 \text{ pc}} \right) \right] \]

as \( \log a - \log b = \log(a/b) \), this simplifies to

\[ m_E - m_J = 5 \log \left( \frac{d_E}{d_J} \right) \]

Putting in \( d_J = 5.2 \text{ AU} = 5.2 d_E \) and \( m_E = -26.7 \) gives \( m_J = -26.7 - 5 \log(1/5.2) = -23.1 \)

6b) Finding the Sun’s absolute magnitude.

(we know that the Earth-Sun distance is 1 AU.) use \( m - M = 5 \log(d/10 \text{ pc}) \)

(convert 1 AU to parsecs (or 10 pc to AU) to do the division. Remember that 1 pc = 206265 AU; alternatively, the length of 1 AU and 1 pc in metres is given in any astronomy textbook (and on the exam constants sheet). So, using apparent magnitude –26.7:

\[ M = m - 5 \log(d/10 \text{ pc}) = -26.7 - 5 \log(1/2062650) = 4.9 \]
4. a) let \( x = \frac{r}{R} \); assume \( P(x) = P_c \cdot \bar{\rho}(x) \) 
where \( \bar{\rho} \) is dimensionless etc.

\[
\frac{dm}{dr} = 4\pi r^2 \rho \implies \frac{M}{R} \frac{d\bar{m}}{dx} = 4\pi x^2 \bar{\rho}(x) \cdot R^2 \rho_c
\]

\( \implies \rho_c \propto \frac{M}{R^3} \) amongst the homologous group of stars.

\[
\frac{dp}{dr} = -\frac{Gm\rho}{r^2} \implies \frac{P_c}{R} \frac{d\bar{p}}{dx} = -\frac{G \bar{m}(x) \bar{\rho}(x)}{x^2} \cdot \frac{m \rho_c}{R^2}
\]

\( \implies P_c \propto \frac{M \rho_c}{R} \propto \frac{M^2}{R^4} \)

\[
P = \frac{R \rho_c T_c}{\mu}
\]

\( \implies T_c \propto \frac{P_c}{\rho_c} \propto \frac{M^2}{R^4} \propto \frac{M}{R} \)

b) Radiative, kramers opacity \( k \propto \rho T^{-7/2} \)

Then

\[
\frac{dT}{dr} = -\frac{3k\rho L}{16\pi acr^2 T^3}
\]

\( \implies \frac{T_c}{R} \propto \frac{(P_c T_c^{-7/4}) \rho_c L}{R^2 T_c^3} \)

\( \implies L \propto T_c^{15/2} R \propto \frac{M^{11/2} R^{-1/2}}{\rho_c^2} \) (*)

\[
\left( T_c \propto \frac{M^2}{R^3} \right)
\]

c) \( \gamma \) reaction chain \( \implies \dot{E} \propto \rho T^4 \)

Then

\[
\frac{d\dot{L}}{dm} = \dot{E}
\]

\( \text{or} \quad \frac{d\dot{L}}{dr} = 4\pi r^2 \rho, \dot{E} \)

\( \implies \frac{\dot{L}}{m} \propto \rho_c T_c^4 \)

\( \implies \dot{L} \propto \rho_c T_c^4 M \propto M^6 R^{-7} \)

\( \text{setting (*) & (2) gives } R \propto M^{1/3} \)
\[ P_c \propto \frac{M}{R^3} \propto M^{1/3} \]
\[ P_c \propto \frac{M^2}{R^4} \propto M^{22/13} \]
\[ T_c \propto \frac{M}{R} \propto M^{12/13} \]
\[ L \propto M^6 R^{-3} \propto M^{41/13} \]

Assumptions of Kramer's opacity and pp reaction chain are approximate for a star of mass \( M \leq 1 M_\odot \).

Also assuming wholly radiative, which is not actually true for any of the M.S. stars in that mass range, but answers will be reasonably good, provided star has substantial radiative interior. But ZAMS stars with \( M \leq 0.5 M_\odot \) will be wholly convective. So let's say \( \alpha = 1 \) for stars with \( 0.5 M_\odot \leq M \leq 1 M_\odot \).

\[ \text{d)} \quad T_{\text{eff}} \propto \left( \frac{L}{R^2} \right)^{1/4} \propto (M^{41/13}/M^{22/13})^{1/4} \propto (M^{69/13})^{1/4} \]

\[
\frac{d \log L}{d \log T_{\text{eff}}} = \frac{d \log L}{d \log M} \times \frac{d \log T_{\text{eff}}}{d \log M} = \frac{71/13}{69/13} = 4 \times \frac{71}{69} = 4.1
\]

So slope of main sequence will be \( \alpha = 4.1 \)

\( \uparrow \) \( T_{\text{eff}} \) plotted to the left!
8. use \( L = 4\pi R^2 T^4 \) and \( \lambda_{\text{max}} T = \text{constant} \) [you are expected to know and be able to write down these equations]

So \( \lambda_{\text{max}} T = \lambda'_{\text{max}} T' \)

so \( T/T' = \lambda'_{\text{max}}/\lambda_{\text{max}} \) so \( T/T' = 750/500 \)

\[ \frac{L}{L'} = \frac{R^2}{R'^2} \times \frac{T^4}{T'^4} \]

\[ \frac{L}{L'} = \frac{1}{(100)^2} \times (750/500)^4 \]

\( L' = 1975 \text{ L} \)

9.(a) use \( L = 4\pi R^2 T^4 \).

\[
T = \left[ \frac{3.8 \times 10^{26}}{(4\pi \times 5.67 \times 10^{-8} \times (7 \times 10^8)^2)} \right]^{\frac{1}{4}}
\]

\[ = 5740 \text{ K to 3 significant figures.} \]

(b) use, from (a), effective surface temperature together with Wien's law,

\[ \lambda_{\text{max}} T = 2.898 \times 10^{-3} \text{ m K.} \]

(The numerical value of the constant would be given in any exam question of this type.)

Substituting for \( T \) gives \( \lambda_{\text{max}} = 2.898 \times 10^{-3}/5740 = 5.0 \times 10^{-7} \text{ m (500 nm).} \)

Note that this calculation assumes that the Sun is a blackbody.

(c) Since 500 is not equal to 470, we conclude that the Sun is not a blackbody. However, 470 and 500 are not hugely different, and the difference is in the right direction (i.e. it suggests that the Sun has a slightly higher surface temperature than expected from Stefan's law). Therefore the Sun is probably a reasonable approximation to a blackbody.

(d) Visible light from the Moon is not emitted by the Moon: it is sunlight reflected from the Moon. But blackbody radiation is radiation emitted by the body in question. Therefore visible light from the Moon is not the Moon's blackbody radiation (which is in fact in the infra-red), and so it cannot tell us the Moon's temperature. If anything, it will tell us the Sun's temperature, since it is reflected sunlight.
1. Ignore small differences between Sun's visual & bolometric magnitudes.
   \[ m = -2.5 \log\left(\frac{L}{d^2}\right) + \text{const} \]

   \[ m \text{ is magnitude at } d = 10\text{pc} \]

   \[ \therefore m - M = 2.5 \log_{10} \left(\frac{1}{10\text{pc}}\right)^2 = 5 \log_{10} \left(\frac{1}{10\text{pc}}\right) \]

   Want distance \( d \) at which Sun would have \( m = 6 \).

   Total \( M \) \( = M_{bol} \) = 4.72

   \[ \therefore d = 10\text{pc} \times 10^{(m-M)/5} \]

   \[ = 10\text{pc} \times 10^{(6-4.72)/5} \]

   \[ = 18\text{pc} \]

2. Parallax 1 arcsec corresponds to star being distance 1pc away.

   For Barnard's star, parallax = 0.55 arcsec

   \[ \therefore \text{ its distance is } \frac{1}{0.55} \text{ pc} = 1.82(2) \text{ pc} \]

   [Parallax motion is periodic with period 1yr: this distinguishes it from proper motion.]

   Proper motion = component of velocity in plane of sky, i.e., perpendicular to line of sight.

   Proper motion of Barnard's star is 10.3 arcsec/yr.

   Since 1AU subtends an angle of 1 arcsec at 1pc, and Barnard's star is 1.82 pc away, deduce that star's velocity 1° to line-of-sight is

   \[ 10.3 \times 1.82 \text{ AU/yr} \]

   \[ = 10.3 \times 1.82 \times 1.5 \times 10^8 \text{ km} \]

   \[ = \frac{3.16 \times 10^7 \text{ km}}{3.16 \times 10^7 \text{ sec}} \]

   \[ = 89 \text{ km/s} \]
Use approximate scalings \( L \propto M^4 \), \( t_{ms} \propto \frac{M}{L} \propto M^{-3} \).

For \( M = 1 M_\odot \), main-sequence lifetime \( t_{ms} \approx 10^{10} \text{yr} \).

By above scaling, then, for \( M = 6 M_\odot \), \( t_{ms} \approx \frac{1}{6^3} \times 10^{10} \text{yr} \)

\( \approx 50 \text{ million yr} \).

(This is correct to within a factor 2 or so, according to more detailed comparisons between observations and stellar evolution calculations.)

1. (a) Star mass \( m_1 \), orbits c.o.m. \( m \) with accel. \( a \), \( \frac{d^2x}{dt^2} = a \).

\( m_1 \) and \( m_2 \) attract each other with force \( F = \frac{G m_1 m_2}{(a_1 + a_2)^2} \).

\( F \) force = \( m_1 \) accel. for star 1: \( \frac{G m_1 m_2}{(a_1 + a_2)^2} = m_1 a_1 L^2 \) \( \text{(1)} \)

Force = \( m_2 \) accel. for star 2: \( \frac{G m_1 m_2}{(a_1 + a_2)^2} = m_2 a_2 L^2 \) \( \text{(2)} \)

\( \frac{m_1}{m_2} = \frac{a_2}{a_1} \) \( \text{(3)} \)

\( \text{(1)} + \text{(2)} \): \( \frac{G (m_1 + m_2)}{(a_1 + a_2)^2} = (a_1 + a_2) L^2 \) \( \implies m_1 + m_2 = \frac{(a_1 + a_2)^3 L^2}{G} \)

If \( \frac{m_1}{m_2} = r \) and \( m_1 + m_2 = S \), then \( r, m_2 + m_2 = S \).

\( \implies m_2 = \frac{S}{1+r} \) and \( m_1 = \frac{S}{1+r} \left( \frac{rS}{1+r} \right) \)
\[ b) \quad \frac{G (m_1 + m_2)}{a^3} = \frac{S^2}{2} \quad \text{where} \quad a = a_1 + a_2 \quad \text{is separation between stars} \]

\[ \therefore \quad a^3 = \frac{G (m_1 + m_2)}{S^2} = \frac{G (m_1 + m_2) P^2}{4 \pi^2} \]

\[ m_1 + m_2 = 3.5 \, M_0 = 3.5 \times 2 \times 10^{30} \, \text{kg} = 7.0 \times 10^{30} \, \text{kg} \]

\[ P = 50 \, \text{yr} = 50 \times 3.16 \times 10^7 \, \text{s} = 1.6 \times 10^9 \, \text{s} \]

\[ G = 6.67 \times 10^{-11} \quad \text{(SI units)} \]

\[ \therefore \quad a = \left( \frac{6.67 \times 10^{-11} \times 7.0 \times 10^{30} \times (1.6 \times 10^9)^2}{4 \pi^2} \right)^{1/3} \]

\[ = 3 \times 10^{12} \, \text{m} \]

\[ = 20 \, \text{AU}. \]

5(a) What is measured is \( v_1 \sin i \) and \( v_2 \sin i \)

\( i = \text{inclination of orbit to line of sight} \)

\[ \therefore \quad \frac{v_2}{v_1} = \frac{100 \, \text{km s}^{-1}}{45 \, \text{km s}^{-1}} = \frac{4}{3} \]

By 4(a), this is mass ratio: \( \frac{m_1}{m_2} = \frac{4}{3} \)

(b) known period \( P = 11 \, \text{days} \)

\[ \therefore \quad R = \frac{2 \pi}{11 \, \text{days}} \]

\[ G \left( \frac{m_1 + m_2}{(a_1 + a_2)^3} \right) = \frac{S^2}{2} \quad \therefore \quad (m_1 + m_2) \sin^3 i = \left( a_1 R \sin i + a_2 R \sin i \right)^3 \]

\[ G \frac{m_1 + m_2}{R^3} = \frac{6.67 \times 10^{-11} \times 2 \pi \times 12 \times 10^{26} \, \text{kg}}{11 \times 24 \times 3600 \, \text{kg} \, \text{Period (in sec)}} \]

\[ \sin^3 i = \frac{12 \times 10^{26}}{6.67 \times 10^{-11}} = 18 \times 10^{36} \, \text{kg} \]

\[ \sin^3 i = 4.13 \, \sin^3 i \]

\[ \text{hence} \quad m_1 \sin^3 i = 7 \times 10^{30} \, \text{kg} \]

\[ m_2 \sin^3 i = 5 \times 10^{30} \, \text{kg} \]
Hence if $i = 90^\circ$ (so $\sin i = 1$), we have

$$m_1 = 7 \times 10^{30} \text{ kg} \quad (= 3.5 \, M_\odot)$$
$$m_2 = 5 \times 10^{30} \text{ kg} \quad (= 2.6 \, M_\odot)$$

If $i = 80^\circ$, however, $\sin i = 0.98$;
there values must then be increased by $(\sin^3 80^\circ)^{-1}$, i.e., by 5%.
So estimates are still correct to the precision quoted above.

6. Collapse whilst remaining close to hydrostatic equilibrium

Use virial theorem to say that energy radiated away
was equal to increase in internal energy.

Perfect gas $\Rightarrow$ internal energy/unit mass $U = \frac{1}{y-1} \frac{P}{\rho} = \frac{1}{y-1} \frac{RT}{M}$

\[ \therefore \text{total internal energy} \quad U = \frac{1}{y-1} \frac{RT \, m_\odot}{M} \]

\[ \therefore \text{change in internal energy} \quad (\text{= energy radiated away}) \]

\[ = \frac{1}{y-1} \frac{R \, m_\odot}{M} \Delta T \]

\[ = R \, m_\odot \, T_{\text{final}} \]

\[ = (8.26 \times 10^3) \times (2 \times 10^{30}) \times (6 \times 10^6) \ J = 10^{44} \ J \]

[Oh - this is same order of magnitude as P.E. released: $\frac{GM^2}{R}$]

Assuming luminosity $= L_\odot = 4 \times 10^{26} \text{ W}$ ($1 \text{W} = 1 \text{J/s}$)
get time for contraction:

\[ \frac{10^{44} \ J}{4 \times 10^{26} \text{ W}} = 2.5 \times 10^{18} \text{ s} \approx 10^7 \text{ yr.} \]
7. You are given the colour index of each star, B - V. We know hot stars are bluer than cool stars, and also know that higher fluxes give smaller magnitudes, so the smaller the B - V, the bluer the star. Hence decreasing order of temperature is simply the increasing order of B - V: Spica, Rigel, Sirius, Canopus, Altair, the Sun, Alpha Centauri A, Arcturus, Aldebaran and Antares.

To determine colours, remember that a colour index of zero corresponds to a white star and that the Sun (B - V = 0.65) is yellow. Therefore Sirius and Rigel are white; the Sun and Alpha Centauri A are yellow.

The only star which is significantly hotter than Sirius and Rigel is Spica, which has a distinctly negative colour index: Spica is blue.

The two stars between Sirius and the Sun have colour indices not large enough to suggest that the stars will be really yellow, but they probably are not brilliant white either: Canopus and Altair are yellowish white.

This leaves the stars which are cooler than the Sun. Antares has an extremely large B - V, and in fact the name “Antares” is Greek for “rival of Mars”, and it is a star that looks coloured to the naked eye. So we can say Antares is red.

The colour index of Arcturus is about halfway between the Sun, which is yellow, and red Antares: Arcturus is orange.

Finally Aldebaran, midway between Arcturus and Antares. Without some experience of looking at colour photos of star fields, it is not obvious which side of the orange/red divide this star will go: Aldebaran is orange or red.

[Additional information relevant to later lectures: Spectral class assignments can be made directly from colour index or estimated from the colour: O and B stars are blue, A stars are white, F yellowish white, G yellow, K orange and M red. Using this we infer that Spica is class B (class O stars are much rarer than class B, so this is a safe guess), Rigel and Sirius are A, Canopus and Altair are F, the Sun and Alpha Centauri A are G, Arcturus K, Aldebaran K or M, Antares M. ]
1. Your plots should look like Fig 1 and Fig 2.

Figure 1 shows the distinction between rocky, terrestrial planet like bodies, including the Moon, Io and Europa, and lower density bodies such as the gas giants and moons like Titan and Callisto that are made up of a larger fraction of volatiles such as water ice, carbon-dioxide, methane, nitrogen etc. Pluto sits squarely among the icy moons. From this diagram you can also see that Phobos and Deimos have properties that are closer to those of small asteroids than those of the icy moons, leading to the idea that they are captured asteroids rather than objects that formed alongside Mars.

Figure 2 shows that the densest material in the Solar System lies in its inner regions, less than 5 AU from the Sun. This relates to the formation processes in the early Solar System, whereby the young Sun was hot enough to boil off all the volatiles from objects that formed close to it. This gives rise to something termed the Snow Line, a point at about 4 AU from the Sun inside which volatiles such as water, methane and carbon dioxide, cannot form into ices. Objects that form inside the snowline, like Earth, thus cannot include large quantities of these materials. Objects that form outside the snowline, such as Jupiter, in contrast, can retain these volatiles, and thus can become much more massive.

2. I don’t think that there is a trick to this one, so, it really is slogging it all out. Please come and see me if you get stuck, but essentially you will have to evaluate

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left( \frac{\cos \theta (1 - e^2)}{1 + e \cos \theta} + e \right)^2 + \left( \frac{\sin \theta \sqrt{1 - e^2}}{1 + e \cos \theta} \right)^2
\]

\[
= \left( \frac{1}{1 + e \cos \theta} \right)^2 \left[ (\cos \theta (1 - e^2) + e(1 + e \cos \theta))^2 + \sin^2 \theta (1 - e^2) \right].
\]

Going through this you should find that it all reduces to 1.

The second part of the question is much quicker, simply plugging in \(a\) and \(b\), one has

\[
1 - \frac{b^2}{a^2} = 1 - \frac{r_0^2}{1 - e^2} \frac{(1 - e^2)^2}{r_0^2} = 1 - (1 - e^2) = e^2.
\]

3. The area element swept out by a radius vector of length \(r\) moving through a small angle \(d\theta\) is a triangle with base length \(r\) and height \(r \cdot d\theta\) so that the area is \(\frac{1}{2} r^2 d\theta\). To get the total area of the ellipse this needs to be integrated from 0 to 2\(\pi\). The angle \(\theta\) changes with time; it takes time \(P\) to sweep out the area of a full ellipse, so changing from \(\theta\) to \(t\) implies a change of integration limits from 0 to \(P\) (or, if you prefer \(T_0\) to \(T_0 + P\)). We can thus write

\[
\pi ab = \int_0^P \frac{1}{2} r^2 \frac{d\theta}{dt} dt.
\]

Noting that \(h = r^2 d\theta / dt\) is constant and that \(h\) is also \(h = \sqrt{r_0 GM}\), we find

\[
\pi ab = \int_0^P \frac{1}{2} h dt = \frac{1}{2} h P = \frac{1}{2} \sqrt{r_0 GM} P.
\]

Re-arranging this and substituting the expression for \(b\) obtained in the previous question, we find

\[
\frac{2\pi a}{P} = \sqrt{r_0 GM} = \sqrt{\frac{r_0 GM}{r_0}} \sqrt{1 - e^2} = \sqrt{GM} \sqrt{\frac{1 - e^2}{r_0}}.
\]

Squaring both sides this becomes

\[
\frac{(2\pi)^2}{P^2} a^2 = GM \left( \frac{1 - e^2}{r_0} \right) = \frac{GM}{a} \Rightarrow \frac{(2\pi)^2}{P^2} a^3 = GM,
\]

which is Kepler’s third law.
Figure 1: Density against $\log(\text{radius})$. Another version of this figure can be found on page 347 of Rothery, McBride and Gilmour.

Figure 2: Orbital semi-major axis vs. density.
4. Rather than calculate the period from Kepler’s law directly, this is easier (and safer!) done by scaling Saturn’s period and distance to the Earth’s. From Kepler’s third law, we know that \( a^3 \propto P^2 \), thus

\[
\left( \frac{P_S}{P_{\oplus}} \right)^2 = \left( \frac{a_S}{a_{\oplus}} \right)^3.
\]

As \( a_{\oplus} \) is 1 AU and \( P_{\oplus} \) is one year, the period for Saturn’s orbit around the Sun is

\[
P_S = (9.5)^{3/2} \text{ yrs} = 29 \text{ yrs}.
\]

5. (a) Flux received from the star is:

\[
F = \frac{L}{4\pi d^2} \, \text{Wm}^{-2}
\]

Total power received by the planet is:

\[
P_r = \pi r^2 F = \pi r^2 \frac{L}{4\pi d^2}
\]

A fraction of \( (1 - \alpha) \) of this is absorbed, where \( \alpha \) is the albedo, so the total power absorbed is:

\[
P_i = \pi r^2 F = \pi r^2 \frac{L(1 - \alpha)}{4\pi d^2}
\]

Power is radiated away as a black body over the full surface area of the planet, thus:

\[
P_o = 4\pi r^2 \sigma T^4
\]

These must balance, so:

\[
4\pi r^2 \sigma T^4 = \pi r^2 F = \pi r^2 \frac{L(1 - \alpha)}{4\pi d^2} \Rightarrow T = \left( \frac{L(1 - \alpha)}{16\pi \sigma d^2} \right)^{1/4}
\]

(b) \( T=292 \text{ K} \)

The albedo of 0.25 is reasonable given that this matches that of Mars and is close to that of Earth. Venus has a larger albedo.

Liquid water is thought necessary for life. It would be expected on this planet.

A greenhouse effect from a sufficiently thick atmosphere could raise the temperature, possibly making it uninhabitable (eg. Venus) if the greenhouse effect is large enough.

6. (a) The flux Mars receives from the Sun is \( \frac{L_{\odot}}{(4\pi d^2)} \), where \( d \) is the mean Sun-Mars distance (1.5 AU). Mars intercepts \( \pi R_M^2 \) of this flux but reflects back 25% (i.e., its albedo is 0.25 and I have used \( R_M \) to indicate the radius of Mars). Assuming black-body radiation, Mars radiates away a flux of \( \sigma T_M^4 \), where \( T_M \) is Mars’ no-atmosphere temperature. Equating the power received and radiated away, we have

\[
(1 - a) \frac{L_{\odot}}{4\pi d^2} (\pi R_M^2) = \sigma T_M^4 (4\pi R_M^2).
\]

The no-atmosphere temperature of Mars is thus

\[
T_M = \left( (1 - a) \frac{L_{\odot}}{16\pi \sigma d^2} \right)^{1/4} \approx 210 \text{ K}.
\]

(b) The escape velocity is obtained by equating the kinetic energy of a particle and the potential energy needed to move this particle from the surface of the planet to infinity, thus

\[
\frac{1}{2} m v_{\text{esc}}^2 = \frac{GMm}{R},
\]
where $M$ and $R$ are the mass and radius of the planet, respectively, while $m$ is the mass of the particle. The escape velocity is thus

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R_M}} = \sqrt{\frac{2 \cdot 6.7 \times 10^{-11} \times 6.4 \times 10^{23}}{3.4 \times 10^6}} \simeq \sqrt{4 \cdot 6.4 \times 10^6} \text{ m s}^{-1} \simeq 5000 \text{ m s}^{-1}.$$  

So 20% of the escape velocity corresponds to about 1000 m s$^{-1}$.

The most probable thermal velocity $v_{\text{th}}$ is given by $v_{\text{th}} = \sqrt{\frac{2kT}{m}}$, where $k$ is Boltzmann's constant and $m$ is the mass of the particular molecule. In the following I will express this in terms of the hydrogen mass, thus $m = am_H$, where $a = 2$ (2 times 1 nucleon/proton) for molecular hydrogen; $a = 18$ (16 nucleons for O, 1 for each H) for water and $a = 44$ (12 nucleons for C) for carbon dioxide. The thermal velocity is thus

$$v_{\text{th}} = \sqrt{\frac{2kT}{m}} = \sqrt{\frac{2 \cdot 1.4 \times 10^{-23} \times 210}{a \cdot 1.7 \times 10^{-27}}} = \sqrt{\frac{1}{a}} \cdot 1860 \text{ m s}^{-1}.$$  

The thermal velocity for $\text{H}_2$, $\text{H}_2\text{O}$ and $\text{CO}_2$ are thus 1300 m s$^{-1}$, 440 m s$^{-1}$ and 280 m s$^{-1}$, respectively, and Mars should be able to retain both, water and carbon dioxide, but not molecular hydrogen.
Problem Sheet 5

1. Example code that does this can be found on Blackboard.

2. First calculate how much energy Jupiter potentially emits through gravitational contraction by using
the information that its total emission (made up of re-radiation and contraction) is about twice that
received from the Sun. The luminosity received from the Sun is

\[
\frac{L_\odot}{4\pi d^2} (\pi R_J^2) = \frac{4 \times 10^{26}}{4} \left( \frac{7 \times 10^7}{5.2 \times 1.5 \times 10^{11}} \right)^2 = 10^{26} (9 \times 10^{-5})^2 = 8 \times 10^{17} \text{ W}.
\]

The power \( P \) released through gravitational contraction is

\[
P = \frac{d}{dt} \frac{GM^2}{R} = -\frac{GM^2}{R^2} \frac{dR}{dt},
\]

where \( \frac{dR}{dt} = \dot{R} \) is the change in radius. Expressing this in terms of the fractional change in radius \( \frac{\dot{R}}{R} \) we have

\[
\frac{\dot{R}}{R} = -P \frac{R}{GM^2},
\]

where \( P \) is has to be the luminosity received from the Sun and \( R \) and \( M \) are the radius and mass of
Jupiter, respectively. Thus

\[
\frac{\dot{R}}{R} = -8 \times 10^{17} \frac{7 \times 10^7}{6.7 \times 10^{-11} (1.9 \times 10^{27})^2} = -\frac{8 \times 7 \times 10^{24}}{6.7 \times 3.6 \times 10^{43}} \approx -2.3 \times 10^{-19} \text{ s}^{-1}.
\]

With \( 3.1 \times 10^7 \text{ s per year, the expected fractional contraction per year is about } 7 \times 10^{-12} \text{ per year. Note that this}
\]
\( \text{corresponds to an actual radius change of less than a 1 mm per year!} \)

3. (a) i. The \textit{solar constant} is the amount of flux (power per unit area) received at the top of the
Earth’s atmosphere.

ii. The \textit{greenhouse effect} is essentially due to greenhouse gases being transparent to visible
radiation while absorbing infrared radiation. This means that a large proportion of the stellar
radiation that peaks in the visible can penetrate the atmosphere and warm the planet that
then emits black-body radiation peaking in the infrared. This ‘back-radiation’ is absorbed by
the greenhouse gases and re-radiated in all directions. This reduces the amount of radiation
escaping back into space significantly. As a consequence the planet can maintain a higher
temperature than expected from (no-atmosphere) equilibrium considerations. In the case of
the Earth, the greenhouse effect is crucial for raising the Earth’s average temperature above
freezing.

iii. The \textit{habitable zone} of a planetary system is where conditions are favourable to the existence
of (carbon-based) life. Typically this means that a planet would be able to maintain liquid
water. In the solar system this is at a distance of between 0.95 and 1.2 AU.

(b) Sun-spots are dark magnetic features that cause a decrease in the solar constant. Small-scale
bright magnetic \textit{faculae} cause an increase. As the number of faculae increases more strongly
with increasing solar activity, the net effect is an increase of the solar ‘constant’ at activity maxi-
mum (though punctuated by short-lived dips due to large sunspots).

(c) The radiation at the top of the Earth’s atmosphere is given by the solar constant, i.e., it is \( S = 1368 \text{ W m}^{-2} \). The Earth intercepts \( \pi R_\oplus^2 S \) of the flux which is distributed over a surface area of
\( 4\pi R_\oplus^2 \), thus the average flux is \( S/4 = 342 \text{ W m}^{-2} \).

The albedo measures the reflectivity of a body; a body with albedo \( a = 1 \) is a perfect reflector.
The albedo \( a \) is defined as the ratio of the reflected to the received flux. In the case of the Earth,
the reflected flux is \( 77 + 30 \text{ W m}^{-2} \), while the incoming flux is \( 342 \text{ W m}^{-2} \), thus

\[
a = \frac{77 + 30}{342} = 0.31.
\]
(d) The black-body flux of a body is $\sigma T^4$, where $\sigma$ is the Stefan-Boltzmann constant. The black-body temperature of a body emitting a flux of $F = 390$ W m$^{-2}$ is

$$T = \left( \frac{F}{\sigma} \right)^{1/4} = \left( \frac{390}{5.7 \times 10^{-8}} \right)^{1/4} = 287 \text{ K}.$$ 

This is mostly emitted in the IR. Using Wien’s law (this would be given, or you would be told that the Sun’s emission peaks at 500 nm), we find that the black-body emission for a body of 287 K peaks at $0.5 \mu\text{m}$.

(e) For proportional scaling, the flux received (and thus also radiated) decreases with distance from the central star as $1/d^2$. If we require water to be available in liquid form, we need the surface temperature to be between 273 and 373 K. As the flux scales with the forth power of the temperature, the temperature and distance are related by $T^4 \propto d^{-2}$, or $d \propto T^{-2}$. We thus have

$$d_{\text{low}} = \left( \frac{T_{\oplus}}{T_{\text{low}}} \right)^2 1 \text{ AU} = 1.1 \text{ AU} \quad \text{and} \quad d_{\text{up}} = \left( \frac{T_{\oplus}}{T_{\text{up}}} \right)^2 1 \text{ AU} = 0.6 \text{ AU}.$$ 

Note that the ‘continuous habitable zone’ is much narrower and reaches from about 0.9 to 1.2 AU. For a system with higher luminosity, the distances need to be scaled up. The flux increases linearly with the luminosity and quadratically with the distance from the host star. Thus to obtain the same flux (which is what defines the temperature) for a star with luminosity $L_*$ we need to be further away by a factor of $\sqrt{L_*/L_\odot}$, and we obtain distances of

$$d'_{\text{low}} = d_{\text{low}} \sqrt{L_*/L_\odot} = d_{\text{low}} \sqrt{5} = 2.5 \text{ AU} \quad \text{and} \quad d'_{\text{up}} = d_{\text{up}} \sqrt{5} = 1.3 \text{ AU}.$$ 

4. For this, we assume that the small body of mass $m$ and radius $r$ is only held together by self-gravity. We will estimate the tidal force (due to the gas giant of mass $M$) on a small test particle with mass $\mu$ that is part of the smaller body $m$. We consider the difference in the gravitational force when the mass is at distance $d$ from the gas giant, and when it is at distance $d - r$, i.e., the difference when the test mass is at the surface of $m$ and at its centre; the tidal force is thus $F_t = \Delta F(d, d - r)$.

The estimate for the Roche limit can be obtained by equating the tidal force to the gravitational force on the test particle $\mu$ due to the smaller object with mass $m$. The Roche limit is where (in this somewhat simplified model) a body held together by self-gravity would be disrupted.

As $r \ll d$, the tidal force can be expanded as follows

$$F_t = \frac{GM\mu}{(d - r)^2} - \frac{GM\mu}{d^2} = \frac{GM\mu}{d^2} \left( \frac{1}{(1 - r/d)^2} - 1 \right) \approx \frac{2GM\mu r}{d^3}.$$ 

The gravitational attraction of the test particle at distance $r$ from the centre of mass of $m$ is $F_g = Gm\mu/r^2$; this provides the balancing force, thus $F_t = F_g$ implies

$$\frac{2GM\mu r}{d_R^3} = \frac{Gm\mu}{r^2} \quad \Rightarrow \quad d_R^3 = \frac{2Mr^3}{m}.$$ 

The rings around the gas giants are all within the Roche limit. Material within $d_R$ is tidally disrupted and would thus be unable to form moons. Where the material is originally from is not so clear. It could be left over from the proto-planetary disk, it could be the remains of a moon that was shattered due to a collision or it could be due to an inward-migrating moon that was ripped apart. (Note that most moons are not only held together by self-gravity but show additional cohesive forces. Depending on the cohesive forces they can stray more or less into the Roche limit).
Finally, note that a more careful calculation yields a somewhat larger Roche limit of

\[ d_R = 2.4 \left( \frac{M r^3}{m} \right)^{1/3}. \]

5. (a) The formula for the escape velocity is

\[ v_{\text{esc}} = \sqrt{\frac{2Gm}{R}}. \]

Thus for Mars it will be:

\[ v_{\text{esc}} = \sqrt{\frac{2Gm_{\oplus}R_{\oplus}}{m_{\oplus}R_{\oplus}}} = v_{\text{esc,\oplus}} \sqrt{\frac{0.1}{0.5}} \approx 0.4 v_{\text{esc,\oplus}}. \]

Mars’ escape velocity is thus about 5200 m s\(^{-1}\) and a factor of 0.4 smaller than the escape velocity from Earth.

5200 m s\(^{-1}\) is still a substantial velocity for a boulder to reach and it is thought that volcanic eruptions are not violent enough to accelerate rocks off the Martian surface. However, meteoroid impacts can be energetic enough (some simulations suggest that impacts that result in crater sizes of about 3 km are sufficient to accelerate fragments above the escape velocity; Head et al 2002, Science 298, 1752). Some of the fragments escaping Mars’ gravitational field could then be nudged towards Earth’s orbit by Jupiter’s gravitational influence and might eventually end up on Earth. To date about 30 meteorites of Martian origin have been found on the Earth, about one third of them through meteorite searches in Antarctica.

(b) The possible resonances for integers smaller than 8 are (2:1), (5:2), (7:3), and (3:1). A (3:1) orbital resonance means that the asteroid will complete 3 orbits while Jupiter only completes one orbit; or, alternatively, the asteroid’s period is only one third of Jupiter’s period. Using Kepler’s third law, \(a^3/P^2 = \text{const}\), we can find the corresponding semi-major axis:

\[ \left( \frac{a_{\ast}}{a_{J}} \right)^3 = \left( \frac{P_{\ast}}{P_{J}} \right)^2 = \frac{1}{9}. \]

The 3 : 1 orbital resonance thus corresponds to a semi-major axis of \(a_{\ast} = 5.2 \text{ AU} / \sqrt{9} = 2.5 \text{ AU}\).
1. (a) The relation between the period $P$ and the semi-major axis $a$ is given by Kepler’s third law:

$$M = \frac{4\pi^2a^3}{GP^2},$$

where $M$ is the total mass of the system and $a = a_p + a_s$ the sum of semi-major axis length of the star and the planet. As the mass of the star far exceeds the mass of the planet, the planet’s semi-major axis also has to exceed the star’s semi-major axis (this is a direct consequence of the centre-of-mass formula, $a_p/a_s = M_s/m_p$). Kepler’s third law thus reduces to

$$M_s = \frac{4\pi^2a_p^3}{GP^2} \Rightarrow a_p = \left(\frac{GM_s P^2}{4\pi^2}\right)^{1/3}$$

and you can use this to calculate $a_p$. Alternatively, as the result is required in AU, you might find it more convenient to scale this to the Earth’s orbit; we know that $M \propto a^3/P^2$ thus, for $a_p$ in AU, we have

$$a_p = \left(\frac{M_s}{M_\odot}\right)^{1/3} \left(\frac{P_p}{P_\oplus}\right)^{2/3} \text{AU} = (1.2)^{1/3} \left(\frac{3.31}{365}\right)^{2/3} \text{AU} = 0.046 \text{AU}.$$ 

(b) We will assume that the luminosity scales as $M^4$, thus the luminosity of $\tau$ Boo, $L_s$, should be about twice the solar luminosity. The flux received by the planet is $F_p = L_s/(4\pi a_p^2)$. We might further assume that about half of the incoming radiation is reflected back (this is quite reasonable, though perhaps a slight underestimate). The star should thus absorb $0.5\pi R_p^2 F_p$ and radiate back $\sigma T^4 4\pi R_p^2$, where $T$ is the no-atmosphere temperature of the planet. Balancing these two terms yields

$$\frac{L_s}{4\pi a_p^2} 0.5\pi R_p^2 = 4\pi R_p^2 \sigma T^4,$$

so that the planetary radius cancels out and

$$T = \left(\frac{0.5L_s}{16\pi\sigma a_p^2}\right)^{1/4} \simeq 1250 \text{ K}.$$ 

To find out whether the planet retains its atmosphere, we need to consider the magnitudes of the escape and the thermal velocities. The escape velocity is obtained by equating the kinetic energy of a particle with the potential energy that would be needed to move this particle from the planetary surface to infinity. Thus

$$v_{\text{esc}} = \sqrt{\frac{2Gm_p}{R_p}}.$$ 

To make further progress we need to estimate the radius of the planet, $R_p$. There are two approaches for this. We can either assume that the density of the planet is very similar to the density of the solar-system gas giants (this is reasonable for a gas giant); or we can use the fact that radii of gas giants are not a strong function of mass and use $R_p \simeq R_J$. Note that in an exam, I would not have expected you to know about the mass-radius relation for gas giants! I would have either specified the planetary radius or given more detail).

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1 In fact, gas giants are transitional objects between bodies where $R$ increases slowly with $M$ and compact objects such as brown dwarfs where the radius decreases for increasing mass ($R \propto M^{-1/3}$ for fully convective objects). There is thus a maximum radius for gas giants and it turns out that this is of the order of 1.1 or 1.2 $R_J$. When a gas giant is strongly irradiated (as is the case here) then its radius will increase, though this effect is only important for lower-mass gas giants.
Let us here assume that the density of the planet is the same as that of a typical gas giant in the solar system, i.e., \( \rho \sim 1000 \) kg m\(^{-3}\). Then \( m_p = (4\pi/3)\rho R_p^3 \) and

\[
R_p = \left( \frac{3m_p}{4\pi\rho} \right)^{1/3} = \left( \frac{3 \cdot 3.9 \cdot 1.9 \times 10^{27}}{4\pi 1000} \right)^{1/3} = 1.2 \times 10^8 \text{ m}.
\]

At 1.7 \( M_J \) this is rather too large, but given the uncertainties of our estimate this does not matter (also, it enters as the square-root only, so the final error is not so bad). The escape velocity is then

\[
v_{\text{esc}} = \sqrt{\frac{2Gm_p}{R_p}} = \sqrt{\frac{2 \cdot 6.7 \times 10^{-11} \cdot 3.9 \cdot 1.9 \times 10^{27}}{1.2 \times 10^8}} \approx 90000 \text{ m s}^{-1}.
\]

For the thermal velocity we take the most probable velocity for a Maxwell-Boltzmann distribution,

\[
v_{\text{th}} = \sqrt{\frac{2kT}{m}} = \sqrt{\frac{2 \cdot 1.4 \times 10^{-23} \cdot 1250}{1.7 \times 10^{-27}}} \approx 4500 \text{ m s}^{-1},
\]

where I have assumed that hydrogen is in atomic rather than molecular form. The thermal velocity is thus more than a factor of 10 smaller than the escape velocity and the gas giant should be able to retain its H and He atmosphere, (note that the atmosphere would be lost over a couple of Gyrs only if \( v_{\text{th}} \) is larger or of the order of 10% \( v_{\text{esc}} \)).

(c) For circular orbits, the maximum radial velocity of the star is \( 2\pi a_s/P \), where \( a_s \) is the star’s semi-major axis. For the observed radial velocity we have to take into account the inclination angle; we only see \( 2\pi a_s \sin i/P \). The star’s semi-major axis is related to the planet’s semi-major axis through \( a_s/a_p = m_p/M_s \), thus

\[
v_{\text{rad}} = \frac{2\pi a_pm_p \sin i}{PM_s} = \frac{2\pi \cdot (0.046 \cdot 1.5 \times 10^{11}) \cdot (3.9 \cdot 1.9 \times 10^{27})}{(3.31 \cdot 24 \cdot 3600) \cdot (1.2 \cdot 2 \times 10^{30})} \approx 470 \text{ m s}^{-1}.
\]

[Note: this is easily detectable even for relatively low inclinations; we can currently detect variations of the order of one m s\(^{-1}\).]

(d) The fraction of light blocked \( \Delta F/F \) is given by

\[
\Delta \frac{F}{F} = \left( \frac{R_p}{R_s} \right)^2 = \left( \frac{1.2 \times 10^8}{1.2 \cdot 7 \times 10^8} \right)^2 \approx 2\%,
\]

or, if a perhaps slightly more realistic planetary radius of \( R_J = 7 \times 10^7 \) is assumed, \( \Delta F/F \approx 1\% \). In either case this is observable from the ground (where the detection limit is about \( 10^{-3} \)) and space (detection limit around \( 10^{-5} \)).

Detections from space are more sensitive as there are no atmospheric effects, in particular ‘seeing’ (= atmospheric scintillation). Seeing is due to turbulent cells in the Earth’s atmosphere; these produce variations in transparancy (making stars ‘twinkle’) and thus background variability so that very low levels of stellar variability can not be detected.

2. The four terrestrial planets only contain negligible amounts of H and He, thus the total mass of all four should go fully towards adding ‘metallicity’ to the star. Looking up the masses of the planets, one finds that Mercury, Venus and Mars have masses of \( 0.05M_\oplus, 0.81M_\oplus \) and \( 0.11 M_\oplus \), where \( M_\oplus \) is the mass of the Earth. The four infalling planets thus roughly add the equivalent of 2 Earth masses in metallicity to the Sun.

(a) If this is distributed throughout the Sun, the relative metal abundance will increase by

\[
\Delta z = \frac{2M_\oplus}{M_\odot} = \frac{2 \cdot 3 \times 10^{-6} M_\odot}{M_\odot} = 6 \times 10^{-6},
\]

where I have used that \( M_\oplus \) is about \( 3 \times 10^{-6} M_\odot \).
If the material is mixed into the convection zone only, then
\[
\Delta z = \frac{2M_\oplus}{0.02M_\odot} = \frac{2 \times 3 \times 10^{-6}M_\odot}{0.02M_\odot} = 3 \times 10^{-4}.
\]

The metallicity of the Sun is \(z \simeq 0.02\), so in either case the infalling planets would only achieve a small perturbation.

You may want to note that the metallicity of other stars is usually expressed as a logarithm of the iron abundance and with respect to the solar value (for stars such as the Sun, the iron abundance is expected to scale reasonably well with metallicity). For example, if you are considering a plot such as the one on slide 11, lecture 24, the x-axis is given as \(\log_{10}(Fe/H) - \log_{10}(Fe/H)_\odot\). Host stars with solar metallicity would thus appear in the bin around 0, stars with twice and half solar metallicity would appear in the bin around +0.3 and −0.3, respectively.

3. For now I will only consider the transit duration as seen by a far-away observer, i.e., we can assume the light rays to be parallel. The derivation that takes into account the Earth’s motion around the Sun and the projections is outlined further on (and as stated on the PS would not be expected).

Venus has to cross a distance corresponding to twice the solar radius. It has a semi-major axis of about \(a_V = 0.72\) AU, so Venus’ period is
\[
\left(\frac{P_V}{P_\oplus}\right)^2 = \left(\frac{a_V}{a_\oplus}\right)^3 = 0.72^3 \quad \Rightarrow \quad P_V = 0.61\text{ yrs}.
\]

The transit time \(\tau\) is the distance Venus has to travel divided by its velocity, thus
\[
\tau = \frac{2R_\odot}{v} = \frac{2R_\odot}{2\pi a_V/P_V} = \frac{(7 \times 10^8) \cdot 0.61 \cdot (3.1 \times 10^7)}{\pi 0.72 \cdot (1.5 \times 10^{11})} \approx \frac{0.6 \cdot 10}{1.5} \times 10^4 \approx 4 \times 10^4 \text{ s} \approx 11\text{ h}.
\]

If we also want to take into account the fact that the Earth moves and that the Sun’s rays are not parallel, we can visualise the situation as shown in the picture below.

\[\text{In the following I will use the rather strange units of } AU\, h^{-1}, \text{ mainly because we know that we expect an answer on the order of several hours, and also because we know the semi-major axes in } AU.\]

The velocity of the Earth and Venus around the Sun in \(AU\, h^{-1}\) are given by
\[
v_E = \frac{2\pi a_E}{P_E} = 7.2 \times 10^{-4}\text{ AU } h^{-1} \quad \text{and} \quad v_V = \frac{2\pi a_V}{P_V} = \frac{0.72}{0.61}v_E = 8.5 \times 10^{-4}\text{ AU } h^{-1}.
\]

The distance across the Sun in units of \(AU\) is \(d = 2R_\odot/1.5 \times 10^{-11} = 9.3 \times 10^{-3}\) AU. The travel time for Venus is \(v_V\Delta t\), the one for the Earth is \(v_\oplus\Delta v\). Using the triangles outlined on the figure, we can solve first for \(x\), where \(x\) is the distance from the Earth to where the hypothetical ingress and egress lines meet, and then for \(\Delta t\).

We can obtain \(x\) (in AU) from \(v_V/v_E = (0.28 + x)/x\) and \(\Delta t\) from \(v_V\Delta t/d = (x + 0.28)/(x + 1)\), or, equivalently, \(v_E\Delta t/d = x/(x + 1)\). The eclipse duration is then about 8 h. It was in fact somewhat
shorter, as Venus did not cross the Sun’s equator, but at a higher (shorter) latitude; the 1639 transit lasted from 14:51 to 22:02 GMT. Apparently Horrocks observed until noon and then again from 15:15 onwards, just missing the very beginning. He would also not have seen the end as it was after Sunset.

4. A star with a measured parallax \( p = 0.02'' \) is 50 pc away. Unfortunately the planet would be too faint to be seen directly and we would have to infer it from its pull onto its host star. The shift \( a_s \) induced by a planet of Earth and Jupiter mass is

\[
\frac{a_s}{a_p} = \frac{m_p}{M_s} \quad \Rightarrow \quad a_s = \frac{m_p}{M_s} a_p,
\]

where \( a_p \) is the distance of the planet from its host stars. Recalling that an Earth mass is about \( 3 \times 10^{-6} M_\odot \) and a Jupiter-mass is about \( 10^{-3} M_\odot \), the numerical values for an Earth-like and Jupiter-like system are

\[
a_{s,E} = 3 \times 10^{-6} \text{AU} \quad \text{and} \quad a_{s,J} = 10^{-3} \cdot 5.2 \text{AU}
\]

As 1 AU corresponds to a parallax of \( 0.02'' \) at a distance of 50 pc, the observed maximum angular displacements are \( 2a_s \cdot 0.02'' \) if we assume favourable conditions (i.e., we are seeing the system edge-on and are able to observe the star for long enough to pick up its maximum displacement perpendicular to the line of sight). For an Earth and Jupiter-like system, we then have

\[
p_{s,E} = 6 \times 10^{-6} \cdot 0.02'' = 1.2 \times 10^{-7} \text{arcsec} \quad \text{and} \quad p_{s,J} = 10.4 \times 10^{-3} \cdot 0.02'' = 2.8 \times 10^{-4} \text{arcsec}
\]

We thus expect to see displacements of 0.12 \( \mu \) arcsec and 280 \( \mu \) arcsec for Earth and Jupiter respectively and both planets are out of reach for Hipparcos. The specifications for GAIA are good enough to pick up a Jupiter-like system easily, though they still lack more than one order of magnitude to be able to pick up an Earth-like system at a distance of 50 pc.

NB: you can of course also work in radians for this question... though take care to translate the specification into radians too!