Advanced Quantum Field Theory
Example Sheet 1

Please email me with any comments about these problems, particularly if you spot an error. Problems with an asterisk (*) may be more difficult.

1. If $B$ is an invertible $n \times n$ matrix and $\theta^i$, $\bar{\theta}^i$, $\eta_i$ and $\bar{\eta}_i$ are independent fermionic variables, show that

$$\mathcal{Z}(\eta, \bar{\eta}) := \int d^n\theta d^n\bar{\theta} \exp (\bar{\theta}^i B_{ij} \theta^j + \bar{\eta}_i \theta^i + \bar{\theta}^i \eta_i) = \det B \exp (\bar{\eta}_i (B^{-1})^{ij} \eta_j).$$

Use this result to obtain an expression for normalized expectation value

$$\langle \bar{\theta}^{i_1} \ldots \bar{\theta}^{i_r} \theta^{j_1} \ldots \theta^{j_s} \rangle := \frac{1}{\mathcal{Z}(0, 0)} \int d^n\theta d^n\bar{\theta} \exp (\bar{\theta}^i B_{ij} \theta^j) \bar{\theta}^{i_1} \ldots \bar{\theta}^{i_r} \theta^{j_1} \ldots \theta^{j_s}$$

and show that it vanishes whenever $r \neq s$ and whenever $r, s > n$. Interpret your answer in terms of Feynman graphs. [The result is known as Wick's theorem for this fermionic theory.]

2. Consider the partition function

$$\mathcal{Z}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx e^{-\frac{1}{2}x^2 - \frac{\lambda}{4!}x^4} \quad (\dagger)$$

for a zero–dimensional QFT with a quartic interaction with $\lambda > 0$.

(a) By expanding the integral in $\lambda$ obtain the $n^{th}$ order perturbative expansion

$$\mathcal{Z}_n(\lambda) = \sum_{\ell=0}^{n} \left( \frac{-\lambda}{4!} \right)^\ell \frac{(4\ell)!}{4\ell (2\ell)! \ell!}$$

and show for $\ell \leq 3$ that the coefficients $a_\ell$ of $\lambda^\ell$ in this expression are the sums of automorphism factors of the relevant loop Feynman graphs. (At two loops there is only one graph, at three loops there are two graphs and at four loops there are four.)

(b) Optional but instructive: Using any computer package, plot $\mathcal{Z}_n(\lambda = \frac{1}{10})$ against $n$ to see that there is a region in $n$ where $\mathcal{Z}_n$ appears to converge, before blowing up as $n$ is increased.
(c) Show that the minimum value of \(a_\ell \lambda^\ell\) occurs when \(\ell \approx \frac{3}{2}\). Hence show that the Borel transform \(BZ(\lambda) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} a_\ell \lambda^\ell\) converges provided \(|\lambda| < \frac{3}{2}\) and that in this case
\[
Z(\lambda) = \int_0^\infty dz \, e^{-z} BZ(z\lambda)
\]
so that \(Z(\lambda)\) may be recovered from its Borel transform.

(d) By expanding \(e^{-\frac{1}{2}x^2}\) in the integral (†) obtain the strong coupling expansion
\[
Z(\lambda) = \frac{1}{2\sqrt{\pi}} \sum_{L=0}^{\infty} \frac{(-1)^L}{L!} \Gamma\left(\frac{L}{2} + \frac{1}{4}\right) \left(\frac{6}{\lambda}\right)^{\frac{L}{2} + \frac{1}{4}}
\]
for \(Z(\lambda)\) as a series in \(1/\sqrt{\lambda}\). For \(\lambda = \frac{1}{10}\) how many terms does one need to obtain the value at which the weak coupling expansion appeared to converge?

3. Consider the action \(S(\phi, \psi, \bar{\psi}) = \frac{m^2}{2} \phi^2 + \frac{M^2}{2} \bar{\psi} \psi + \lambda \bar{\psi} \phi \psi\) describing a real bosonic variable \(\phi\) coupled to two independent fermionic (Grassmann) variables \(\psi, \bar{\psi}\).

(a) Treating this as a zero–dimensional QFT, write down the Feynman rules for the propagators and the interaction.

(b) Integrate out the fermions to show that the effective action for \(\phi\) has an infinite expansion
\[
S_{\text{eff}}(\phi) = \sum_{n=0}^{\infty} \frac{\lambda_n}{n!} \phi^n
\]
in terms of an infinite series of couplings \(\lambda_n\) whose values you should find. Which correlation functions can be computed using \(S_{\text{eff}}\)?

(c) Working to order \(\lambda^2\), compute \(\frac{1}{2} \langle \phi^2 \rangle\) first by using the original action and then by using the effective action \(S_{\text{eff}}\). Check your results agree.

4. Let \(M\) be an \(N \times N\) Hermitian matrix and consider the integral
\[
\mathcal{Z}(a; N) = \int d^{2N}M \exp\left(-\frac{1}{2} \text{tr}(M^2) - \frac{a}{N} \text{tr}(M^4)\right)
\]
where \(a\) is a coupling constant. The measure \(d^{2N}M\) represents an integral over the real and imaginary parts of each entry of \(M\).

(a) Represent the propagator as a “double line” where one line edge represents the rows and the other edge represents the columns of \(M\). What are the Feynman rules for this action?

(b) Show that \(\mathcal{Z}(a; N)/\mathcal{Z}(0; N)\) can be reduced to an integral over the eigenvalues \(\{\lambda_i\}\) of \(M\). [You may use without proof that the measure \(d^{2N}M\) is invariant under \(M \rightarrow U^{-1}MU\) for any unitary matrix \(U\).]
(c) Show that $Z(a; N)/Z(0; N)$ admits a perturbative expansion of the form

$$\ln \frac{Z(a; N)}{Z(0; N)} = \sum_{g=0}^{\infty} N^{2-2g} \left( \sum_{n=0}^{\infty} (-a)^n F_{g,n} \right)$$

where $F_{g,n}$ is a combinatoric number, independent of $N$ and $a$. (You are not required to find an explicit expression for $F_{g,n}$.)

(d) (*) Show that $F_{g,n}$ may be interpreted as the number of ways to cover a genus $g$ Riemann surface with $n$ squares. [For help with this part of the question, you may wish to consult the first few sections of D. Bessis, C. Itzykson & B. Zuber, Quantum Field Theory Techniques in Graphical Enumeration, Adv. Applied Maths 1, 109-157, (1980).]

5. Consider the Quantum Mechanics of a particle moving on $\mathbb{R}^n$ with Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$. Obtain (Euclidean time) path integral expressions for the following Heisenberg picture transition functions:

(a) $\text{Tr}_\mathcal{H}(P e^{-TH})$, where $P$ is the parity operator $P : \hat{x}^a \to -\hat{x}^a$.

(b) $\langle \psi_f | e^{-TH} | \psi_i \rangle$, where $\psi_{i,f}(x) = \langle x | \psi_{i,f} \rangle$ are arbitrary states in the Hilbert space.

where $T$ is the proper time on the worldline.

Suppose $n = 1$ and the worldline action includes the potential term $\frac{1}{2}m^2\omega^2 x^2$. Given that the heat kernel for the (Euclidean) harmonic oscillator is

$$\langle x | e^{-TH} | y \rangle = \sqrt{\frac{m\omega}{2\pi \sin \omega t}} \exp \left( -m\omega \frac{(x^2 + y^2) \cos \omega t - 2xy}{2\sin \omega t} \right),$$

evaluate your expressions for (a) and (b) explicitly in the case that $|\psi_{i,f}\rangle$ are the ground state of the harmonic oscillator. Check that they agree with what you expect from QM, working directly in the energy basis.

6. In a 1d QFT, let $x$ be a real bosonic scalar field and let $\psi$ and $\bar{\psi}$ be fermionic fields. Consider the action

$$S = \int d\tau \left[ \frac{1}{2} \left( \frac{\partial x}{\partial \tau} \right)^2 + \bar{\psi} \frac{\partial \psi}{\partial \tau} + \frac{1}{2} \lambda^2 h'(x)^2 - \lambda h''(x) \bar{\psi}\psi \right]$$

where $\lambda$ is a coupling constant and $h(x)$ is a smooth function of $x(\tau)$ (and $h'$ the derivative of this function).

(a) Show that this action is invariant under the transformations

$$\delta x = \epsilon \bar{\psi} - \bar{\epsilon} \psi$$
$$\delta \psi = \epsilon (-\dot{x} + \lambda h'(x))$$
$$\delta \bar{\psi} = \bar{\epsilon} (\dot{x} + \lambda h'(x))$$
where $\epsilon$ and $\bar{\epsilon}$ are constant fermionic parameters and $\dot{x} = \partial x / \partial \tau$. Find the conserved charges $Q$ and $\bar{Q}$ associated to these two symmetries.

(b) Show that $Q\bar{Q} + \bar{Q}Q = 2H$ where $H$ is the Hamiltonian associated to the above Lagrangian. Hence or otherwise show that the energies of the system are non-negative, and that the ground state $|\Psi\rangle$ obeys $Q|\Psi\rangle = \bar{Q}|\Psi\rangle = 0$.

(c) Put this theory on a circle and take $x(\tau)$, $\psi(\tau)$ and $\bar{\psi}(\tau)$ each to be periodic. Show that the partition function $Z = \int Dx\ D\psi\ D\bar{\psi}\ e^{-S}$ with these boundary conditions is independent both of the radius of the circle and of $\lambda$, provided $\lambda \neq 0$.

(d) What is the operator expression to which this partition function corresponds? [Hint: Think carefully about the fermionic boundary conditions.]

(e) (*) Argue that the only field configurations that contribute to $Z$ have $\dot{x} = h'(x) = 0$. By expanding the action in a neighbourhood of these configurations, evaluate $Z$ in the case that $h(x)$ is a polynomial of degree $n$ with isolated roots.
Advanced Quantum Field Theory
Example Sheet 2

Please email me with any comments about these problems, particularly if you spot an error. Problems with an asterisk (*) may be more difficult.

1. Show that under the redefinition $g_i \rightarrow g'_i(g_j)$ of the couplings of a theory at scale $\Lambda$, the $\beta$-functions transform as

$$\beta_i \rightarrow \beta'_i = \frac{\partial g'_i}{\partial g_j} \beta_j.$$ 

Show that in a theory with a single coupling $g$, the first two terms in the $\beta$–function $\beta(g) = ag^3 + cg^5 + \mathcal{O}(g^7)$ are invariant under any coupling constant redefinition of the form $g \rightarrow g' = g + \mathcal{O}(g^3)$. Show that it is possible to choose this redefinition so as to remove all terms except these first two.

2. In a theory with a single coupling $g$ with $\beta(g) = -ag^3 - bg^5 + \mathcal{O}(g^7)$, solve the Callan–Symanzik equation for the running coupling to find

$$\frac{1}{g^2} = a \ln \frac{\Lambda^2}{\mu^2} + b \left( \ln \frac{\Lambda^2}{\mu^2} \right) + \mathcal{O}(1/\ln(\Lambda^2/\mu^2)),$$

or equivalently

$$\mu^2 = \Lambda^2 e^{-\frac{1}{ag^2} \left( ag^2 \right)^{-\frac{b}{a}}} \left( 1 + \mathcal{O}(g^2) \right).$$

Now suppose that $g' = g + cg^3 + \mathcal{O}(g^5)$. Show that $g'^2$ can be expressed in terms of $\mu'^2$ as above, where $\ln(\mu'^2/\mu^2) = c/b$.

3. Let $\psi^i$ denote a (fermionic) Dirac spinor field transforming in the fundamental representation of a $\text{U}(N)$ gauge group, and let $\bar{\psi}_j$ denote the Dirac conjugate spinor transforming in the antifundamental. Let $(A_{\mu})^i_{\ j}$ denote the gauge field for this interaction. Write down all possible $\text{U}(N)$ gauge invariant local operators involving these fields that are relevant or marginal in the cases where the space–time has dimension $d = 4$, $d = 3$ and $d = 2$.

4. Consider a four dimensional theory whose only couplings are a mass parameter $m^2$ and a marginally relevant coupling $g$. 
(a) Write down generic expressions for the $\beta$-functions in such a theory to lowest non-trivial order. (You should be able to identify the values of the classical contributions to the $\beta$-functions, and the sign of the leading-order quantum correction to $\beta(g)$.)

(b) Sketch the RG flows for this theory.

(c) Suppose that $g(\Lambda') = 0.1$ when the cut-off $\Lambda'$ is fixed at $10^5$ GeV. If $m^2(\Lambda')$ is measured to be 100 GeV, what value of $m^2(\Lambda)$ would be needed at the higher scale $\Lambda = 10^{19}$ GeV?

(d) Suppose you changed your value of $m^2(\Lambda)$ by one part in $10^{20}$. What would be the change in $m^2(\Lambda')$?

5. Furry’s theorem states that $\langle 0 | T \{ \tilde{A}_{\mu_1}(k_1) \cdots \tilde{A}_{\mu_n}(k_n) \} | 0 \rangle = 0$ when $n$ is odd. It is a consequence of charge conjugation invariance.

(a) In scalar QED, charge conjugation swaps $\phi$ and $\bar{\phi}$. How must the photon field $A_\mu$ transform if the action is to be invariant?

(b) Prove Furry’s theorem using the path integral.

(c) Does Furry’s theorem hold for off-shell photons with $k_\mu k^\mu \neq 0$?

(d) Prove Furry’s theorem in QED.

6. In the Lorenz gauge $\partial^\mu A_\mu = 0$, the classical equation of motion for the photon that follows from the QED action is $\Box A_\mu = e j_\mu = e \bar{\psi} \gamma_\mu \psi$.

(a) Obtain the Dyson–Schwinger equation

$$\Box(x) \langle A_\mu(x) A_\nu(y) \psi(x_1) \bar{\psi}(x_2) \rangle = e \langle j_\mu(x) A_\nu(y) \psi(x_1) \bar{\psi}(x_2) \rangle - \delta^4(x-y) \delta_{\mu\nu} \langle \psi(x_1) \bar{\psi}(x_2) \rangle$$

for the photon in the quantum theory. What is the corresponding equation for the electron?

(b) Use the Dyson–Schwinger equation to related the QED correlation function $\langle j_\mu(x) \psi(x_1) \bar{\psi}(x_2) \rangle$ to the exact electron–photon vertex function.

(c) Assuming the path integral measure is invariant, show that the Ward identity

$$p_\mu M^\mu(p, k_1, k_2) = M_0(k_1 + p, k_2) - M_0(k_1, k_2 - p)$$

obtained in lectures implies that the exact electron–photon vertex obeys

$$p^\mu \langle \tilde{A}_\mu(p) \psi(k_1) \bar{\psi}(k_2) \rangle = 0$$

when the electrons are on-shell, whether or not $p^2 = 0$. What is the significance of this result?
Advanced Quantum Field Theory
Example Sheet 3

Please email me with any comments about these problems, particularly if you spot an error. Problems with an asterisk (∗) may be more difficult.

1. Consider the theory given by the action

\[ S[\phi] = \int d^d x \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 + \frac{g}{3!} \phi^3 + \frac{\lambda}{4!} \phi^4. \]

where \( \phi \) is a real scalar field.

(a) Determine all connected one loop graphs, complete with their appropriate symmetry factors, which contribute to

\[ \langle \phi(x)\phi(y) \rangle, \quad \langle \phi(x)\phi(y)\phi(z) \rangle \quad \text{and} \quad \langle \phi(x)\phi(y)\phi(z)\phi(w) \rangle, \]

expressing your answer in terms of integrals over \( d \)-dimensional loop momenta. [You are not required to evaluate the integrals.]

(b) Now set \( \lambda = 0 \) so that just the cubic interaction remains. Determine the momentum space correlation function \( \int \prod_{i=1}^3 d^4 x_i e^{ip_i \cdot x_i} \langle \phi(x_1)\phi(x_2)\phi(x_3) \rangle \) to one loop accuracy.

2. Consider the theory of a (bare) scalar field \( \phi_0(x) \) and (bare) fermionic Dirac field \( \psi_0 \), with action

\[ S[\phi_0, \psi_0] = \int d^4 x \frac{1}{2} \partial_\mu \phi_0 \partial^\mu \phi_0 + \frac{1}{2} m_0^2 \phi_0^2 + \bar{\psi}_0 (i\partial + \mu_0) \psi_0 + g_0 \phi_0 \bar{\psi}_0 \gamma_5 \psi_0 + \frac{\lambda_0}{4!} \phi_0^4 \]

in four dimensional Euclidean space, where \( \gamma_5 = +\gamma_1\gamma_2\gamma_3\gamma_4 \) is the product of Dirac matrices obeying \( \{\gamma^\mu, \gamma^\nu\} = 2 \delta^{\mu\nu} \) in Euclidean signature.

(a) Show that \( (\gamma_5)^2 = +1 \) and that \( \{\gamma_5, \gamma^\mu\} = 0 \). Hence show that the action is invariant under the global transformation

\[ \phi_0 \rightarrow -\phi_0 \quad \text{and} \quad \psi_0 \rightarrow e^{-i\pi\gamma_5/2} \psi_0. \]

Assuming that the path integral measure is also invariant under this transformation, show that renormalization cannot generate any vertices involving odd powers of the scalar field unless they are accompanied by an odd power of \( \bar{\psi}_0 \gamma_5 \psi_0 \) as in the original action.
(b) Write the action in terms of renormalized fields and counterterms for the fields, masses and couplings. Use dimensional regularization and the on-shell renormalization scheme to evaluate these counterterms to 1-loop accuracy. Show that all 1-loop amplitudes are finite when expressed in terms of the renormalized couplings.

3. Scalar QED describes the interactions of a photon with a complex scalar field. In \( d \) dimensions it is defined by the action

\[
S[A, \phi] = \int d^d x \frac{1}{4} F^{\mu \nu} F_{\mu \nu} + \frac{1}{2} (D^\mu \phi)^* D_\mu \phi + \frac{m^2}{2} \phi^* \phi
\]

where \( D_\mu \phi = \partial_\mu \phi + ie A_\mu \phi \).

(a) Show that, not including counterterms, there are two distinct 1 loop Feynman graphs that contribute to vacuum polarization in scalar QED. One of these diagrams leads to an integral that is independent of the external momentum. What is its role?

(b) By considering vacuum polarization, show that when \( d = 4 \), the 1-loop \( \beta \)-function for the dimensionless coupling \( g \) corresponding to the charge \( e \) is

\[
\beta(g) = \frac{g^3}{48\pi^2}
\]

in the \( \overline{\text{MS}} \) scheme. How does the theory behave at scales far below the mass of the scalar?

4. Consider a (neutral) scalar field \( \phi \) with potential \( V(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{1}{6} \Lambda^{\epsilon/2} g(\Lambda) \phi^3 \) in dimension \( d = 6 - \epsilon \). Here \( \Lambda \) is an arbitrary mass scale so that the coupling \( g(\Lambda) \) is dimensionless.

(a) Draw the one-loop one particle irreducible graph which contributes to the propagator at order \( g^2 \).

(b) Using dimensional regularisation, show that the divergent part of the corresponding integral for the six dimensional theory is

\[
-\frac{1}{\epsilon} \frac{g^2}{(4\pi)^3} \left( m^2 + \frac{1}{6} p^2 \right),
\]

where \( p \) is the external momentum. Also compute the divergence corresponding to the one particle irreducible one-loop graph that gives a \( g^3 \) correction to three point function, and find the one loop divergence for the one point function.
(c) Show that in six dimensions all these divergences may be cancelled by introducing the counterterm action

\[ S_{\text{ct}}[\phi] = \int d^d x \, \mathcal{L}_{\text{ct}} := \frac{1}{\epsilon} \frac{1}{6(4\pi)^3} \left[ \int d^d x \, \frac{1}{2} g^2 (\partial \phi)^2 + \Lambda^{\epsilon} V''(\phi)^3 \right]. \]

Check that \( \mathcal{L}_{\text{ct}} \) has dimension \( d \).

(d) Determine the \( \beta \)-function for the coupling \( g \) and show that \( \beta(g) < 0 \) at small \( g \). Does the theory have a continuum limit in perturbation theory? Do you expect this to survive non-perturbatively?
Advanced Quantum Field Theory

Example Sheet 4

Please email me with any comments about these problems, particularly if you spot an error. Questions marked with an asterisk may be more challenging.

1. Let \( t_A \) be the generators of a Lie algebra \( \mathfrak{g} \), \([t_A,t_B] = if_{AB}^C t_C\), and let \( c^A \) be anticommuting variables. Show that

\[
Q := c^A t_A - \frac{1}{2} f_{BC}^A c^B c^C \frac{\partial}{\partial c^A}
\]

satisfies \( Q^2 = 0 \). Suppose \( t_A = 0 \) and also that \( f_{ABC} = k_{CD} f_{AB}^D \) is completely antisymmetric, where \( k_{CD} \) is the Killing form on \( \mathfrak{g} \). If \( X = f_{ABC} c^A c^B c^C \) show that \( QX = 0 \) but that \( X \neq QY \).

2. Consider a gauge-fixed action for a free (Abelian) gauge field \( A_\mu \) of the form

\[
S = \int d^Dx \left( -\frac{1}{4} F_{\mu\nu}^2 + h \partial^\mu A_\mu + \frac{\xi}{2} h^2 + \bar{c} \partial^2 c \right)
\]

where \( h \) is an auxiliary bosonic field and \((c,\bar{c})\) are anticommuting ghost and antighost fields.

(a) Verify that this action is invariant under the BRST transformations \( \delta A_\mu = \epsilon \partial_\mu c \), \( \delta c = 0 \), \( \delta \bar{c} = -\epsilon h \), \( \delta h = 0 \) and that \( \delta \) is nilpotent.

(b) Show that the action can be written in the form

\[
S = -\int d^Dx \left( \frac{1}{2} \Phi^T \Delta \Phi + \bar{c} (\partial^2 c) \right)
\]

where \( \Phi = \begin{pmatrix} A_\mu \\ h \end{pmatrix} \), \( \Phi^T \) is its transpose and where

\[
\Delta = \begin{pmatrix} -\partial^2 \delta^\mu_\nu + \partial_\mu \partial_\nu & \partial_\nu \\ -\partial^\mu & -\xi \end{pmatrix}.
\]
3. For a gauge theory coupled to scalars the single particle states are

$$|A^A(p)\rangle, \qquad |\phi_i(p)\rangle, \qquad |c^A(p)\rangle, \qquad |\bar{c}^A(p)\rangle,$$

where $A$ runs over a basis of the adjoint representation and $i$ similarly indexes the $R$ representation. These states have non-zero scalar products

$$\langle A^A_{\mu}(p)|A^B_{\nu}(p')\rangle = \eta_{\mu\nu}\delta^{AB}\delta_{pp'}, \quad \langle \phi_i(p)|\phi_j(p')\rangle = \delta_{ij}\delta_{pp'},$$

$$\langle c^A(p)|\bar{c}^B(p')\rangle = \langle \bar{c}^A(p)|c^B(p')\rangle = \delta^{AB}\delta_{pp'},$$

where $\delta_{pp'} := (2\pi)^{D-1}2p^0\delta^{(D-1)}(p - p')$. The non-zero action of the BRST charge $Q$ is given by

$$Q|A^A_{\mu}(p)\rangle = \alpha p_{\mu}|c^A(p)\rangle, \quad Q|\phi_i(p)\rangle = \sum_A v_{iA}|c^A(p)\rangle,$$

$$Q|\bar{c}^A(p)\rangle = \beta p^\mu|A^A_{\mu}(p)\rangle + \sum_i \bar{v}_{Ai}|\phi_i(p)\rangle,$$

while the ghost charge $Q_{gh}$ acts non-trivially as

$$Q_{gh}|c^A(p)\rangle = i|c^A(p)\rangle, \quad Q_{gh}|\bar{c}^A(p)\rangle = -i|\bar{c}^A(p)\rangle.$$

Verify that this is compatible with $Q$ and $Q_{gh}$ being Hermitian if $\alpha$, $\beta$, $v_{iA}$ and $\bar{v}_{Ai}$ are related appropriately. Assume a basis has been chosen so that $\sum_i \bar{v}_{Ai}v_{iB} = \delta_{AB}\rho_A$ and so is diagonal. Find the conditions under which the BRST charge $Q^2 = 0$. Use this to determine the possible physical single particle states.

4. Consider a gauge invariant Lagrangian density of the form

$$\mathcal{L}(A) = -\frac{1}{4} \text{tr} (F^\mu{}^\nu X(D^2)F_{\mu\nu}) ,$$

where $D_{\lambda}F_{\mu\nu} = \partial_{\lambda}F_{\mu\nu} + [A_{\lambda}, F_{\mu\nu}]$ and where $X(D^2) = 1 + (-D^2)^r/\Lambda^{2r}$ for some scale $\Lambda$. The full quantum Lagrangian with gauge fixing and ghost fields is

$$\mathcal{L}_q(A, c, \bar{c}) = \mathcal{L}(A) - \frac{1}{2\xi} \text{tr} (\partial^\mu A_{\mu} X(\partial^2) \partial^\nu A_{\nu}) + \text{tr} (\bar{c} X(\partial^2) \partial^\mu D_{\mu} c) .$$
(a) Show that the Feynman rules require that the gauge and ghost propagators are each proportional to $p^{-2-2r}$.

(b) Show that there must be vertices with $n$ gauge field legs $n = 3, \ldots, 2r + 4$ and with $2r + 4 - n$ powers of momentum, but that there is just a single vertex involving both ghost and gauge fields with $2r + 1$ momentum factors.

(c) Hence show that, in four dimensions, the superficial degree of divergence of an $\ell$-loop Feynman graph with $E_A$ external gauge field lines and $E_{gh}$ ghost field lines is

$$\delta = 4 - E_A - E_{gh} - 2r(\ell - 1).$$

5. *Consider pure (= no charged matter) electrodynamics with Lagrangian $L = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu}$.

Let $W_\gamma[A] := \exp \left( i \oint_\gamma A_\mu \, dx^\mu \right)$ be a Wilson loop around a closed curve $\gamma$.

(a) Show that

$$\langle W_\gamma[A] \rangle = \exp \left[ -\frac{e^2}{8\pi^2} \oint_\gamma dx^\mu \oint_\gamma dy^\mu \frac{1}{(x - y)^2} \right].$$

(b) Now suppose $\gamma$ is a large rectangle with space-like width $L$ and time-like length $T$. Compute $\langle W_\gamma[A] \rangle$ in the limit $T \gg L$. By comparing your result to the usual expression for time evolution, show that the potential between two point-like charges at fixed separation $L$ in electrodynamics is $V(L) = -e^2/4\pi L$.

(c) In Feynman gauge, the propagator for a non-Abelian gauge field is

$$\langle A^R_\mu(x) A^C_\nu(y) \rangle = i\eta_{\mu\nu} \delta^{BC} \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-ip \cdot (x - y)}}{p^2}.$$ 

Compute the expectation value of a Wilson loop in pure SU($N$) Yang-Mills theory to lowest non-trivial order in the coupling $g^2$. [Your result should depend on a choice of the representation $R$ of $\mathfrak{su}(N)$.]  

(d) Show that, to this order, the Coulomb potential of non-Abelian gauge theory is $V(L) = -g^2 C_2(R)/4\pi L^2$, where $C_2(R)$ is the quadratic Casimir of the $R$ representation.
\[ \int d^n \bar{\sigma} \exp \left( \bar{\sigma} \cdot B \cdot \sigma + \bar{\eta} \cdot \sigma + \bar{\sigma} \cdot \eta \right) \]

Consider first \[ \int d^n \bar{\sigma} \exp \left( \bar{\sigma} \cdot B \cdot \sigma \right), \quad d^n \bar{\sigma} = n \, d^n \bar{\sigma} \]

\[ = \int d^n \bar{\sigma} \frac{1}{n!} \left( \bar{\sigma} \cdot B \cdot \sigma \right)^n \]

\[ = \frac{1}{n!} \prod_{i=1}^{n} \text{E}_{\bar{\eta}_i} \prod_{i=1}^{n} \text{E}_{\sigma_i} \prod_{i=1}^{n} B_{ij} \quad \text{Bin}_{ij} \]

\[ = \det B \sum_{\text{perm}} \int d^n \bar{\sigma} \ldots d^n \sigma \]

Now \[ \bar{\sigma} \cdot B \cdot \sigma + \bar{\eta} \cdot \sigma + \bar{\sigma} \cdot \eta = \bar{\sigma} \cdot B \cdot \sigma - \bar{\eta} \cdot (B')^{-1} \cdot \eta \]

when \[ \sigma' = \sigma + B' \cdot \eta \]

\[ \bar{\sigma}' = \bar{\sigma} + \bar{\eta} \cdot B' \]

\[ \Rightarrow \int d^n \bar{\sigma}' d^n \sigma' \exp \left( \bar{\sigma}' \cdot B' \cdot \sigma' \right) \exp \left( - \bar{\eta}_i (B')^{-1} \cdot \eta_i \right) \]

\[ = \det B \exp \left( - \bar{\eta}_i (B')^{-1} \cdot \eta_i \right) \]
\[ \langle \theta_i \bar{\theta}_j \rangle = \frac{1}{\text{det} B} \int \prod_{i=1}^{n} d\bar{\theta}_i d\theta_i \exp \left( -\bar{\theta}_i B_{ij} \theta_j \right) \]

\[ = \frac{1}{\text{det} B} \int \prod_{i=1}^{2n} d\bar{\theta}_i d\theta_i \exp \left( \bar{\theta}_i B_{ij} \theta_j + \bar{\theta}_i \bar{\theta}_j + \bar{\theta}_i \eta_{ij} \right) \]

\[ = -\frac{2}{\eta_{ij}} \exp \left( -\bar{\theta}_i B_{ij} \theta_j \right) \]

\[ = -B_{ij} \]

\[ \Rightarrow \langle \theta_{i_1} \theta_{i_2} \ldots \theta_{i_n} \bar{\theta}_{j_1} \bar{\theta}_{j_2} \ldots \bar{\theta}_{j_s} \rangle \sim \frac{1}{\eta_{i_1}} \frac{1}{\eta_{i_2}} \cdots \frac{1}{\eta_{i_n}} \frac{1}{\eta_{j_1}} \frac{1}{\eta_{j_2}} \cdots \frac{1}{\eta_{j_s}} \exp \left( -\bar{\theta}_{i_1} B_{i_1 j_1} \theta_{j_1} \right) \ldots \exp \left( -\bar{\theta}_{i_n} B_{i_n j_s} \theta_{j_s} \right) \]

up to numerical factors

\[ \Rightarrow \langle \theta_{i_1} \theta_{i_2} \ldots \theta_{i_n} \bar{\theta}_{j_1} \bar{\theta}_{j_2} \ldots \bar{\theta}_{j_s} \rangle = 0 \text{ unless } n = s \]

\[ \text{Wick's th. only fully contracted terms survive} \]

Also if \( n \leq s \) then the integral

\[ \int d\bar{\theta} d\theta \exp \left( \bar{\theta}_i B_{ij} \theta_j \right) \bar{\theta}_i \cdots \bar{\theta}_i \theta_i \cdots \theta_i \]

\[ \text{Trivially vanishes} \]

\[ \int d\bar{\theta} \theta_i \cdots \theta_i = \begin{cases} 1 \text{ for } n = u, \text{otherwise} \end{cases} \]
This integral can actually be integrated exactly in terms of a modified Bessel function.

\[ Z(\gamma) = \left( \frac{3}{2\pi} \right)^{1/2} \exp \left( \frac{3}{4\gamma} \right) K_{1/4} \left( \frac{3}{4\gamma} \right), \text{ singular at } \gamma < 0. \]

Now expand integral in \( \gamma \to \) shift \( x \to \sqrt{2\gamma} \)

\[ Z = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} dx \exp(-x^2) \exp \left( -\frac{\gamma x^4}{6} \right). \]

Expand \( \exp(-\frac{\gamma x^4}{6}) \) in a power series up to \( o\left(\frac{1}{x^6}\right) \)

for each integral in the series we use

\[ \int_{-\infty}^{+\infty} dx \exp(-x^2) x^{2n} = \frac{\Gamma(1+2n)}{\Gamma(n+1)} \]

\[ Z_N = \frac{1}{(2\pi)^{1/2}} \sum_{n=0}^{N} \frac{\Gamma(\frac{1}{2}+2n)}{\Gamma(1+n)} \left( -\frac{\gamma}{6} \right)^n \]

\[ \Gamma \left( \frac{1}{2} + n \right) = (2n)! \sqrt{\pi} \]

\[ \Rightarrow Z_N = \sum_{n=1}^{\infty} \frac{(-\gamma)^n}{4^n n!} \frac{(4!)}{2^{2n} (2n)! n!} \]

\[ \ln Z_N(\gamma) = \frac{-1}{8} \gamma + \frac{1}{12} \gamma^2 - \frac{11}{96} \gamma^3 + o(\gamma^4) \]

Now check diagrammatically:

\[ o(\gamma^3): \quad \infty \quad \Rightarrow \text{coeff} = \frac{1}{8} \]

\[ s = 2 \cdot 2 \cdot 2 = 8 \]
$O(x^2):$

\[ s = 2 \cdot 2 \cdot 2 \cdot 2 = 16. \]

$\Rightarrow \text{Coeff} = \frac{1}{16} + \frac{1}{48} = \frac{1}{12}.\]

$O(x^3):$

\[ s = 3! \cdot 2 \cdot 2 \cdot 2 = 24.\]

\[ s = 3! \cdot 2^3 = 48.\]

\[ s = 3! \cdot 2^3 = 48.\]

\[ \text{Coeff} = \frac{11}{96}.\]

(4) plots \to see later.
(c) \[ a_k x^k = 2 \quad (k!) \quad \implies d \ln |a_k x^k| = 0. \]

\[ \frac{d}{de} \left( \ln 2^e \right) \cdot \frac{1}{e!} \]

\[ \frac{d}{de} \left( \ln 2^e - \ln 2^e + \ln (4e^4) \right) = 0 \]

\[ \ln 2 - \ln 2^2 + \ln (4e^4) \frac{1}{(2e)^2} = 0 \]

\[ \ln 2 - \ln 2^2 + \ln (4e^4) \frac{1}{4e^4} = 0 \]

\[ \ln 2 - \ln \frac{1}{e^3} \]

\[ \ln 2 \quad \frac{1}{2} \]

To show that the Borel transformation converges, consider d'Alembert's ratio test.

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1 \implies \text{series cgt} \]

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1 \implies \text{series dgt} \]

\[ = 1 \implies \text{Test fails.} \]

for the Borel sum, we get

\[ \lim_{l \to \infty} \left| \frac{a_l}{(l+1)!} \right| \]

\[ \lim_{l \to \infty} \left| \frac{(4l+3)}{(l+1)^2} \right| \]

\[ \lim_{l \to \infty} \left| \frac{(4l+1)}{4l} \right| \]

\[ \lim_{l \to \infty} \left| \frac{2l}{3} \right| < 1 \implies |\lambda| < \frac{3}{2} \]

\[ Z(\lambda) = \sum_{l=0}^{\infty} z e^{-z} a_l (z^2)^l = \sum_{l=0}^{\infty} \frac{\lambda^l a_l}{l!} \int_0^\infty dz e^{-z} z^l \]

\[ \left( \frac{\lambda}{(l+1)!} \right) = \lambda^l 
\]

\[ \sum_{l=0}^{\infty} \lambda^l z^l = Z(\lambda). \]
(a) Strong coupling expansion \( \left( \frac{1}{x} \right) \), shift \( x \rightarrow \left( \frac{24}{x} \right)^{1/4} \).

\[
\phi(\gamma) = \frac{1}{\sqrt{2\pi}} \left( \frac{24}{\gamma} \right)^{1/4} \int_{-\infty}^{+\infty} dx \ e^{-x^2} \exp \left[ - \left( \frac{6}{x^2} \right) x^2 \right]
\]

\[
= \frac{1}{\sqrt{2\pi}} \left( \frac{3}{2\gamma} \right)^{1/4} \sum_{n=0}^\infty \frac{\Gamma \left( \frac{1}{4} + \frac{n}{2} \right)}{\Gamma(1+n)} \left( -\left( \frac{6}{\gamma^2} \right) \right)^n
\]

\[
= \frac{1}{2\sqrt{\pi}} \sum_{n=0}^\infty \frac{(-1)^n}{n!} \frac{\Gamma \left( \frac{n+1}{4} \right)}{\Gamma \left( \frac{n}{2} + \frac{1}{4} \right)} \left( \frac{6}{\gamma^2} \right)^{n/2 + 1/4}
\]

\( (\text{QED}) \)

→ SU graphs.
Question (3)

\[ S(\phi, \psi, \bar{\psi}) = \frac{m^2 \phi^2}{2} + M \bar{\psi} \psi + \lambda \phi \bar{\psi} \psi \]

a) There's a propagator for each field

\[ \langle \phi \phi \rangle = \frac{1}{m^2} \]

denote \( \langle \phi \phi \rangle \longrightarrow \) \( \frac{1}{m^2} \)

\[ \langle \bar{\psi} \bar{\psi} \rangle = \frac{1}{M} \]

denote \( \langle \bar{\psi} \bar{\psi} \rangle \longrightarrow \) \( \frac{1}{M} \)

and one interaction written \( \longrightarrow \lambda = -2 \)

b) Integrate out the fermions

\[ \int d\psi d\bar{\psi} \exp \left( -M \bar{\psi} \psi + 2 \phi \bar{\psi} \psi \right) \]

\[ = \int d\psi d\bar{\psi} \left( 1 - \bar{\psi} \psi (M + 2\phi) + \ldots \right) \]

\[ = -\left( (M + 2\phi) \right) \]

The full path integral is \( \int d\phi \exp \left( -\frac{m^2 \phi^2}{2} (M + 2\phi) \right) \)

\[ = -\int d\phi \exp \left( \frac{m^2 \phi^2}{2} \right) \exp \log(M + 2\phi) \]

\[ \Rightarrow S_{\text{eff}} = \frac{m^2 \phi^2}{2} - \log(M + 2\phi) \]

\[ = \frac{m^2 \phi^2}{2} - \log \left( 1 + \frac{2\phi}{M} \right) - \log(M) \]
\[ S_{\text{eff}} = \frac{n^2 \phi^2}{2} + \sum_{n=1}^{\infty} \left( \frac{(-i)^{n} (\lambda \phi)^{n}}{n!} \right) - \log M \]

\[ = \frac{n^2 \phi^2}{2} + \sum_{n=1}^{\infty} \frac{\lambda_n \phi^n}{n!} - \log M \]

with \( \lambda_n = (-1)^n \frac{(n-1)!}{(n\lambda)^n} \)

\[ \Rightarrow \text{The more massive the fermions better this expansion become.} \]

With this effective action we can calculate correlation functions of the form \( \langle \phi^k \rangle \ \forall k \in \mathbb{Z}^+ \)

c) Define \( Z_0 = Z(\lambda = 0) = \int d\phi \, d\psi \, d\bar{\psi} \exp\left( -\frac{m^2}{2} \phi^2 + M \bar{\psi} \psi \right) \)

\[ Z_0 = - \frac{1}{2T} \frac{M}{m} \]

First calculate using the original action (tadpoles ignore)

\[ \langle \phi^2 \rangle = \frac{1}{2} \langle (\phi^2) (2 \lambda \bar{\psi} \psi)^2 \rangle_0 = \frac{\lambda_2}{2} \langle (\bar{\psi} \psi)^2 \rangle_0 = 0 \]

since \( \psi^2 = 0 \)

\[ \frac{2Z_0}{\lambda^2} = \frac{2Z_0}{2Z_0} \]

Using \( S_{\text{eff}} \), \( \langle \phi^2 \rangle = 0 \)

\[ \frac{2Z_0}{\lambda^2} - \frac{2Z_0}{2Z_0} \]

From \( \lambda_2 - \lambda_2 = \frac{\lambda_2}{2} - \frac{\lambda_2}{2} = 0 \)

\[ \frac{M^2}{M^2} \]

\[ \Rightarrow \text{This occurs because interactions of order } \lambda^n, \ n > 1 \]

vanish since \( \int d\phi \, d\bar{\psi} \exp(-\lambda \phi \bar{\psi}) = -\lambda \phi \)

\[ \Rightarrow \text{Implies non-trivial cancellations when using the effective action.} \]
\[ S = \int dz \left[ \frac{1}{2} \left( \frac{\partial \bar{\psi}}{\partial z} \right)^2 + \bar{\psi} \frac{\partial^2 \psi}{\partial z^2} + \frac{1}{2} \lambda^2 \phi(x)^2 + \lambda \phi(x) \bar{\psi} \psi \right] \]

\[ \Rightarrow S_S = \int dz \left[ \frac{\partial \bar{\psi}}{\partial z} \frac{\partial \psi}{\partial z} + \bar{\psi} \frac{\partial^2 \psi}{\partial z^2} + \frac{1}{2} \lambda^2 \phi(x)^2 + \lambda \phi(x) \bar{\psi} \psi \right] \]

\[ \Rightarrow S_S = \int dz \left[ -\bar{\psi} \frac{\partial^2 \psi}{\partial z^2} + \bar{\psi} \frac{\partial \psi}{\partial z} + \frac{1}{2} \lambda^2 \phi(x)^2 + \lambda \phi(x) \bar{\psi} \psi \right] \]

\[ = \int \left[ -\bar{\psi} \frac{\partial \psi}{\partial z} + \bar{\psi} \phi(x) + \frac{1}{2} \lambda^2 \phi(x)^2 + \lambda \phi(x) \bar{\psi} \psi \right] dz \]

\[ \Rightarrow S_S = 0 \]

\( \Rightarrow S_S = 0 \)

To get the changes we take the variational parameter \( e \rightarrow e(t) \), then

\[ \delta S = -\int (\dot{e} \bar{\psi} + \dot{\bar{\psi}} \psi) dt \]

\[ \Rightarrow \delta = \bar{\psi} (-\dot{x} + \lambda \dot{h}) \quad \text{and} \quad \bar{\delta} = \psi (\dot{x} - \lambda \dot{h}) \]

\[ \delta e \partial S = \int d \bar{\psi} \partial \psi + \cdots \]

\[ = \int \bar{\psi} \delta \psi \left( -\dot{x} + \lambda \dot{h} \right) + \text{d}(\mathcal{L}) \]

\[ = -\int \bar{\psi} \psi \left( \dot{\bar{x}} - \lambda \dot{\phi} \right) + \text{d}(\mathcal{L}) \]

\[ \Rightarrow \delta \]
\[
\frac{\partial L}{\partial \dot{\psi}} = \dot{p} = \frac{\partial \bar{L}}{\partial \dot{\bar{\psi}}} = \dot{\bar{p}} = \psi (x + \chi(x))
\]

\[
\begin{align*}
\delta = \bar{\psi} (\bar{\psi} + \chi'(x)) & , \bar{\delta} = \psi (\psi + \chi'(x)) \\
\{ \delta, \bar{\delta} \} &= \{ -\bar{\psi} (-\bar{p} + \chi') , \psi (\bar{p} + \chi') \} \\
&= -\{ \bar{\psi} , \bar{p} \} p^2 + \bar{\psi} \chi'' + \{ \psi , \bar{\psi} \} \bar{p}^2 - \bar{\psi} \chi' \bar{p} - \psi \chi' \bar{p} + \{ \psi , \psi \} \bar{p}^2 \\
&= \bar{p}^2 - \chi' \bar{p}^2 - \bar{\psi} \chi' \bar{p} - \psi \chi' \bar{p} + \psi \bar{\psi} \chi' \bar{p} + \bar{\psi} \chi' \bar{p} \\
&= \bar{p}^2 - \chi' \bar{p}^2 + \bar{\psi} \chi' \bar{p} - \psi \chi' \bar{p} \\
&= \bar{p}^2 - \chi' \bar{p}^2 + \chi (\bar{\psi} - \psi) \bar{p} \\
&= 2 \bar{H}.
\end{align*}
\]

\[
\bar{H} = q \bar{p} - \bar{L}
\]

\[
\begin{align*}
\bar{L} &= \frac{1}{2} \bar{\psi}^2 + \frac{1}{2} \bar{p}^2 + \frac{1}{2} (\bar{\psi} \psi + \bar{\psi} \bar{\psi}) + \chi'' (\bar{\psi} \psi - \bar{\psi} \bar{\psi}) \\
\Rightarrow \bar{\Delta} = \frac{\partial \bar{L}}{\partial \bar{\psi}} = \bar{\psi} , \ \bar{\Lambda} = \frac{\partial \bar{L}}{\partial \bar{p}} = -\bar{p} , \ \bar{\Delta} = \frac{\partial \bar{L}}{\partial \psi} = -\psi \\
\Rightarrow \bar{H} &= \frac{\bar{p}^2}{2} - \frac{\chi' \bar{p}^2}{2} + \frac{\chi (\bar{\psi} - \psi)}{2}
\end{align*}
\]

\[
\begin{align*}
\langle \psi | \bar{H} | \psi \rangle &= 2 E \psi \\
&= \langle \psi | \bar{\Delta} | \psi \rangle + \langle \psi | \bar{\Lambda} | \psi \rangle = |\bar{\psi} \psi|^2 + |\bar{\psi} \bar{\psi}|^2 > 0.
\end{align*}
\]

\[
E \psi = 0 \Rightarrow \bar{\psi} \bar{\psi} = \bar{\psi} \psi = 0. \text{ ground state.}
\]
\[
Z = \int D\varphi \bar{D}\bar{\varphi} e^{-S}
\]

Independent of \( \varphi \):\( d e^{-S} = -e^{-S} \int d^2 \varphi \frac{d}{d^2 \varphi} = -e^{-S} \int d^2 \varphi \left( \frac{\partial}{\partial \varphi} \left( i \frac{\partial}{\partial \varphi} \right) \right) = \frac{2 \pi}{2 \pi} \int d^2 \varphi \frac{\partial}{\partial \varphi} \left( i \frac{\partial}{\partial \varphi} \right) = 0.\]

Where we can verify that

\[
\frac{\partial}{\partial \varphi} \left( i \frac{\partial}{\partial \varphi} \right) = -\int_{\Sigma} \{ \varphi, \varphi^* \} = 0.
\]

The last equality follows by noting that in a QFT with a symmetry, all correlation functions of quantities that are variations of other fields under the symmetry vanish. Or

\[
\left\langle f \right\rangle = \int e^{-S} = \int S g e^{-S} = \int S (g e^{-S}) = 0 \quad \text{as } g \to 0.
\]

Independence of \( \text{radius } \beta \):

Consider \( \beta \): \( T e^{-\beta H} \)

\[
\frac{dZ}{d\beta} = -\int \{ \bar{\varphi}, \varphi \} = 0 \quad \text{hence } Z \text{ is independent of the radius } \beta.
\]

\[
Z = \int D\varphi \bar{D}\bar{\varphi} e^{-S} \quad \text{corresponds to the periodic Witten index } \Rightarrow \text{Tr} (-1)^F = \text{Tr} (-1)^F e^{-\beta H}.
\]

\( F = \) Fermion No. operator
loosely speaking the diluted index counts the number of bosonic ground states minus the number of fermionic ground states which is an invariant for a SUSY theory.

(c) Follows from localisation in SUSY QFT - If action invariant under SUSY ⇒ path integral localises to regions where the super symmetrical variations of fermionic fields vanish.

A rough justification: -

If \( \delta A \neq 0 \) anywhere then we can use the SUSY transformation of \( U, \bar{U} \) to get rid of one of the fermionic fields in the action and then using the rules of Grassman integration it can be shown that the path integral vanishes. However, if \( \delta A = 0 \) the above change of variables to eliminate one of the fermionic fields fails.

This can also be seen from the deformation invariance of SUSY QFTs which we used in part (c) of this question. Since the theory is invariant under \( h \rightarrow \lambda h \) in the \( \lambda \rightarrow \infty \) \( e^{-S} \) is very small except in the vicinity of the critical points of \( h \) (i.e \( \delta h = 0 \)).

So \( S + S = 0 \Rightarrow \frac{dx}{dc} = h'(x) = 0 \).

Say critical pts - \( x_1, \ldots, x_n \).

Expanding the action in the neighbourhood of these pts -

for \( x_i \) with \( \xi_i = x - x_i + \text{h.c.p. terms} \quad (\text{quadratic terms}) \)

\[
S = \int_0^L dc \left[ \frac{\xi_i}{2} \left( \delta \frac{d}{dc} + h''(x_i) \xi_i \right) + \bar{\xi}_i \left( \delta + h''(x_i) \xi_i \right) \right]
\]
\[ e^S = \prod_{\mathbf{x}_i} e^{-S(\mathbf{x}_i)} \text{ is gaussian and we} \]

\[ \text{periodic} \quad \text{get} \]

\[ \frac{\det \left( \partial^2 + h''(\mathbf{x}_i) \right)}{\det \left( -\partial_x^2 + h''(\mathbf{x}_i)^2 \right)} \]

\[ = \prod_{n \in \mathbb{Z}} \left( \frac{i n + h''(\mathbf{x}_i)}{n^2 + h''(\mathbf{x}_i)^2} \right) \quad \text{(fourier modes indexed by \( n \) cancel)} \]

\[ = \prod_{n \in \mathbb{Z}} \frac{(-i + h'')(i + h'')}{(i + h'')(-i + h'')(i + h'')(i + h'')(i - h'') \cdots} \]

\[ = \frac{h''}{|h''|^2} \]

\[ \Rightarrow \text{Tr} \left( -1 \right)^F e^{-\beta H} = \sum_{f=1}^N \text{sign} \left( h''(\mathbf{x}_i) \right) \]

\[ \uparrow \]

Sum over all \( \mathbf{x}_i \)
6) From the previous question

\[ m_0^2 = \frac{m^2 + B}{1 + A} = m^2 \left( 1 - \frac{1}{e^{2q^2}} \right) \left( \frac{1 + \frac{1}{e^{2q^2}}}{1 + \frac{q^2}{384x^3}} \right) + O(q^4) \]

\[ = m^2 \left( 1 + \frac{5q^2}{256x^3} \right) + O(q^4) \]

\[ g_0 = \mu \frac{e^{1/2}}{1 + e} = \mu \frac{e^{1/2}}{1 + \frac{q^2}{256x^3}} \left( 1 - \frac{1}{e} \frac{q^2}{256x^3} \right) + O(q^5) \]

\[ \beta_g = (g_0 g - 1) f_1 = \frac{3g^3}{256x^3}, \text{ decreasing \( \Rightarrow \) asymptotic freedom} \]

\[ \gamma_m^2 = g_0 \frac{d_g}{d y} = \frac{5g^2}{128x^2} \]

4) multiple couplings \( g_i \), change variable \( g \rightarrow g'(g) \).

\[ \beta'(g') = \mu \frac{d g'}{d \mu} \left( \frac{d g}{d g'} \right) \left( \frac{d g}{d \mu} \right) - \frac{d g'}{d \mu} \frac{d g}{d g'} \beta(g) \]

for \( \beta(g) = b_1 g^3 + b_2 g^5 + O(g^7) \) and \( g'(g) = g + a g^3 + O(g^5) \)

\[ \Rightarrow \beta'(g') = \left( 1 + 3a g^2 + O(g^4) \right) \left( b_1 (g^1 - a g^3)^3 + b_2 g^5 + O(g^7) \right) \]

\[ = b_1 g_1^3 + b_2 g_1^5 + O(g_1^7) \Rightarrow 1\text{st}, 2\text{nd} \text{ terms invariant} \]

when we inverted the perturbative series term by term

\[ e_1 \Rightarrow x = \alpha y + \beta y^2 + O(y^3) \]

\[ \Rightarrow y(x) = \frac{x}{\alpha}, \quad y(\alpha) = x = \alpha y_0 + \beta \left( \frac{x}{\alpha} \right)^2 \]

\[ \Rightarrow y(\alpha^2) = \frac{x}{\alpha} - \frac{\beta}{\alpha^2} x^2 + O(x^3) \]
Consider higher order terms:

\[ \beta(g) = b_1 g^3 + b_2 g^5 + b_3 g^7 + O(g^8) \]
\[ g' = g^a + a_1 g^3 + a_2 g^4 + a_3 g^5 + O(g^6) \]

\[ \Rightarrow \beta(g') = \left(1 + 3a_1 g^2 + 4a_2 g^3 + 5a_3 g^4\right) \left(b_1 g^3 + b_2 g^5 + b_3 g^7 + O(g^8)\right) \]
\[ = b_1 g^3 + (b_2 + 3a_1 b_1) g^5 + 4a_2 b_1 g^6 + \]
\[ + (b_3 + 3a_1 b_2 + 5a_3 b_1) g^7 + O(g^8) \]

\[ = b_1 \left(g' - a_1 g^{13} - a_2 g^{14}\right)^3 + \left(b_2 + 3a_1 b_1\right) \left(g' - a_1 g^{13}\right)^5 \]
\[ + 4a_2 b_1 g' \left(g' - a_1 g^{13}\right)^3 + \left(b_3 + 3a_1 b_2 + 5a_3 b_1\right) g' + O(g^8) \]
\[ = b_1 g' + b_2 g' + (4a_1 b_1 - 3a_2) g' + \]
\[ + (b_3 + 3a_1 b_2 + 5a_3 b_1 - 3a_2, b_1 + 3a_2 b_1 + 3a_3, g') + O(g^8) \]

\[ g' \to 0 \text{ if } a_2 = 0 \]
\[ g' \to 0 \text{ for appropriate choice of } a_1, a_3 \]

\[ \text{\Rightarrow first 2 coefficients are universal, others are scheme dependent.} \]
\[ B(q) = -b_1 q^3 - b_2 q^5 + O(q^7) \]

\[ \frac{d}{dq} B(q) = \frac{1}{q} = b_1 \ln \frac{q^2}{\Lambda^2} + b_2 \ln \left( \frac{q^2}{\Lambda^2} \right) + O\left( \frac{1}{q^2} \right) \]

\[ \frac{d}{dq} \frac{1}{q} = -b_1 q^{-3} - b_2 q^{-5} = B(q) \]

\[ \frac{d}{dq} \left( \frac{1}{q^2} \right) = -2b_1 + \frac{2}{q^2} \]

\[ \frac{d}{dq} \ln(q^2) = b_1 \ln(q^2) \]

\[ \Lambda^2 = \mu^2 \exp\left( -\frac{1}{b_1 q^2} \right) \left( b_1 q^2 \right)^{-b_2/b_1} \]

Now suppose \[ \bar{q} = q + \alpha q^3 + O(q^5) \]

\[ \frac{1}{\bar{q}^2} = \frac{1}{q^2} \frac{1}{(1 - 2\alpha q^2)} = 1 - 2\alpha q^2 \]

\[ = b_1 \ln \frac{\mu^2}{\Lambda^2} + b_2 \ln \left( \frac{\mu^2}{\Lambda^2} \right) - 2\alpha \]

\[ = b_1 \ln \frac{\mu^2}{\Lambda^2} + b_2 \ln \left( \frac{\mu^2}{\Lambda^2} \right) - 2\alpha + b_3 \ln \left( \frac{\mu^2}{\Lambda^2} \right) \]

\[ \ln \left( \frac{\mu^2}{\Lambda^2} + \Lambda^2 \right) = \ln \left( \frac{\mu^2}{\Lambda^2} \right) + \ln \left( 1 + \frac{\Lambda^2}{\mu^2} \right) \]

\[ = b_1 \ln \left( \frac{\mu^2}{\Lambda^2} \right) + b_2 \ln \left( 1 + \frac{\Lambda^2}{\mu^2} \right) \]

\[ \bar{q}^2 = b_1 \ln \frac{\mu^2}{\Lambda^2} + b_2 \ln \left( \frac{\mu^2}{\Lambda^2} \right) + O\left( \frac{1}{\ln \frac{\mu^2}{\Lambda^2}} \right) \]

Thus, \[ b_1 \ln \frac{\mu^2}{\Lambda^2} = 2\alpha \] change of scale changes the scale.
The dimension of $\psi, \bar{\psi}$ and $A$ depend on their kinetic term

$$\int d^4x \left( \bar{\psi} \psi + \bar{\psi} \left( m + \frac{g A_{\mu} \sigma^{\mu \nu} A_{\nu}}{4} \right) \right)$$


Gauge-invariant operators have to be built out of $D \psi, D \bar{\psi}, F_{\mu \nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$ etc...

@ $D=4$ $[\psi] = 3/2$, $[A] = 1$, $[g] = 0$ $\rightarrow$ dimensionless

Ups: $\bar{\psi} \slashed{D} \psi = \bar{\psi} \left( \gamma^{\mu} \partial_{\mu} + m \right) \psi$

In $\bar{\psi} \psi$ where $m \bar{\psi} \psi$ is the mass "coupling",

$T \left( F_{\mu \nu} (\ast F)_{\mu \nu} \right) = F_{\mu \nu} F_{\rho \sigma} \epsilon^{\mu \nu \rho \sigma}$ $\rightarrow$ topological term

@ $D=3$ $[\psi] = 1$, $[A] = 1/2$, $[g] = 1/2$ $\rightarrow$ Relevant!

Ups: $\bar{\psi} \slashed{D} \psi, m \bar{\psi} \psi, T \left( F_{\mu \nu} F^{\mu \nu} \right) = T \left( \frac{1}{2} \partial_{\mu} A_{\nu} + g a_{\mu} A_{\nu} \right) \left( \partial_{\nu} A_{\mu} + g a_{\nu} A_{\nu} \right)$

has relevant and marginal pieces

@ $D=2$ $[\psi] = \frac{1}{2}$, $[A] = 0$, $[g] = 1$

Ups: $\bar{\psi} \slashed{D} \psi, m \bar{\psi} \psi, T \left( F_{\mu \nu} F^{\mu \nu} \right); T \left( \ast F \right) \rightarrow$ only non $U(1)$ piece

$\rightarrow$ Topological

Gauge invariance greatly restricts the operators and can add to the Lagrangian.
4D theory with relevant coupling $m^2$ and marginal coupling $g$

a) To lowest order the most general β-function are:

$$\beta_{m^2} = \lambda \frac{\partial m^2}{\partial \lambda} = -2m^2 + ag$$

I → classical contribution from non-dimension analysis

$$\beta_g = \lambda \frac{\partial g}{\partial \lambda} = 0 + b m^2$$

b) To solve it we first diagonalize the equation

$$\lambda \frac{\partial (m^2)}{\partial \lambda} = \begin{pmatrix} -2 & a \\ b & 0 \end{pmatrix} \begin{pmatrix} m^2 \\ g \end{pmatrix}$$

$M$ has eigenvalues $\lambda^\pm = -1 \pm \sqrt{1 + ab}$

Let $g^\pm$ be eigenvector of $M$, $\lambda \frac{\partial g^\pm}{\partial \lambda} = \lambda g^\pm$

The solution is $g^\pm(\lambda) = \begin{pmatrix} \lambda^\pm \\ \mu \end{pmatrix}$

$\mu$ is a mass scale coming from the integration constant

Write $g = \alpha g^+ + \beta g^-$, $\alpha \beta - \gamma \delta = 1$

$$m^2 = \gamma g^+ + \delta g^-$$

Where we rescaled the solution to absorb a constant factor

$$g = \frac{\gamma g^+ + \lambda g^-}{\lambda^+ - \lambda^-}$$

$$m^2 = -\left( \frac{\alpha g^+ + \lambda g^-}{\lambda^- \gamma} \right)$$
and \( g_+ = \frac{1}{(\lambda_+ - \lambda_-)(\lambda_+^2 + \lambda_-^2)} \)
\( g_- = \frac{(-m^2 + a g)}{\lambda_-} \)

So in terms of the scale \( \mu \)
\[
g(\Lambda) = g(\mu) \left[ \frac{\lambda_+ (\Lambda)}{\mu} \right]^2 - \lambda_- \left( \frac{\Lambda}{\mu} \right)^2 \\
+ m^2(\mu) \frac{\lambda_+ + \lambda_-}{\lambda_+ - \lambda_-} \left[ \left( \frac{\Lambda}{\mu} \right)^2 + \left( \frac{\Lambda}{\mu} \right)^2 \right]
\]
\[
m^2(\Lambda) = \frac{m^2(\mu)}{\lambda_+ - \lambda_-} \left[ \lambda_+ \left( \frac{\Lambda}{\mu} \right)^2 + \lambda_- \left( \frac{\Lambda}{\mu} \right)^2 \right] \\
+ g(\mu) \frac{a}{\lambda_+ - \lambda_-} \left[ \left( \frac{\Lambda}{\mu} \right)^2 - \left( \frac{\Lambda}{\mu} \right)^2 \right]
\]

For simplicity, take \( b = 0 \). The flow equations are now
\[
g(\Lambda) = g(\mu) \\
m^2(\Lambda) = \left( \frac{\Lambda}{\mu} \right)^2 m^2(\mu) - g(\mu) a \left( 1 - \left( \frac{\Lambda}{\mu} \right)^2 \right)
\]

Suppose that at the UV (\( \Lambda \approx 10^{13} \text{ GeV} \)) we make a small change \( \Delta m^2(\Lambda) \). What is the effect on the IR physics (\( \mu \approx 10^5 \text{ GeV} \))?

\[
\Delta m^2(\Lambda) = \left( \frac{10^5}{10^{13}} \right)^2 \Delta m^2(\mu) = 10^{-20}
\]
\[ \Delta m^2(\mu) = 10^9 \text{ GeV} \rightarrow \text{there's a huge sensitivity of the IR couplings on their UV values.} \]

This is referred to as the fine-tuning problem.
Scalar QED

\[ L = (\partial \mu \phi^*) (\partial \mu \phi) - \frac{1}{2} \phi \phi^* - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} \]

\[ D \mu \phi = \partial \mu \phi - ie A_\mu \phi \]

Now under C: \( \phi \leftrightarrow \phi^* \)

Consider the following interaction terms in \( L \):

\[ C \left( ie A_\mu \phi^* \partial_\mu \phi - ie A_\mu \phi \partial_\mu \phi^* \right) \]

\[ = ie A_\mu \phi^* \partial_\mu \phi - ie A_\mu \phi \partial_\mu \phi^* \]

for invariance \( A_\mu^C = -A_\mu \).

(b) \[ \langle 0 | \bar{T} A_{\mu_1} \cdots A_{\mu_n} | 0 \rangle \]

\[ = \langle 0 | C A_{\mu_1} \cdots C A_{\mu_n} | 0 \rangle \]

If this is a symmetry of the theory then it must leave the vacuum invariant

\[ \langle 0 | 0 \rangle = 1 \]

\[ \langle 0 | C \rangle = \langle 0 | C^2 \rangle = \langle 0 | 1 \rangle \]

and it must leave n-point correlators invariant

\[ \langle 0 | A_{\mu_1} \cdots A_{\mu_n} | 0 \rangle (-1)^n \]

\[ = - \langle 0 | A_{\mu_1} \cdots A_{\mu_n} | 0 \rangle \]

if \( n \) is odd.

hence odd correlators vanish.

(c) This holds at the level of the path integral so true for off shell photons etc.

\[ \rightarrow \text{true for sub diagrams! (leads to simplification)} \]
To see this, consider \( \text{L} \) ge d \( \alpha \) \( \psi \gamma^\mu \psi A^\mu \).

Now under charge conjugation \( \psi \gamma^\mu \psi \rightarrow -\psi \gamma^\mu \psi \)

\[
\hat{c} \psi \gamma^\mu \psi \hat{c}^{-1} = \frac{1}{2} \left( \gamma^\mu \right)_{\alpha \beta} \left[ \hat{c} \psi \gamma^\mu (x) \hat{c}^{-1} , \hat{c} \psi \gamma^\mu (x) \hat{c}^{-1} \right] \\
= -\frac{1}{2} \left( \gamma^\mu \right)_{\alpha \beta} \left[ \left( \psi (x) \gamma^\mu \right) x_{\alpha} , c \left( \psi (x) \gamma^\mu \right) x_{\beta} \right] \\
= -\frac{1}{2} \left( c^{-1} \gamma^\mu c \right) x_{\alpha} y_{\beta} \left[ \psi x_{\alpha} , \bar{\psi} y_{\beta} \right]
\]

But \( c^t y_m c = -y_m \)

\[
\hat{c} \psi \gamma^\mu c^t = \frac{1}{2} \left( \gamma^\mu \right)_{\alpha \beta} \left[ \psi x_{\alpha} , \bar{\psi} x_{\beta} \right] \\
= -\psi \gamma^\mu \psi
\]

where \( \hat{c} \psi \hat{c}^t = \eta_c \gamma^\mu \hat{c} \psi \hat{c}^t \)

\( \hat{c} \psi \hat{c}^t = \eta_c \psi \hat{c} \gamma^\mu \hat{c} = -\eta_c \psi \gamma^\mu \hat{c} \)

\( \eta, \bar{\eta} = \text{phase} \)

Then \( L \) ge d \( c \rightarrow -\psi \gamma^\mu \psi A^\mu \)

\[
\Rightarrow A^\mu c = -A^\mu c
\]

\( \Rightarrow <\delta^T A^\mu - A^\mu \delta^T > = 0 \quad n = \text{odd} \quad \text{by} \ (b) \)
\( \psi_{\alpha \nu} = F_{\mu \nu} F^{\mu \nu} + \bar{\psi} \partial \psi = \frac{1}{2} \gamma_{\mu} A_{\nu} \gamma^{\mu} A_{\nu} + \bar{\psi} (\partial + e A) \psi \)
\( \frac{4e}{4e} \)
\[ = - \frac{1}{2} A^\mu \nabla A_\mu + \bar{\psi} \gamma^\mu \psi + e A^\mu \bar{\psi} \gamma^\mu \psi + \partial_\mu (\cdots) \]

\( \Rightarrow \) integration by parts + gauge condition \( \partial_\mu A^\mu = 0 \)

Consider the 3-point function \( \langle A_\mu(y) \psi(\alpha) \bar{\psi}(\alpha_2) \rangle \). Do a field redefinition \( A \rightarrow A + SA \), the path integral is invariant, so:

\[ \int \mathcal{D}A, \psi, \bar{\psi} \mathcal{L} = \int \mathcal{D}A, \psi, \bar{\psi} \mathcal{L}' = \frac{1}{4} \int \mathcal{D}A, \psi, \bar{\psi} \left( A_\mu \gamma^\mu A_\nu \gamma^{\mu} A^{\mu \nu} + e A_\mu \bar{\psi} \gamma^\mu \psi + e \bar{\psi} \gamma^\mu \psi A_\mu \right) \]

\[ = \int \mathcal{D}A, \psi, \bar{\psi} \left( A_\mu \gamma^\mu A_\nu \gamma^{\mu} A^{\mu \nu} + e A_\mu \bar{\psi} \gamma^\mu \psi + e \bar{\psi} \gamma^\mu \psi A_\mu \right) \]

\[ = \int \mathcal{D}A, \psi, \bar{\psi} \left( A_\mu \gamma^\mu A_\nu \gamma^{\mu} A^{\mu \nu} + e A_\mu \bar{\psi} \gamma^\mu \psi + e \bar{\psi} \gamma^\mu \psi A_\mu \right) \]

\[ = \int \mathcal{D}A, \psi, \bar{\psi} \left( A_\mu \gamma^\mu A_\nu \gamma^{\mu} A^{\mu \nu} + e A_\mu \bar{\psi} \gamma^\mu \psi + e \bar{\psi} \gamma^\mu \psi A_\mu \right) \]

\[ = \int \mathcal{D}A, \psi, \bar{\psi} \left( A_\mu \gamma^\mu A_\nu \gamma^{\mu} A^{\mu \nu} + e A_\mu \bar{\psi} \gamma^\mu \psi + e \bar{\psi} \gamma^\mu \psi A_\mu \right) \]

\[ = \int \mathcal{D}A, \psi, \bar{\psi} \left( A_\mu \gamma^\mu A_\nu \gamma^{\mu} A^{\mu \nu} + e A_\mu \bar{\psi} \gamma^\mu \psi + e \bar{\psi} \gamma^\mu \psi A_\mu \right) \]

\[ = \int \mathcal{D}A, \psi, \bar{\psi} \left( A_\mu \gamma^\mu A_\nu \gamma^{\mu} A^{\mu \nu} + e A_\mu \bar{\psi} \gamma^\mu \psi + e \bar{\psi} \gamma^\mu \psi A_\mu \right) \]

\[ = \int \mathcal{D}A, \psi, \bar{\psi} \left( A_\mu \gamma^\mu A_\nu \gamma^{\mu} A^{\mu \nu} + e A_\mu \bar{\psi} \gamma^\mu \psi + e \bar{\psi} \gamma^\mu \psi A_\mu \right) \]

\[ = \int \mathcal{D}A, \psi, \bar{\psi} \left( A_\mu \gamma^\mu A_\nu \gamma^{\mu} A^{\mu \nu} + e A_\mu \bar{\psi} \gamma^\mu \psi + e \bar{\psi} \gamma^\mu \psi A_\mu \right) \]

\[ = \int \mathcal{D}A, \psi, \bar{\psi} \left( A_\mu \gamma^\mu A_\nu \gamma^{\mu} A^{\mu \nu} + e A_\mu \bar{\psi} \gamma^\mu \psi + e \bar{\psi} \gamma^\mu \psi A_\mu \right) \]

\[ = \int \mathcal{D}A, \psi, \bar{\psi} \left( A_\mu \gamma^\mu A_\nu \gamma^{\mu} A^{\mu \nu} + e A_\mu \bar{\psi} \gamma^\mu \psi + e \bar{\psi} \gamma^\mu \psi A_\mu \right) \]

\[ = \int \mathcal{D}A, \psi, \bar{\psi} \left( A_\mu \gamma^\mu A_\nu \gamma^{\mu} A^{\mu \nu} + e A_\mu \bar{\psi} \gamma^\mu \psi + e \bar{\psi} \gamma^\mu \psi A_\mu \right) \]

\[ = \int \mathcal{D}A, \psi, \bar{\psi} \left( A_\mu \gamma^\mu A_\nu \gamma^{\mu} A^{\mu \nu} + e A_\mu \bar{\psi} \gamma^\mu \psi + e \bar{\psi} \gamma^\mu \psi A_\mu \right) \]
for the electron we do a field redefinition \( \Psi \rightarrow \Psi + \delta \Psi \) and follow the same steps to find

\[
\langle \bar{\Psi}(\alpha_1) A_{\mu}(y) \Psi(\alpha_2) \rangle - e \langle \bar{\Psi}(\alpha_1) A_{\mu}(y) \Psi(\alpha_2) \bar{\Psi}(\alpha_3) \rangle
\]

\[= \frac{1}{2} \delta^{\mu}(\alpha_1 - \alpha_2) \langle A_{\mu}(y) \bar{\Psi}(\alpha_2) \rangle
\]

(b) \( e \bar{\chi} \chi \) vertex \( = \langle A_{\mu}(\alpha) \bar{\Psi}(\alpha_2) \Psi(\alpha_3) \rangle \)

Consider the 2-point function \( \langle \bar{\Psi}(\alpha_1) \bar{\Psi}(\alpha_2) \rangle \) and do a field redefinition \( A_{\mu} \rightarrow A_{\mu} + 5A_{\mu} \). An analogous computation to (a) gives

\[\delta^{\mu}(\alpha - \alpha_1) \langle A_{\mu}(\alpha) \bar{\Psi}(\alpha_2) \rangle = e \langle \bar{\mu}(\alpha_2) \bar{\Psi}(\alpha_2) \rangle \]

(c) From the lecture, we have the Ward identity

\[\partial^\mu \langle \bar{\mu}(\alpha) \Psi(\alpha) \bar{\Psi}(\alpha_2) \rangle = -\delta_\mu^{\alpha - \alpha_1} \langle \bar{\Psi}(\alpha_1) \bar{\Psi}(\alpha_2) \rangle
\]

\[+\delta_\mu^{\alpha - \alpha_2} \langle \bar{\Psi}(\alpha_1) \bar{\Psi}(\alpha_2) \rangle
\]

Putting \( \Psi, \bar{\Psi} \) on-shell amounts to doing the LSZ reduction on both sides of this equation. In momentum space this is

\[
\lim_{K_1 \rightarrow 0} \lim_{K_2 \rightarrow 0} K_1^2 K_2^2 \frac{d}{d^4 s} \langle \Psi(1) \bar{\Psi}(2) \rangle = \frac{d}{d^4 s} \langle \bar{\Psi}(1) \bar{\Psi}(2) \rangle
\]

\[
= \lim_{K_1 \rightarrow 0} \lim_{K_2 \rightarrow 0} K_1^2 K_2^2 \left( M_0(K_1 + p, K_2) - M_0(K_1, K_2 - p) \right) = 0
\]

Since \( M_0(K_1 + p, K_2) \) has the \( 1/K_1^2 \) singularity and \( M_0(K_1, K_2 - p) \) has no \( 1/K_2^2 \) singularity.
This is a consequence of the terms originating from contact terms \( \propto S(x-y) \). They don't contribute to scattering amplitude.

Do the LSZ reduction on the vertex function:

\[
\langle \not\! p | A_\mu(p) | i \rangle = \lim_{K_i^2 \to 0} \lim_{K_i^2 \to 0} K_i^2 K_i^2 \langle A_\mu(p) \bar{\psi}(K_i) \psi(K_i) \rangle
\]

So \( p^2 p^n \langle \not\! p | A_\mu(p) | i \rangle = e \ p^n \langle \not\! p | j_\mu(p) | i \rangle = 0 \)

\[
\Rightarrow p^n \langle \not\! p | A_\mu(p) | i \rangle = 0
\]

The same calculation shows that this is actually valid for general initial and final states.

It is a consequence of gauge invariance \( p \cdot A = 0 \).
\[ J = \frac{1}{2} \left( \mu \phi \partial^\mu \phi - m^2 \phi^2 - g \phi^3 - \frac{3}{4} \phi^4 \right) \]

a) \( \langle \phi(a) \phi(y) \rangle \)

\[ \frac{1}{2} q^2 \int \frac{d^0 k}{(k^2 + m^2)(k^2 + m^2)} \]

\[ \frac{1}{2} q^2 \int \frac{d^0 \ell}{(\ell^2 + m^2)} \]

\( \langle \phi(a) \phi(y) \phi(z) \rangle \)

\[ \frac{1}{2} q^2 \int \frac{d^0 k}{(k^2 + m^2)(k^2 + m^2)(k^2 + m^2)} \]

\[ \frac{1}{2} q^2 \int \frac{d^0 \ell}{(\ell^2 + m^2)(\ell^2 + m^2)} \]

\[ \frac{1}{2} q^2 \int \frac{d^0 \ell}{(\ell^2 + m^2)} \]

\( \langle \phi(a) \phi(y) \phi(z) \phi(w) \rangle \)

\[ \frac{1}{2} q^2 \int \frac{d^0 k}{(k^2 + m^2)(k^2 + m^2)(k^2 + m^2)(k^2 + m^2)} \]

\[ \frac{1}{2} q^2 \int \frac{d^0 \ell}{(\ell^2 + m^2)(\ell^2 + m^2)(\ell^2 + m^2)(\ell^2 + m^2)} \]

\[ \frac{1}{2} q^2 \int \frac{d^0 \ell}{(\ell^2 + m^2)(\ell^2 + m^2)(\ell^2 + m^2)(\ell^2 + m^2)} \]
\[ \sum_{\text{other channels}} \]
\[ \sum_{\text{other channels}} \]
\[ \sum_{\text{other channels}} \]
\[ b) \lambda = 0, \text{ only 3-pft interaction} \]
\[ \text{tree-level} \quad \rightarrow - \frac{g^3}{2} \quad \frac{1}{(P_1^2 + m^2)(P_2^2 + m^2)(P_3^2 + m^2)} \]
\[ \text{1-loop} \quad \rightarrow - \frac{g^3}{2} \quad \frac{1}{\Pi (P_1^2 + m^2)} \quad \frac{1}{(l^2 + m^2)(l + P_1^2 + m^2)(l - P_3^2 + m^2)} \]
\[ \rightarrow \frac{g^3}{3} \quad \frac{1}{\Pi (P_1^2 + m^2)} \quad \frac{1}{(l^2 + m^2)(l + P_2^2 + m^2)(l + P_3^2 + m^2)} \]
\[ + P_2 \leftrightarrow P_1 \quad + P_3 \leftrightarrow P_1 \]
I have dropped terms containing tadpoles since they always sum to zero if we are expanding around the correct vacuum state. Otherwise several more diagrams have to be included, e.g.

\[ \text{for } \langle \phi(x) \phi(y) \rangle \]

\[ \text{for } \langle \phi(x) \phi(y) \phi(z) \rangle \]

\[ \text{for } \langle \phi(x) \phi(y) \phi(z) \phi(w) \rangle \]
2) \[ S = \int d^4x \left( \frac{1}{2} \nabla \phi \cdot \nabla \phi + m^2 \phi^2 + \bar{\psi} (i\gamma^0 + \mu_0) \psi \right) \]

\[ + \frac{g_0}{4!} \phi \bar{\psi} \gamma_5 \psi \phi^4 + \frac{g_2}{4!} \phi^6 \]

Define \( \psi_5 = \psi_1 \psi_2 \psi_3 \psi_4 \); where \( \delta \psi_4 \psi_3 = 2 \delta \psi \)

\[ \psi_5, \psi_3 = \psi_1 \psi_2 \psi_4 \psi_5 \]

\[ \psi_5 = -\psi_1 \psi_2 \psi_3 \psi_4 \]

\[ \psi_5^2 = -\psi_1 \psi_2 \psi_3 \psi_4 \psi_5 \psi_4 \psi_5 = \psi_1 \psi_2 \psi_3 \psi_4 \psi_5 \psi_4 = \psi_1 \psi_2 \psi_3 \psi_4 \psi_5 \psi_4 = 1 \]

Under a parity transformation \( \mathcal{P} \) is a pseudoscalar

\[ \mathcal{P} \psi \mathcal{P} = -\psi \]

and so is \( \bar{\psi} \gamma_5 \psi \) because of the \( \psi_5 \)

\[ \mathcal{P} \bar{\psi} \gamma_5 \psi \mathcal{P} = -\bar{\psi} \gamma_5 \psi \]

The only relevant or marginal interactions which are

invariant under parity are \( \phi^4 \) and \( \phi \bar{\psi} \gamma_5 \psi \).

We assume that the \( R^0 \) flavons preserve this symmetry

no no parity breaking terms like \( \phi \) or \( \phi^3 \) are generated.

b) We split the Lagrangian as \( S_0 + S_1 + S_{\text{ext}} \)

\[ S_0 = \frac{1}{2} \nabla \phi \cdot \nabla \phi + m^2 \phi^2 + \bar{\psi} (i\gamma_0 + \mu_0) \psi \]

\[ S_1 = g_0 \bar{\psi} \gamma_5 \psi + \lambda \phi^4 \]

\[ S_{\text{ext}} = \sum (Z_4 - 1) (\bar{\psi} \frac{1}{2} \gamma_0 \phi + m^2 \phi^2) + (Z_4 - 1) \bar{\psi} (i\gamma_0 + \mu_4) \phi \]

\[ \frac{1}{2} \left( \frac{1}{2} \bar{\psi} \gamma_5 \phi \frac{1}{2} - \frac{1}{2} \bar{\psi} \gamma_5 \phi \frac{1}{2} - g_0 \bar{\psi} \gamma_5 \phi \frac{1}{2} \right) \frac{1}{2} \phi \]

\[ \text{where: } \phi = Z_4^{1/2} \phi \]

\[ m = m_0 + S \phi \]

\[ \mu = \mu_0 + S \phi \]

\[ g = Z_4^{1/2} + S \phi \]

\[ \psi = Z_4^{1/4} \psi \]

\[ M = \mu_0 + S \phi \]

\[ \frac{1}{2} \]
The coefficients of the counter-terms are fixed order by order in perturbation theory by requiring that the cancel terms which diverge when the cut-off is removed. Here we'll use the on-shell regularization scheme. While not the easiest its physical interpretation is quite transparent.

In the appendix I collected some formulas and identities which are used often in the calculations below.

Let's start with the 2-point function of two scalars \( \langle \phi \bar{\phi} \rangle \). In momentum space the amplitude to one loop is

\[
M(p) = \frac{-\theta - \bar{\theta} \gamma_5}{\gamma_5} + \frac{p}{\gamma_5} - \frac{p}{\gamma_5} + \frac{- (Z_\phi - 1)(p^2 + m^2)}{\gamma_5} - \lambda \bar{\phi} \gamma_5 \phi V_1(p) - (\alpha p^2) V_2(p) + 8m^2 Z_\phi
\]

Note that we are working in d'alembertian, the dimension of space-time is \( D = 4 + \varepsilon \) so we take \( \lambda = \frac{2}{2^\varepsilon} \) and \( \alpha \rightarrow \frac{2}{2^\varepsilon} \) so that \( \lambda, \alpha \) remain dimensionless.

\[
\bar{\phi} \gamma_5 \phi V_1(p) = \bar{\phi} \gamma_5 \frac{1}{(2\pi)^d \varepsilon^2 + m^2} = \bar{\phi} \gamma_5 \frac{1}{(2\pi)^2} \left( -1 + \frac{\varepsilon}{2} \right) \frac{1}{m^2} \left( \frac{4\pi^2}{m^2} \right)^{2/2} \\
= \frac{m^2}{(4\pi)^2} \left( -2 + (\varepsilon - 1) \right) \left( 1 + \varepsilon \ln \left( \frac{4\pi^2}{m^2} \right) \right) + O(\varepsilon^2)
\]

\[
= - \frac{m^2}{(4\pi)^2} \left( \frac{2 + \varepsilon + \ln \left( \frac{m^2}{\tilde{m}^2} \right)}{\varepsilon} \right)
\]

where \( \tilde{m}^2 = 4\pi \varepsilon^{\varepsilon/2} \bar{\mu}^2 \)
The next contribution takes a bit more work.

\[ \tilde{\tilde{m}}^{-2} V_\alpha(p) = \tilde{\tilde{m}}^{-2} \int d^4 q \frac{T_V \left[ (\epsilon + q + M)\gamma_5 (\epsilon + M)\gamma_5 \right]}{(2\pi)^4 \left( (\epsilon + q)^2 + M^2 \right) \left( \epsilon^2 + M^2 \right)} \]

We simplify the numerator using standard \( \gamma \)-matrix identities.

\[ T_V \left[ (\epsilon + q + M)\gamma_5 (\epsilon + M)\gamma_5 \right] = T_V \left[ (\epsilon + q + M)(-\epsilon + M) \right] \]
\[ = -4 \frac{1}{\epsilon \cdot (\epsilon + q) - M^2} \]

For the denominator, use a Feynman parameter.

\[ \frac{1}{((\epsilon + q)^2 + M^2)(\epsilon^2 + M^2)} = \int_0^1 dx \left( \frac{1}{((\epsilon + q)^2 + M^2)x + (1-x)(\epsilon^2 + M^2)(1-x)} \right)^2 \]
\[ = \int_0^1 dx \frac{1}{(q^2 + A)^2} \quad \text{with} \quad q = \epsilon + x\epsilon p \]
\[ A = \epsilon x p^2 (1-x) + M^2 \]

So,

\[ \tilde{\tilde{m}}^{-2} V_\alpha(p) = \tilde{\tilde{m}}^{-2} \int d^4 q \frac{1}{(2\pi)^4} \left( \frac{q^2}{q^2 + A} \right) \left( \frac{1}{(q^2 + A)^2} \right) \]

The linear term cancels by Lorentz invariance after integration.

\[ -4 \tilde{\tilde{m}}^{-2} \int d^4 q \frac{\Gamma(-1 + \epsilon/2)\Gamma'(2 - \epsilon/2)A}{(4\pi)^2 - \Gamma(2 - \epsilon/2)A} \left( \frac{A}{(4\pi)^2 \Gamma(\epsilon/2) \left( \frac{4\pi}{A} \right)^{\epsilon/2}} \right) \]
\[ - A \left( \frac{\Gamma(\epsilon/2) \left( \frac{4\pi}{A} \right)^{\epsilon/2}}{(4\pi)^2} \right) \]
\[-y \int_0^{(4\pi)^2} \left[ A \left( -\frac{4}{\varepsilon} + (2\varepsilon - 1) \right) \left( \frac{e^{-\frac{4\pi^2}{A^2}}}{2} \right) - A \left( \frac{2 - \gamma}{\varepsilon} \right) \left( \frac{e^{-\frac{4\pi^2}{A^2}}}{2} \right) \right] + \mathcal{O}(\varepsilon) \]

\[= \frac{4}{(4\pi)^2} \int_0^{(4\pi)^2} \ln A \left( \frac{6 + 3 \ln \left( \frac{\mu^2}{A} \right) + 1}{\varepsilon} \right) \]

\[= \frac{4}{(4\pi)^2} \left( \frac{1}{6} \left( \frac{p^2 + 6\mu^2}{p^2 + M^2} \right) + \frac{1}{6} \ln \left( \frac{\mu^2}{A} \right) \right) \]

Now recall that we can add up all the contributions to the full propagator as

\[S(p) = \frac{1}{p^2 + m^2 + M(p)} \]

requiring that the propagator has a simple pole at the physical mass \( m \) implies \( M(-m^2) = 0 \).

We just calculated \( M(p) \) to one-loop, setting \( p^2 = -m^2 \) gives

\[M(p) = -(Z_\sigma - 1)(p^2 + m^2) + \lambda m^2 Z_\sigma + \lambda m^2 \left( \frac{2 + 1 + \ln \left( \frac{\mu^2}{m^2} \right)}{(4\pi)^2} \right) - \frac{1}{8} \frac{4}{(4\pi)^2} \left( \frac{1}{6} \left( \frac{2M^2 - m^2}{2} \right) + \frac{1}{6} \ln \left( \frac{m^2}{(mM(m-1) + M^2)} \right) \right) \ln \left( \frac{\mu^2}{m^2 + (m-1)M^2} \right) \]

\[= 0 \quad \Rightarrow \text{We have the freedom of choosing} \]

\[Z_\sigma \text{ such that this is true.} \]
We also impose wavefunction normalization, the residue of $S(p)$ at $p^2 = -m^2$ equal to one, so

$$\left. \partial^2 M(p) \right|_{p^2 = -m^2} = 0 \quad \text{This condition fixes } Z_f, \quad \text{and}$$

$$\partial\frac{p^2}{p^2 = -m^2} \quad \text{from } M(-p^2) = 0 \quad \text{the fixed } S_m^2$$

Now let us move on to the fermion 2-point function

$$\Sigma(p) = \sum_{\text{fermions}} - \left( Z_f - z \right) \left( \gamma + M \right) + g^2 F(p)$$

$$+ Z_f Z$$

$$F(p) = \frac{-e^2}{2} \int \frac{d^6 l}{(2\pi)^6} \frac{\gamma \cdot (p + l + M)}{(l^2 + m^2)(l^2 + m^2)}$$

$$= -\frac{e^2}{2} \int \frac{d^6 l}{(2\pi)^6} \frac{(q + B)^2}{(q^2 + B)^2}$$

$$= -\frac{e^2}{2} \int \frac{d^6 l}{(2\pi)^6} \frac{1}{(q^2 + B)^2}$$

$$= -\frac{e^2}{2} \int \frac{d^6 l}{(2\pi)^6} \frac{\Gamma(\epsilon/2)}{(4\pi)^2}$$

$$= -\frac{e^2}{2} \int \frac{d^6 l}{(2\pi)^6} \frac{\Gamma(\epsilon/2)}{(4\pi)^2} \left( \frac{2 - \epsilon}{2} \right) \left( 1 + \frac{\epsilon - \epsilon}{2} \right) \left( \frac{4\pi^2}{B} \right)$$

$$= -\frac{e^2}{2} \int \frac{d^6 l}{(2\pi)^6} \frac{\Gamma(\epsilon/2)}{(4\pi)^2} \left( \frac{2 - \epsilon}{2} \right) \left( 1 + \frac{\epsilon - \epsilon}{2} \right) \left( \frac{4\pi^2}{B} \right)$$

$$= -\frac{e^2}{2} \int \frac{d^6 l}{(2\pi)^6} \frac{\Gamma(\epsilon/2)}{(4\pi)^2} \left( \frac{2 - \epsilon}{2} \right) \left( 1 + \frac{\epsilon - \epsilon}{2} \right) \left( \frac{4\pi^2}{B} \right)$$
The fermionic propagator is given by

\[ S(x) = \frac{1}{P + M + \Sigma(x)} \]

It must have a simple pole at \( P = -M \)

\[ \Sigma(-M) = 0 \] and the residue must be zero:

\[ \frac{d\Sigma}{dP} \bigg|_{P=-M} = 0 \]

These two conditions fix \( S(x) \) and \( Z_P \).

Hint: we'll compute \( \langle \phi \phi \rangle \) to one loop, this should fix \( S(x) \).

\[ M(\phi \to \phi \phi) = P_1 \rightarrow P_2 + P_3 \rightarrow P_2 + P_3 \rightarrow P_2 + P_3 \]

\[ -g^2 \delta^5 + g^3 \delta^5 - g^3 \Gamma(P_1, P_2) \]

\[ \Gamma(P_1, P_2) = \int \frac{d^4k}{(2\pi)^4} \frac{(-i\gamma_\sigma (P_1 + k + M) \gamma_5 (P_2 + k + M) \gamma_\sigma)}{((-k + \alpha_3 P_3 + (\alpha_3 - 1) P_1 + M)^2 + m^2)} \]

\[ = \frac{d^4k}{(2\pi)^4} \frac{(-i\gamma_\sigma (P_2 + k + M)(-\gamma_5 - \alpha_3 P_3 + (\alpha_3 - 1) P_1 + M)}{((-k + \alpha_3 P_3 + (\alpha_3 - 1) P_1 + M)^2 + m^2)} \]

\[ = 2 \int \frac{d\alpha_1, d\alpha_2, d^4k}{(2\pi)^4} \frac{(-i\gamma_\sigma (P_2 + k + M)(-\gamma_5 - \alpha_3 P_3 + (\alpha_3 - 1) P_1 + M)}{((-k + \alpha_3 P_3 + (\alpha_3 - 1) P_1 + M)^2 + m^2)} \]

\[ q = k + \alpha_1 P_1 + \alpha_2 P_2 \]

\[ N = P_1^2 \alpha_1 (1 - \alpha_3) + P_2^2 \alpha_2 (1 - \alpha_3) - P_1 \gamma_5 P_2 (1 - \alpha_1 - \alpha_3 + 2 \alpha_1 \alpha_2) \]

Linear terms in \( g \) were discarded.
\[ C = \alpha_1 (1 - \alpha_1) P_1^2 + \alpha_2 (1 - \alpha_2) P_2^2 - 2 \alpha_1 \alpha_2 P_1 \cdot P_2 + (\alpha_1 + \alpha_2) M^2 + \alpha_2 m^2 \]

\[ \Gamma(p_1, p_2) = 2 \chi_5 \int d^4x \, d^4 \bar{\psi} \left[ - \frac{\Gamma(\varepsilon/2) \Gamma(3 - \varepsilon/2)}{(4\pi)^2} \left( \frac{4\pi \bar{\mu}^2}{\varepsilon} \right)^{3/2} \right] \]

\[ + \frac{N \Gamma(1 - \varepsilon/2)}{(4\pi)^2} \left( \frac{4\pi \bar{\mu}^2}{\varepsilon} \right)^{1/2} \]

\[ \bar{\mu} \text{ were added by dimensional reasons} \]

\[ = \chi_5 \int d^4x \, d^4 \bar{\psi} \left[ - \frac{1}{\varepsilon} + 1 - 2 \ln \left( \frac{\mu^2}{\varepsilon} \right) + N \right] \]

\[ = - \frac{2 \chi_5 + \chi_5}{(4\pi)^2} \left( \frac{1}{\varepsilon} + \frac{1}{(4\pi)^2} \right) \int d^4x \, d^4 \bar{\psi} \left[ \frac{N - 2 \ln \left( \frac{\mu^2}{\varepsilon} \right)}{(4\pi)^2} \right] \]

We are free to pick a renormalization condition for the vertex function in the end physical observables won't depend on the choice. An unphysical but convenient condition is to define \( g \) to be the amplitude at zero momentum \( M(0,0) = -\chi_5 g \). This gives \( \chi_5 S g = g^3 \Gamma(0,0) \).

The only counter-term left to fix is \( S g \). We need to study the scattering of 4 scalar fields at one loop.

\[ M_4 = \sum_{\text{perm.}} \left( \frac{\hat{p}_1 \cdot \hat{p}_2}{p_1^2} \right) + \sum_{\text{perm.}} \left( \frac{\hat{p}_3 \cdot \hat{p}_4}{p_3^2} \right) + \sum_{\text{perm.}} \left( \frac{\hat{p}_1 \cdot \hat{p}_4}{p_1^2} \right) + \sum_{\text{perm.}} \left( \frac{\hat{p}_2 \cdot \hat{p}_3}{p_2^2} \right) + \sum_{\text{perm.}} \left( \frac{\hat{p}_3 \cdot \hat{p}_1}{p_3^2} \right) + \sum_{\text{perm.}} \left( \frac{\hat{p}_4 \cdot \hat{p}_2}{p_4^2} \right) \]

\[ + \sum_{\text{perm.}} \left( \frac{\hat{p}_1 \cdot \hat{p}_3}{p_1^2} \right) + \sum_{\text{perm.}} \left( \frac{\hat{p}_2 \cdot \hat{p}_4}{p_2^2} \right) + \sum_{\text{perm.}} \left( \frac{\hat{p}_3 \cdot \hat{p}_2}{p_3^2} \right) + \sum_{\text{perm.}} \left( \frac{\hat{p}_4 \cdot \hat{p}_1}{p_4^2} \right) \]

\[ - \frac{g^4}{8} V_4 \]
Let's first consider the scalars running in the loop.

\[ P_3 \]
\[ P_2 \]
\[ P_1 \]
\[ P_4 \]

\[ \frac{1}{\sqrt{2}} \int \frac{d^4l}{(2\pi)^4} \frac{1}{(l^2 + m^2)(l + P_1 + P_2)^2 + m^2} \]

\[ = \int_0^1 dx \int \frac{d^4l}{(2\pi)^4} \frac{1}{(l^2 + D_x^2)^2} \]
\[ \xi = l + (P_1 + P_2) \alpha \]
\[ D_x = \sqrt{\xi(1 - \alpha) + m^2} \]
\[ \lambda = (P_1 + P_2)^2 \]
\[ = \int_0^1 dx \Gamma\left(\frac{3}{2}\right) \left(\frac{4\pi^2}{D_x}\right)^{-3/2} \cdot \frac{1}{(2\pi)^2} \int_0^1 dx \ln \left(\frac{\mu^2}{D_x^2}\right) \]

We should add the contributions from the other configuration:

\[ \frac{1}{2} \int \frac{d^4l}{(2\pi)^4} \ln \left(\frac{\mu^2}{D_x^2}\right) \]

So \( V_4^0 = 6 + \int \frac{dx}{(2\pi)^2} \ln \left(\frac{\mu^2}{D_x^2}\right) \]
\[ \ln \left(\frac{\mu^2}{D_x^2}\right) \]

The diagrams with fermions running in the loop take more effort to evaluate.

\[ = - \frac{1}{\sqrt{2}} \int \frac{d^4l}{(2\pi)^4} \left( \frac{1}{(l^2 + M)^{1/2}} \right) \sum (l - P_1 - M) \cdot \sum (l + P_2 + M) \cdot \sum (l + P_1 + M) \cdot \sum (l + P_2 + M) \cdot (l + P_1 + M) \cdot (l + P_2 + M) \]

Let's take care of the numerator first. Antisymmetrize the \( Y_0 \)'s through the \( Y_2 \)'s. Now recall that \( Tr(Y_i Y_j Y_k) = 0 \) and \( Tr(Y_j Y_k) = 0 \). So the numerator splits into a \( Tr(Y_2) \) part and a \( Tr(Y_1) \) part.
The numerator is:

\[ N = 4 \sum \frac{l \cdot (l-P_4) (l+P_1) \cdot (l+P_2) - l \cdot (l+P_1) (l-P_4) \cdot (l+P_2)}{l \cdot (l+P_1) (l-P_4) \cdot (l+P_2)} \]

\[ + 4 M^2 \left[ l \cdot (P_4 - l) + l \cdot (l+P_1 + P_2) - l \cdot (l+P_1) \right] \]

\[ - (l-P_4) \cdot (l+P_1 + P_2) + (l-P_4) \cdot (l+P_1) - (l+P_1) \cdot (l+P_1 + P_2) \]

\[ + 4 M^4 \]

Now we need to introduce Feynman parameters for the denominator.

\[ \frac{1}{D} = 3! \int \frac{d\alpha_1 d\alpha_2 d\alpha_3}{(q^2 + F)^4} \]

\[ q = l + P_1 \alpha_2 + (P_1 + P_2) \alpha_3 - P_4 \alpha_1 = l + K \]

\[ F = \frac{1}{P_4} \left[ \alpha_1 + (P_1 + P_2)^2 \alpha_2 + P_3 \alpha_3 + (-P_4 \alpha_1 + (P_1 + P_2) \alpha_2 + P_1 \alpha_3)^2 + M^2 \right] \]

Substituting \( l \) for \( q \) in the numerator, the terms \( \sim M^2 \) are

\[ 4 M^2 \left[ M^2 - 2 q^2 + K \cdot (K-P_4) + P_2 \cdot (K+P_4) - (P_1 - K) \cdot (P_1 + P_2 - K) \right] \]

and I discarded terms that integrate to zero (\( \alpha q \))

In the term coming from \( Tr(q^4) \) there are two kinds of contractions that integrate to zero: \( \alpha q \) and \( \alpha q^3 \).

There are also terms like

\[ Tr \left[ \int \frac{d^3q}{(2\pi)^3} \frac{q^4}{(q^2 + F)^3} \right] \]

by Lorentz symmetry this integrates to
\[ \frac{T_{\mu \nu} S^{\mu \nu} \cdot 4}{D (2\pi)^D (q^2 + F)^3} \]

Adding up all the terms:

\[ 4 \int \frac{d^D q}{(2\pi)^D} \left( \frac{q^2 + q^2 G_7 + H}{q^2 + F} \right) \]

\[ G_7 = -2 M^2 + (p_1 + p_2 - k) \cdot (p_1 - k) + k \cdot (k + p_4) + (p_i + k) \cdot (p_1 - k) \]
\[ - k \cdot (p_1 + p_2 - k) - (k + p_4) \cdot (p_1 + p_2 - k) - k \cdot (p_1 - k) \]
\[ + \frac{4}{9} \left[ (p_1 + p_2 - 2 k) \cdot (p_1 - p_4 - 2 k) - (2 k - p_4) \cdot (2 p_1 + p_2 - 2 k) \right] \]
\[ + (p_1 - 2 k) \cdot (p_1 + p_2 - p_4 - 2 k) \]

\[ H = M^2 \left[ k \cdot (k - p_4) + p_2 \cdot (k + p_4) - (p_1 - k) \cdot (p_1 + p_2 - k) \right] \]
\[ + k \cdot (k + p_4) \cdot (p_1 + p_2 - k) \cdot (p_1 - k) + k \cdot (p_1 + p_2 - k) \cdot (p_4 + k) \cdot (p_1 - k) \]
\[ + k \cdot (p_1 - k) \cdot (k + p_4) \cdot (p_1 + p_2 - k) \]

Now the integrals are easy to perform.

\[ \int \frac{d^D q}{(2\pi)^D} \left( \frac{q^2 + q^2 G_7 + H}{q^2 + F} \right) = \frac{\Gamma(\varepsilon/2) \Gamma(4 - \varepsilon/2)}{(4\pi)^2} \left( \frac{4r^2 - 1}{F} \right)^{\varepsilon/2} \]
\[ + \frac{G}{F} \frac{\Gamma(1 + \varepsilon/2) \Gamma(3 - 3/2)}{3! (4\pi)^2} \left( \frac{4r^2 - 1}{F} \right)^{\varepsilon/2} + \frac{H}{F^2} \frac{\Gamma(4 - \varepsilon/2)}{3! (4\pi)^2} \left( \frac{4r^2 - 1}{F} \right)^{\varepsilon/2} \]
\[ \approx \frac{1}{3! (4\pi)^2} \left[ \frac{12}{\varepsilon} + \frac{2G}{F} + \frac{6H - 5}{F^2} + 6 \ln \left( \frac{M^2}{F} \right) \right] \]
So we have

\[ V_4^{\mu} = \frac{1}{4\pi^2} \left[ \frac{1}{2} + \int_0^1 \frac{d\xi}{\xi} \ln \left( \frac{2G + 6H - \xi + 6 \ln \left( \frac{\mu^2}{F} \right)}{F} \right) \right] + 5 \text{ non-equivalent permutations of the external momenta.} \]

We can write as a renormalization prescription that

\[ \mu(0,0,\epsilon) = -\lambda, \text{ this fixes the counter-term} \]

\[ \lambda = -\lambda^0 V_4^{\mu} + q^4 V_4^{\mu}. \]

Now we have fixed the value of all the counter-terms.

The amplitudes are manifestly finite as \( \mu^2 \to \infty \) and \( \epsilon \to 0 \).

For example, consider the 3-point vertex \( \langle \bar{\psi} \psi \psi \rangle \)

\[ M(p_1, p_2) = -q y_5 + q^3 \Gamma(0,0) - q^3 \Gamma(p_1, p_2) \]

\[ = -q y_5 + q^3 \int_0^1 d\xi \ln \left( \frac{N(0,0) - N(p_1, p_2)}{C\xi} \right) \frac{\ln \left( \frac{\mu^2}{c(0,0)} \right)}{c(p_1, p_2)} \]

\[ = -q y_5 + q^3 \int_0^1 d\xi \ln \left( \frac{N(0,0) - N(p_1, p_2)}{C\xi} \right) \frac{\ln \left( \frac{\mu^2}{c(p_1, p_2)} \right)}{c(0,0)} \]

And analogously for the 4-point vertex. This implies that all 1-loop amplitudes are finite. We can build one-loop amplitudes using these vertices and attaching trees to them. Together with the other 1PI diagrams, there are easily shown to be finite by power counting.
Useful formulas for problem 2

\[
\int \frac{d^q \mathbf{q} (q^2)^a}{(2\pi)^D (q^2 + A)^b} \propto \left( \frac{\Gamma(b-a-D/2)}{(4\pi)^{D/2}} \right)^{a} \Gamma(a+D/2) \mathcal{A}^{-(b-a-D/2)}
\]

\[
\frac{1}{A_1 \cdots A_n} \int \frac{dn_1 \cdots dn_n}{[A_1 n_1 + \cdots + A_n n_n]^{n}}
\]

\[
\Gamma(x/2) \approx 2 - \frac{x}{2} \quad A^{x/2} = e^{\frac{x}{2} \ln A} \approx 1 + \frac{x}{2} \ln A
\]

\[
\{ y^{\mu}, y^{\nu} \} = 2 \delta^{\mu\nu} \quad \text{Tr}(y^{\mu} y^{\nu}) = 4 \delta^{\mu\nu}
\]

\[
\text{Tr}(y^{\mu}) = 0 \quad \text{Tr}(y^{\mu} y^{\nu} y^{\rho} y^{\sigma}) = 4 \left( \delta^{\mu \rho} \delta^{\nu \sigma} - \delta^{\mu \sigma} \delta^{\nu \rho} + \delta^{\mu \nu} \delta^{\rho \sigma} \right)
\]

\[
\text{Tr}(y^{\mu} y^{\nu} y^{\rho}) = 0
\]

\[
\text{Tr}(y^{\mu} y^{\nu} y^{\rho} y^{\sigma}) = 0
\]
\[ l = -\frac{1}{4} F_{\mu\nu}^2 - \phi^\dagger (\partial^2 + m^2) \phi - i e \, A_\mu \left[ \phi^\dagger \partial_\mu \phi - \partial_\mu \phi^\dagger \phi \right] + e^2 A_\mu \phi^\dagger \phi \phi \]

Feynman rules:
- Scalar propagator \( \rightarrow \frac{i}{p^2 - m^2 + i\epsilon} \)
- Photon propagator \( \rightarrow \frac{\epsilon}{p^2 + i\epsilon} \)

\[ \phi^\dagger \phi \text{ symmetry factor from } A_\mu \text{ fields.} \]

\[ -i e A_\mu (\phi^\dagger \partial_\mu \phi - \partial_\mu \phi^\dagger \phi) \text{ vertex } \]

\[ (-ie)(i)(i)(\pm p_\mu) = ie(\pm p_\mu) \text{ derivative} \]

\[ \times \int (i \hbar \text{ vol}) \]

\[ \int [-\frac{i}{(k+q)^2 - m^2} (2k+q)^{\alpha} \left( \frac{i}{k^2 m^2} \right)] = M_1 \]

\[ \int \frac{2ie^2}{(2\pi)^4} \int \frac{d^4k}{(k^2 - m^2)} = \rightarrow M_2 \]

\[ \Rightarrow M_1 + M_2 = e^2 \int \frac{d^4k}{(2\pi)^4} \frac{(2k+q)^{\alpha}}{(k+q)^2 - m^2} \frac{(2k+q)^{\nu}}{(k^2 - m^2)} - 2 \frac{q^{\mu\nu}[(k+q)^2 - m^2]}{(k+q)^2 - m^2} \]

\[ = \tilde{T}^\mu\nu(q) \]

both diagrams needed to make sure everything is gauge invariant
\[
\text{denominator } \Rightarrow \\
\frac{1}{[(k+q)^2 - m^2][k^2 - m^2]} = \int_0^1 \frac{dx}{x [(k+q)^2 - m^2] + (1-x) [k^2 - m^2]} \]
\[
= \int_0^1 \frac{dx}{(k^2 + 2nx q^2 + 2nx^2 - m^2)^2}
\]

Change variables \( l \rightarrow k + nxq \)

\[
\Rightarrow \mathcal{A}^{\mu\nu}(q) = e^{2} \int dx \int d^4 l \frac{4k^\mu l^\nu + (1-2x)^2 q^\mu l^\nu - 2 q^\mu (l^2 + (1-2x)^2 - m^2)^2}{(2\pi)^4} \\
\quad \times [l^2 - \Delta]^2
\]

\( \Delta = -x(1-x)q^2 + m^2 \) and ignore terms odd in \( l \).

Now in \( d \)-dimensions under the integral sign \( \frac{d^d l}{d} \)

\[
\Rightarrow \mathcal{A}^{\mu\nu}(q) = e^{2} \int dx \int d^d l \frac{q^\mu q^\nu I_1(\Delta) + (1-2x)^2 q^\mu q^\nu - 2 [(1-2x)^2 q^2 - m^2] q^\mu q^\nu I_2(\Delta)}{(2\pi)^d}
\]

where \( I_1(\Delta) = \frac{(-i)}{(4\pi)^2} \left( \frac{d}{d-2} \right) \frac{1}{2} \Gamma\left( \frac{1-d}{2} \right) \)

\[
\Delta^{1-d/2}
\]

\[
= \frac{-2i}{(4\pi)^2} \Gamma\left( \frac{2-d}{2} \right) \Delta^{2-d/2}
\]

\( I_2(\Delta) = \int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2 - \Delta)^2} = \frac{i}{(4\pi)^2} \Gamma\left( \frac{2-d}{2} \right) \Delta^{2-d/2} \)
\[ \Rightarrow \mathcal{A}^{\mu \nu}(q) = \frac{i e^2}{(4\pi)^2} \int dx \frac{\Gamma(2-d/2)}{\Delta^2-d/2} \left\{ \begin{array}{c} -2\Delta - 2(1-x)^2 \frac{q^2 + 2m^2}{\Delta^2} \\ \frac{1}{2} \frac{1}{\Delta^2} \left\{ -(1-2x) \frac{q^2 g_{\mu \nu} - q^\mu q^\nu}{\Delta^2} \right. \\ \left. + (1-2x)^2 \frac{q^\mu q^\nu}{\Delta^2} \right\} \end{array} \right\} \]

\[ = \frac{i e^2}{(4\pi)^2} \int dx \frac{\Gamma(2-d/2)}{\Delta^2-d/2} \left\{ \begin{array}{c} -2(1-x) (1-2x) \frac{q^2}{\Delta^2} \\ \frac{1}{2} \frac{1}{\Delta^2} \left\{ -(1-2x)^2 \left[ \frac{q^\mu q^\nu}{\Delta^2} \right] \right. \\ \left. - \frac{1}{4} \frac{1}{\Delta^2} \left( 1-2x \right)^2 \frac{q^\mu q^\nu}{\Delta^2} \right\} \end{array} \right\} \]

\[ \rightarrow 0 \text{ as odd } \]

under \( x \rightarrow 1-x \)

\[ \Rightarrow i \mathcal{A}^{\mu \nu}(q) = (q^2 g_{\mu \nu} - q^\mu q^\nu) i \mathcal{A}(q^2) \]

when \( i \mathcal{A}(q^2) = -\frac{e^2}{(4\pi)^2} \int_0^1 (1-2x)^2 \frac{\Gamma(2-d/2)}{\Delta^2-d/2} \left\{ \begin{array}{c} \text{const} - \log \Delta \\ \frac{1}{\Delta^2} \left( 1-2x \right)^2 \text{const} - \log \Delta \end{array} \right\} \]

\[ \text{const} = \frac{2}{\epsilon} - \gamma_e + \log 4\pi \]

for \( |m^2| < -q^2 \) we just get

\[ \mathcal{A}^{\mu \nu} \sim -\frac{e^2}{48\pi^2} (q^2 g_{\mu \nu} - q^\mu q^\nu) \left( \frac{2}{\epsilon} + \text{finite} \right) \]

Now the \( q^\mu q^\nu \) term can be gotten rid of by choosing the \( \xi \) gauge parameter in the photon propagator appropriately.
Now if we look at the Lagrangian \( \text{L}_0 \) and counterterms, we have,

\[
\text{L}_0 = -\frac{1}{4} F^{\mu \nu} F_{\mu \nu} - (\partial \nu \phi^2)^2 - \frac{1}{2} \partial^\mu \phi \partial^\nu \phi^2 - \frac{1}{2} m^2 \phi \phi
\]

(see gauge fixing + F-P ghosts for more on the \( \phi^4 \) term)

\[
\text{L}_1 = -\left( 2m-1 \right) \frac{m^2 \phi^2}{2} - \left( Z_3 - 1 \right) \frac{\partial^\mu \phi \partial^\nu \phi^2}{2} - \left( Z_3 - 1 \right) \frac{1}{4} F^{\mu \nu} F_{\mu \nu}
\]

\[- \frac{1}{\phi^2} \left( i e A^\mu \phi^2 \partial^\nu \phi - \partial^\nu \phi \phi + e^2 A^\mu \phi^2 \phi^2 \right)
\]

\[
\text{L}_{\text{bare}} = \text{L}_0 + \text{L}_1
\]

\[
\Rightarrow Z_1 e^2 \mu^2 A_\mu \phi^2 = e^2 \left( A_\mu \right)^2 \phi^2 \left( \text{RHS is in e} \right)
\]

\[
\Rightarrow e_0^2 = e^2 \mu^2 Z_1 Z_2^{-1} Z_3^{-1}
\]

Again gauge invariance \( \Rightarrow \) Ward identity \( \Rightarrow Z_2 = Z_1 \)

\[
\left( e_0^2 = e^2 \mu^2 Z_3^{-1} \right) \Rightarrow \alpha_0 = \frac{e \mu}{Z_3^{-1}} \left( \alpha = \frac{e^2 \mu}{Z_3} \right)
\]

\[
\Rightarrow \frac{d}{dx} \left( \alpha_0 = \alpha \mu Z_3^{-1} \right)
\]

\[
\Rightarrow 0 = \frac{d}{dx} - \frac{d}{dx} Z_3 \cdot \frac{dx}{dx}
\]

\[
\Rightarrow \frac{d}{dx} \left[ \frac{1}{\alpha} = \frac{1}{6 \alpha} \right] = -e
\]

Where \( Z_3 = 1 - \frac{e^2}{4 \pi^2} \cdot \frac{2}{2} = 1 - \frac{e^2}{6 \pi} \cdot \frac{2}{2} \) \( \Rightarrow \) we evaluated \( \frac{e^2}{6 \pi} \) in \( m^2 \)

\[
\Rightarrow \frac{dx}{dx} = \frac{\alpha^2}{d\mu} \Rightarrow \frac{dx}{dx} = \frac{e^2}{6 \pi} \cdot \frac{2}{2}
\]
5) $\nu(\phi) = \frac{m^2 \phi^2}{2} + \frac{1}{2!} \mu \epsilon^{1/2} g \phi^3$, $d = 6 - e$

1st function

$$= -\frac{\mu \epsilon^{1/2} g}{2} \int \frac{d^4k}{(2\pi)^d} \frac{1}{k^2 + m^2}$$

$$= -\frac{\mu \epsilon^{1/2} g}{2} \Pi^{(0)} = -\frac{\mu \epsilon^{1/2} g}{2 \bar{g} (4\pi)^{3-\epsilon+2}} \Gamma(-2 + \epsilon/2)$$

from (1.4)

$$= -\frac{1}{2} m \epsilon^{1/2} \log m^2/m^2 \left( \frac{1 + 3 - \epsilon/2}{\epsilon} \right) + O(\epsilon)$$

$$= -\frac{g m^4}{128 \bar{g}^3} \left( \frac{1 + 3 - \epsilon/2}{\epsilon} \right) \frac{1}{2} \log m^2 + O(\epsilon)$$

dgt pt part $= -\frac{1}{\epsilon} \frac{g m^4}{128 \bar{g}^3}$

2nd function

$$= \frac{\mu \epsilon^{1/2} g}{2} \int \frac{d^4k}{(2\pi)^d} \frac{1}{(k^2 + m^2)(\ell - k)^2 + m^2}$$

just like the 1st integral in the previous question.

read off dgt part

$$-\frac{1}{\epsilon} \frac{g^2}{64 \bar{g}^3} \int \frac{d\ell}{\epsilon} (m^2 + \alpha(1-\alpha)\ell^2) = -\frac{1}{\epsilon} \frac{g^2}{64 \bar{g}^3} \left( \frac{m^2 + \ell^2}{6} \right)$$
\[ \begin{align*}
\text{3st:} & \quad P_2 \\
& \quad P_1 \quad P_{1-k} \quad P_3 \\
& \quad (k^2 + m^2)(P_{1-k}^2 + m^2)(P_{2-k}^2 + m^2) \\
& = \quad -2g^3 \int \frac{d^4k}{(k^2 + m^2)(P_{1-k}^2 + m^2)(P_{2-k}^2 + m^2)} \\
& = \quad -2g^3 \int d^4k \int d\alpha \int d\beta \int d\gamma \\
& \quad \frac{6(1-\alpha - \beta - \gamma)}{\left[ \alpha(k^2 + m^2) + \beta(P_{1-k}^2 + m^2) + \gamma \left( \frac{1}{(P_{2-k}^2 + m^2)} \right) \right]^3} \\
\text{Complete squares:} & \quad (\alpha + \beta + \gamma)^2 \left[ k^{'2} + m^2 + (\alpha \beta + \beta \gamma) p_1^2 + (\alpha + \beta)^2 p_2^2 - 2\beta \gamma p_1 p_2 \right] \\
& \quad \frac{1}{\alpha + \beta + \gamma} \\
\text{the } k' \text{ integral is } I^{(3)}_d \\
& = \quad -2g^3 \int \frac{d^4k}{(k^2 + m^2)(P_{1-k}^2 + m^2)(P_{2-k}^2 + m^2)} \\
& = \quad -\frac{g^3}{128\pi^3} \left( \frac{2}{\epsilon} - 2 - \log \frac{m^{'2}}{m^2} + \log \frac{\mu^2}{m^2} \right) + O(\epsilon) \\
\text{det part:} & \quad -\frac{1}{\epsilon} \cdot \frac{g^3}{64\pi^3} \\
\end{align*} \]
This adds, \( E = L + \frac{\varepsilon}{128 \pi^2} \mu \cdot g^2 \cdot \frac{e^{\frac{1}{2}}}{m} \) to 1 pt.

\[ A + A = \frac{1}{e} \frac{1}{64 \pi^2} \cdot g^2 \left( m^2 + \frac{1}{6} b^2 \right) \to 2 \text{ pt} \]

\[ c = \frac{1}{e} \frac{1}{(4\pi)^3} \mu e^2 g^3 \to 3 \text{ pt} \]

\[ \phi = 2 - \phi_2, \quad [m] = 1 \quad \Rightarrow [\phi] = 0 - e \]

Note that \( L_{\text{bare}} = L_0 + L_{\phi e} \).

\[
\begin{align*}
L_{\phi e} = & \quad \frac{1}{2} (\phi')^2 + \frac{1}{2} m^2 \phi^2 + g \frac{\phi^3}{3!} + \frac{(2g-1) (2m \phi)}{2!} \\
& + \frac{(2m-1) m^2 \phi^2 + (2g-1) \phi^3 g^2 + Z \phi}{3!}
\end{align*}
\]

What we have computed in this question with all the divergent bits is just \( Z_\phi, 2 \varepsilon, 2 \varepsilon, 2m, 2g \), to plug them back into the lagrangian you still need to multiply them with \( \frac{1}{3!}, \frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!} \) respectively.

So these are no fudge symmetry factors hanging around.
From the previous question

\[ m_0^2 = \frac{m^2 + B}{1 + A} = m^2 \left( 1 - \frac{1}{e} \frac{g^2}{(4\pi)^3} \right) \left( 1 + \frac{1}{e} \frac{g^2}{384\pi^3} \right) + O(g^4) \]

\[ = m^2 \left( 1 + \frac{5g^2}{384\pi^3} \right) + O(g^4) \]

\[ g_0 = \frac{\mu e^{1/2}}{(1 + A)^{3/2}} = \frac{\mu e^{1/2}}{(1 + A)^{3/2}} \left( 1 - \frac{1}{e} \frac{g^2}{(4\pi)^3} \right) \left( 1 + 3 \frac{1}{e} \frac{g^2}{384\pi^3} \right) \]

\[ = \frac{\mu e^{1/2}}{(1 + A)^{3/2}} \left( 1 - \frac{3g^2}{256\pi^3} \right) + O(g^5) \]

\[ \beta_2 = \left( \frac{gd}{dq} \right) f_1 = -\frac{3g^2}{256\pi^3}, \text{ decreasing } \Rightarrow \text{ asymptotic freedom} \]

\[ \frac{d_\mu^2}{dq} = \frac{gd}{dq} b_1 = \frac{5g^2}{192\pi^3} \]
Define \( I_d^{(n)} = \frac{1}{(4\pi)^d} \int \frac{dk}{(k^2 + m^2)^n}, \quad d - 2n \geq 0. \)

\[
\Rightarrow I_d^{(n)} = \frac{1}{(2\pi)^d \Gamma(n)} \int_0^\infty d\alpha \int_0^\infty d\alpha \alpha^{n-1} e^{-\alpha (k^2 + m^2)}
\]

\[
= \frac{1}{(4\pi)^{d/2} \Gamma(n)} \int_0^\infty d\alpha \alpha^{n-d/2-1} e^{-\alpha m^2}
\]

\[
= \frac{\Gamma(n-d)}{(4\pi)^{d/2} \Gamma(n)}
\]

Poles for \( n-d = 0, -1, -2, \ldots \)

No poles for odd \( d \)!
\[ Q = \frac{c^a c_a - \frac{1}{2} f^a_{\,bc} c^b c^c \, 2}{2} \]

\[ Q^2 \text{ must hold as an operator statement } Q^2 A = 0 \; \forall A \]

\[ \left( \frac{c^a c_a - \frac{1}{2} f^a_{\,bc} c^b c^c \, 2}{2} \right) \left( \frac{c^{\,cd} - \frac{1}{2} f^{\,cd}_{\,ef} c^e c^f \, 2}{2} \right) A \]

\[ = \frac{c^a c_a c^{\,cd} A - \frac{1}{2} f^a_{\,bc} c^{\,cd} c^b c^c \, 2}{2} \]

\[ = \frac{1}{4} \frac{c^a c_a c^{\,cd} \, 2}{2} \left( c^{\,cd} 2 A \right) \]

\[ = \frac{1}{4} \frac{c^a c_a c^{\,cd} \, 2}{2} \left( c^{\,cd} 2 A \right) \]

\[ = \frac{1}{4} \frac{c^a c_a c^{\,cd} \, 2}{2} \left( c^{\,cd} 2 A \right) \]

The first 4 terms cancel among themselves, last term is zero since \( \frac{1}{2} \) is antisymmetric.

The fifth term cancel due to the Jacobi identity.

Now \( c_a = 0 \) \( \Rightarrow X = f_{abc} c^a c^b c^c \)

\[ Q X = \frac{-1}{2} f^a_{\,bc} c^b c^c \, 2 \left( f^{\,cd}_{\,ef} c^e c^f \right) = -\frac{3}{2} f^a_{\,bc} c^b c^c \, f^{\,cd}_{\,ef} c^e c^f \]

\[ = 0 \; \text{by Jacobi identity} \]

For \( Q Y = X = f_{abc} c^a c^b c^c = -\frac{1}{2} f^a_{\,bc} c^b c^c \, 2 \)

\[ J Y = -2 c_a \rightarrow \text{no non-trivial solution for } c \text{ Gramm} \]

\[ \{ \text{Jacobi } \} \; f_{def} c^e c^f c^g = f_{def} f_{abcd} c^a c^b c^c c^d = 0 \]

\[ \therefore f_{def} f_{abcd} c^a c^b c^c c^d = 0 \]
\[ S = \int d^4x \left( -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \not{D} A_{\mu} + \frac{1}{2} h^2 + 2 \not{D}_{A}^{2} c \right) \]

a) \[ \delta A_{\mu} = \epsilon \partial_{\mu} c \quad ; \quad \delta h = 0 \]
\[ \delta c = 0 \quad ; \quad \delta h = 0 \]

\[ S_S = \int d^4x \left( -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} c \partial_\mu \partial_\nu c - \frac{1}{2} \not{D} A_{\mu} \not{D} A_{\nu} c + \epsilon \not{D} A_{\mu} \not{D} A_{\nu} c - \epsilon h \not{D} c \right) \]
\[ = 0 \]
\[ S_a^2 = \epsilon \not{D} c = 0 \quad ; \quad S_a^2 c = - \epsilon S h = 0 \quad \Rightarrow \text{multeplet} \]

b) \[ \int d^4x \not{A} \not{A} = \int d^4x \frac{1}{2} \left( A^\mu A^\mu - \frac{1}{2} \not{D} A \not{D} A \right) \left( -\partial^\mu \partial_\nu + \partial_\nu \partial^\mu \right) \left( A_\nu \right) \]
\[ = \int d^4x \left( -A^\mu \partial_\nu A_\nu + A_\mu \partial_\nu A_\nu + A_\mu \partial_\nu h - h \partial_\nu A - \frac{1}{2} h^2 \right) \]

So the Lagrangians differ only by boundary terms.

c) We can verify that these expressions give the matrix of \( \Delta \)

\[ \Delta = \begin{pmatrix} -\partial^\mu \partial_\nu & 0 & 0 \\ 0 & -\partial^\mu \partial_\nu & 0 \\ 0 & 0 & -\partial^\mu \partial_\nu \end{pmatrix} \]

\[ \Delta(p) = \int d^4x \mathcal{E}^{\mu
u} \Delta = \begin{pmatrix} S^\mu p^\nu - p^\mu p^\nu / p^2 & i p^\nu \\ -i p^\mu & -i p^\nu \end{pmatrix} \]

\[ = \begin{pmatrix} S^\mu p^\nu - p^\mu p^\nu - (1 - \gamma^5) p^\mu p^\nu + (1 - \gamma^5) p^\mu p^\nu & p^\nu (p^2 - p^\mu p^\nu) \\ -i p^\mu + i p^\mu (1 - \gamma^5) + i p^\mu \gamma^5 & p^2 \end{pmatrix} \]

It is evident for the ghost propagator.
\[ d \langle \delta Y \rangle = 0 \quad \forall Y \]

\[
\langle \delta (\overline{h}(\alpha) \overline{c}(0)) \rangle = \langle \delta \overline{h}(\alpha) \overline{c}(0) \rangle + \langle \delta \overline{c}(\alpha) \overline{c}(0) \rangle
\]

\[
= \begin{pmatrix}
\langle \delta A_\mu (\alpha) \overline{c}(0) \rangle \\
\langle \delta h (\alpha) \overline{c}(0) \rangle
\end{pmatrix} + \begin{pmatrix}
\langle A_\mu (\alpha) \delta \overline{c}(0) \rangle \\
\langle h (\alpha) \delta \overline{c}(0) \rangle
\end{pmatrix}
\]

\[
= \begin{pmatrix}
e \langle \partial_\mu (\alpha) \overline{c}(0) \rangle \\
0
\end{pmatrix} + \begin{pmatrix}
- e \langle A_\mu (\alpha) h(0) \rangle \\
- e \langle h(\alpha) h(0) \rangle
\end{pmatrix} = 0
\]

\[
\Rightarrow \langle h(\alpha) h(0) \rangle = 0
\]

\[
\langle \partial_\mu (\alpha) \overline{c}(0) \rangle = \langle A_\mu (\alpha) h(0) \rangle
\]

\[
\Rightarrow \text{ off-diagonal element of } \Lambda^{-1}
\]

\[
\partial_\mu \langle c(\alpha) \overline{c}(0) \rangle \xrightarrow{\downarrow} \Lambda^{-1}\partial_\mu \langle c(\alpha) \overline{c}(0) \rangle
\]

\[
\partial_\mu \left( - \frac{i}{p^2} \right)
\]

\[
\frac{i}{p^2}
\]
Q_{brst}
\langle z^b(p) | A^\mu(p) | \bar{0} \rangle = \frac{-i}{2} \epsilon^{|\mu|} B^{ab} \delta_{pp'}
\langle z^b(p) | \phi^* \rangle = \frac{1}{2} \epsilon^{*|\mu|} B^{ab} \delta_{pp'}
\langle z^b(p) | \phi(p) | \bar{0} \rangle = \frac{1}{2} \epsilon^{*|\mu|} B^{ab} \delta_{pp'}
\Rightarrow \epsilon^{* \mu} = B^{ab}
\langle z^b(p) | \phi^* \rangle = \frac{1}{2} \epsilon^{*|\mu|} B^{ab} \delta_{pp'}
\Rightarrow \epsilon^{* \mu} = (B^{ab})^*
\sum_a \tilde{\nu}_a \tilde{\gamma}_b = 8 \delta_{pp'}
Q^2 | z^b(p) \rangle = \text{the only non-trivial one since } Q^2 | 0 \rangle = 0
Q^2 | z^b(p) \rangle = Q \left( \frac{1}{2} p^\mu A^\mu(p) + \sum_a \tilde{\nu}_a \phi_a(p) \right)
= 1/(2m^2) C^a(p) + \sum \tilde{\nu}_a \tilde{\gamma}_b C^b(p)
= (1/(2m^2) p^2 + g_a) | C^a(p) \rangle \Rightarrow p_a = -(2m^2)
\sum_a \tilde{\nu}_a \tilde{\gamma}_b = (-1/2m^2) S_{ab}
The single particle states are (killed by BRSs)

\[ \psi_c \sim 0 \]

* \[ \psi^* A^\mu(p) \] with \( p^2 = 0 \)

\[ Q \psi^* A^\mu(p) = \psi^* p^\mu \psi_c(p) = 0 \Rightarrow \psi^* \mu = 0 \quad \text{gauge freedom} \]

\[ \psi^* A^\mu(p) \neq Q_1^1 \]

* \[ \sum \psi_i ^* \phi_i (p) \]

\[ Q \sum \psi_i ^* \phi_i (p) = \sum \psi_i ^* \omega_{ia} (c_a(p)) = 0 \quad \text{if} \quad \sum \psi_i ^* \omega_{ia} = 0 \]
\[ L(A) = -\frac{1}{4} \text{tr} \left( F^\mu_\nu (D^3)^{-1} F^\nu_\mu \right), \quad X(D^3) = 1 + (-D^3)^r \]

\[ \Lambda^{2r} \]

\( ^a \) Propagators are defined by the kinetic term in the Lagrangian:

\[ \mathcal{L}_K = \int d^4p A_\mu p^\mu X(D^3) p^2 A^\mu \]

\[ D_\mu (p^2) \sim p^2 \rightarrow (D_\mu)^{-1} \sim \frac{1}{p^2} \]

Analogously for the ghost propagator.

\( ^b \) \[ X(D^3) = 1 + (-D^3)^r = 1 + (-\partial^\mu (\partial+\Lambda^2))^r \]

\[ \Lambda^{2r} \]

\[ F \times (D^3) F = (dA + A^2) \times (D^3) (dA + A^2) \]

\( ^* \) \[ dA \times (D^3) dA \sim \text{kinetic operator}, \text{ all others are } \text{vertices.} \]

\[ [A, A] (1 + \partial^{2m} A^{2k}) [A, A] \Rightarrow \{ \begin{array}{c} 2k+3 \text{ gauge legs} \\ n = \# \text{ gauge legs, } d = \# \text{ momenta} \\ n = 2k+1, \quad n = 3, \ldots, 2r+3 \\ d = 2m+1 = 2r-n+4 \end{array} \]

\[ [A, A] (1 + \partial^{2m} A^{2k}) [A, A] \Rightarrow \{ \begin{array}{c} 2k+3 \text{ gauge legs} \\ n = 2k+4, \quad n = 4, \ldots, 2r+4, \quad d = 2r-n+4 \end{array} \]

\[ \varepsilon (\frac{\partial^2}{\Lambda^2})^r [A^\mu, c] \Rightarrow \{ \begin{array}{c} n = 1 \\ d = 2r+1 \end{array} \]
c) degree of divergence

\[ D = xL + \sum_{n=3}^{2k+4} (2r+4-n) V_n - (2+2r) \cdot I \]

Each vertex \( V_n \) has \( 2r+4-n \) powers of momenta

and each propagator \( I \) has \( -2-2r \) powers of momenta

\[ \text{Vertex \#}_n \sum_{n=3}^{2k+4} V_n - I + L = 1 \quad \text{and \# of legs} \sum_{n=3}^{2k+4} nV_n = E + 2I \]

For \( d=4 \)

\[ D = (2r+4)\sum_{n=3}^{2k+4} nV_n - (2+2r)I \]

\[ = 4L + (2r+4)(1-L+I) - E - 2I - (2+2r)I \]

\[ = 4 - 2r(L-1) - E \]
\[ \langle \omega, [A] \rangle = \exp \left[ -ie \oint dx^m A_m(x) \right] \]

\[ \Rightarrow \langle \omega, [A] \rangle = \int \det A_m \exp \left[ iS[A]\pi - ie \oint dx^m A_m(x) \right]. \]

\( A_m \) = free, SCAM quad

\( \Rightarrow \) gaussian integral which we can solve exactly.

\[ \Rightarrow \langle \omega, [A] \rangle = \exp \left[ -\frac{1}{2} \left( -ie \oint dx^m \right)^2 \right] \]

\[ \Rightarrow \langle \omega, [A] \rangle = \exp \left[ -\frac{1}{2} \left( -ie \oint dx^m \right)^2 \right] \]

\[ \frac{1}{(2\pi)^4} \int \frac{d^4k}{(k^2 + i\epsilon)^4} e^{-ik \cdot (x-y)} \]

\[ \text{Momentum integral:} - \]

\[ = \frac{i}{(2\pi)^4} \int \frac{d^4k}{k^2 + i\epsilon} \]

\[ = \frac{i}{(2\pi)^4} \int d\omega \sin \theta \int_0^\infty dk \frac{e^{-ik(\omega-x-y)}}{k^2 + i\epsilon} \]

\[ = -\frac{i}{4\pi^2} \int_0^\infty dk \frac{e^{-i(k_\parallel^2)(x-y)}}{k_\parallel^2 + i\epsilon} \]

\[ = -\frac{i}{4\pi^2} \int_0^\infty dk \frac{e^{-i(k_\parallel^2)(x-y)}}{k_\parallel^2 + i\epsilon} \]

\[ J_1(k_\parallel |x-y|) \]

\[ \Rightarrow J_1(0) \to Bessel fn. \]

\[ \int_0^\infty J_1(0) dx = 1. \]

\( \text{Subs (B) in (A) \Rightarrow Q.E.D.} \)
integral over \(x, y\) independently over loop \(\gamma (x-y)\rightarrow 0\) \(\Delta t\), but we want to show here that the dependence of \(\langle \omega, \gamma \rangle\) on the geometry of the loop \(\rightarrow \Delta t\) free.

\(\langle \omega \rangle\) 

(\(\text{e.g.}\))

\(\langle \omega \rangle = \exp \left[ -2 \epsilon^2 \int \frac{e^2 \delta z}{\epsilon} \int \frac{e^4 \delta w}{\epsilon} \frac{g_{\mu\nu}}{2 \pi^2} \left[ (x-y)^2 - i \epsilon \right] \right]

\(\langle \omega \rangle = \exp \left[ -2 \epsilon^2 \frac{T}{8 \pi^2} \int_0^\infty \int_0^\infty \frac{dy^0}{T} \left\{ \frac{1}{(x-y)^2 - i \epsilon} \right\} \right]

\(\Rightarrow T \gg R \Rightarrow -\frac{\epsilon^2}{\pi} \frac{T}{8 \pi^2} \int_0^\infty \frac{dy^0}{(y^0)^2 - R^2 - i \epsilon}

The exponent \(\frac{\epsilon^2}{\pi} \frac{T}{8 \pi^2} \int_0^\infty \frac{dy^0}{(y^0)^2 - R^2 - i \epsilon}

\(= \frac{\epsilon^2}{\pi} \frac{T}{8 \pi^2} \int_0^\infty \frac{dy^0}{(y^0)^2 - R^2 - i \epsilon} \left( y^0 - R - i \epsilon \right) \left( y^0 + R + i \epsilon \right)

\(= \frac{\epsilon^2}{\pi} \frac{T}{8 \pi^2} \int_0^\infty \frac{dy^0}{(y^0)^2 - R^2 - i \epsilon} \left( 2 \pi i \cdot \frac{1}{2R} \right)

\(= \frac{i \epsilon^2}{\pi} \frac{T}{4 \pi R} \left( \text{because} \langle \omega \rangle = e^{-i \nu C R T} \right)

\(-i \nu C R T = \frac{i \epsilon^2}{\pi} \frac{T}{4 \pi R} \left( \text{because} \langle \omega \rangle = e^{-i \nu C R T} \right) \Rightarrow V C R = -\frac{\epsilon^2}{4 \pi R} \)
non abelian wilson loop:

\[ w = tr \left\{ P \exp \left( -i g \oint \frac{dx}{x} A^i_{m} \right) \right\} \]

group generators in sect. 7.

\[ tr \left( \left( 1 - g^2 C_2 (x) \right) \frac{d}{dx} f \frac{d}{dy} A^a_{m} \right) \sim tr \left( e^{i g \oint A^i_{m} \frac{dx}{x}} \right) \]

compared to the abelian case.

\[ e^2 \rightarrow g^2 C_2 (x) \]

\[ v(x) = \frac{g^2 C_2 (x)}{4 \pi R} \]

On one hand, many arguments to ignore the dgl.

dglt. terms like this (to get the correct L dependence etc).

Enpact while the divergent pieces

look like

(look like)

\[ c_3 \]

\[ c_2 \]