Non-Abelian Gauge Theory

Generalise EM: gauge group $U(1) \to$ compact Lie group, G (usually $SU(N)$)
basis $\{t^A\} \in L(G)$

Lie bracket $[t^A, t^B] = i f^{AB}{}_C t^C$ $\quad : L(G) \otimes L(G) \to L(G)$

Killing form $(t^A, t^B) = \frac{\delta^{AB}}{2}$ $\quad : L(G) \otimes L(G) \to \mathbb{R}$ $\quad$ generally $(t^A, t^B) = \mathrm{tr}(t^A t^B)$
fun rep of $SU(N)$

Let ψ transform in the fundamental rep of $SU(N)$ $\Rightarrow$ covariant derivative $D_\mu \psi = \partial_\mu \psi - i \underbrace{A^A_\mu t_A}_{A_\mu} \psi$

gauge transformations $\begin{cases} \psi(x) \to U\psi(x) \\ A_\mu \to U A_\mu U^{-1} + i U \partial_\mu U^{-1} \end{cases}$ $\Rightarrow$ $D_\mu \psi \to U(D_\mu \psi)$ ✓ $\quad$ use $\partial_\mu (UU^{-1}) = 0$

Field tensor $F_{\mu\nu}\psi = i[D_\mu, D_\nu]\psi$ $\quad$ and $[A_\mu, A_\nu] = A^B_\mu A^C_\nu [t_B, t_C]$
$= (\partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu])\psi$ ✓

$i[D_\mu, D_\nu]\psi \to U i[D_\mu, D_\nu]\psi = U F_{\mu\nu}\psi = (U F_{\mu\nu} U^{-1})(U\psi)$ $\quad \therefore F_{\mu\nu} \to U F_{\mu\nu} U^{-1}$ $\quad$ not gauge inv. ✗

Yang-Mills action $S_{YM}[A] = \frac{1}{2g^2}\int d^4x\, (F^{\mu\nu}, F_{\mu\nu}) = \frac{1}{2g^2}\int d^4x\, F^{\mu\nu A} F^B_{\mu\nu} (t_A, t_B)$ $\quad$ gauge inv. ✓
(g: coupling const.)

with $\begin{cases} D_\mu F^{\mu\nu} = 0 & \text{EoM} \\ D_{[\mu} F_{\nu\kappa]} = 0 & \text{Bianchi} \end{cases}$

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construct
connection $A^{(\tau)}_\mu \to U A^{(\tau)}_\mu U^{-1} + i U \partial_\mu U^{-1}$ $\quad$ $A^{(\tau)}_\mu = \tau A^{(1)}_\mu + (1-\tau) A^{(0)}_\mu$ $\quad \forall \tau \in [0,1]$
adjoint $(A^{(1)}_\mu - A^{(0)}_\mu) \to U(A^{(1)}_\mu - A^{(0)}_\mu)U^{-1}$

space of all gauge fields $=$ ∞-dim affine space modelled on adjoint-valued covectors, $\mathcal{A}$. ($\exists$ measure DA)

partition function $Z_{YM} = \int_{\mathcal{A}/\mathcal{G}} d\mu\, e^{-S_{YM}[A]}$ $\quad$ want to path integrate over $\mathcal{A}/\mathcal{G}$ (space of gauge trans.)
BUT non-Abelian theory $\mathcal{A}/\mathcal{G}$ not linear

$\delta A_\mu = D_\mu \lambda = \partial_\mu \lambda - i[A_\mu, \lambda]$ depends on A_μ

i) locally $\mathcal{A} = (\mathcal{A}/\mathcal{G}) \times \mathcal{G}$ $\Rightarrow$ $DA = d\mu_{\mathcal{A}/\mathcal{G}} \times d\mu_{\mathcal{G}}$

ii) $\Sigma = \{ f^A(A_\mu(x)) = 0 \mid A_\mu \in \mathcal{A} \}$ $\Rightarrow$ $DA \, |\det| \, \delta(f^A(x))$

iii) locally $d\mu_{\mathcal{G}} = D\lambda$ $\therefore$ for any A_μ $\exists\, \lambda$: $\int D\lambda \left| \det \frac{\delta f^A(x)}{\delta \lambda^B(y)} \right| \delta(f^A(x)) = 1$

$$\therefore \int_{\mathcal{A}} DA \, e^{-S_{YM}[A]} \underbrace{\left| \det \frac{\delta f^A(x)}{\delta \lambda^B(y)} \right|}_{\text{FP det}} \delta(f^A(x)) = \int_{\mathcal{A}/\mathcal{G} \times \mathcal{G}} d\mu_{\mathcal{A}/\mathcal{G}} \, d\mu_{\mathcal{G}} \, e^{-S_{YM}[A]} \left| \det \frac{\delta f^A(x)}{\delta \lambda^B(y)} \right| \delta(f^A(x)) = \int_{\mathcal{A}/\mathcal{G}} d\mu_{\mathcal{A}/\mathcal{G}} \, e^{-S_{YM}[A]}$$

ensures cont from only 1 A_μ in each gauge equiv class.

iv) $\delta(f^A(x)) = \int Dh \exp\left(i \int d^4x \, h_A(x) f^A(x) \right)$ $\quad$ h_A bosonic

$$\det\left(\frac{\delta f^A(x)}{\delta \lambda^B(y)} \right) = \int Db \, Dc \exp\left(-\int d^4x \, b_A(x) \frac{\delta f^A(x)}{\delta \lambda^B(y)} c^B(y) \, d^4y \right)$$

c^A, b_A fermionic

$$\therefore Z_{YM} = \int_{\mathcal{A}} DA \, Db \, Dc \, Dh \exp\left(\frac{-1}{4g^2} \int d^4x \, F^{\mu\nu A} F^A_{\mu\nu} - \int d^4x \, d^4y \, b_A(x) \frac{\delta f^A(x)}{\delta \lambda^B(y)} c^B(y) + i \int d^4x \, h_A(x) f^A(x) \right)$$

Ex. Lorenz gauge $\boxed{\partial^\mu A^A_\mu(x) = 0}$ set condition $f^A(A_\mu(x)) = \partial^\mu A^A_\mu(x)$

ghosts couple to A^A_μ

$$\delta A_\mu = D_\mu \lambda \Rightarrow \frac{\delta f^A(x)}{\delta \lambda^B(y)} = \delta^A_B \delta^{(4)}(x-y) \partial^\mu D_\mu \rightarrow Z_{YM} = \int DA \, Db \, Dc \, Dh \, e^{-S_{YM}[A] \underbrace{- \int d^4x \, b_A \partial^\mu D_\mu c^A + i \int d^4x \, h_A \partial^\mu A^A_\mu}_{\text{NOT gauge inv}}}$$

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i) ghosts couple to gauge fields - appear in Feynman diagram loops $\therefore$ violates spin stat. (unitarity)

ii) action appears to depend on gauge fixing functional $f^A[A_\mu(x)]$

iii) gauge inv. breaking $\Rightarrow$ we can generate all non gauge inv counter terms in the quantum theory

BRST - global nilpotent field transformations

const. ferm. param. $\quad \lambda(x) = \varepsilon c(x)$

$\delta A_\mu = \varepsilon D_\mu c \qquad \delta b = -i \varepsilon h$

$\delta h = 0 \qquad \delta c = \frac{i}{2} \varepsilon [c, c]_+$

fermionic BRST charge Q : $\delta \Phi = [\varepsilon Q, \Phi]_- = \varepsilon Q \Phi - \Phi \varepsilon Q = \varepsilon [Q, \Phi]_\pm$ for ferm/bos field

bosonic fields $\quad [Q, [Q, A_\mu]]_+ = 0 \qquad [Q, [Q, h]_-]_+ = 0$

fermionic fields $\quad [Q, [Q, c]_+]_- = 0 \qquad [Q, [Q, b]_+]_- = 0$

δ is nilpotent on each field individually.