Summary Sheet 1: Complex Numbers

- A **complex number** is a numerical expression that contains the factor i such that $i^2 = -1$, e.g. $1 + 2\,i$.

- The standard form of a complex number is $z = x + i\,y$, where x is the **real part** of z, i.e. $\mathrm{Re}(z) = x$, and y is the **imaginary part**, i.e. $\mathrm{Im}(z) = y$.

- Two complex numbers $z_1 = x_1 + i\,y_1$ and $z_2 = x_2 + i\,y_2$ are equal if and only if their real and complex parts are each equal, i.e. if and only if $x_1 = x_2$ and $y_1 = y_2$.

- A complex number can be represented graphically by a point in the **complex plane** (or an (**Argand diagram**). In **rectangular coordinates**, the real part is plotted as the abscissa (the x-axis) and the imaginary part as the ordinate (the y-axis).

- A complex number can also be represented in **polar coordinates** (r, θ) as $x = r\cos\theta$ and $y = r\sin\theta$, in which $r = \sqrt{x^2 + y^2}$ is the **modulus** (i.e. the magnitude) of z and $\theta = \tan^{-1}(y/x)$ is the **argument** or **phase** of z. The modulus measures the distance from the origin to the point and the argument measures its orientation with respect to the x-axis, with positive angles measured in the counterclockwise direction. The polar representation of $z = x + i\,y$ is $z = r(\cos\theta + i\,\sin\theta)$.

- **Euler's formula** is

$$\cos\theta + i\,\sin\theta = e^{i\,\theta}\,,$$

  from which we have $z = r\,e^{i\,\theta}$.

- The complex conjugate z^* of a complex number $z = x + i\,y$ is $z^* = x - i\,y$. In polar form, with $z = r\,e^{i\,\theta}$, the complex conjugate is $z^* = r\,e^{-i\,\theta}$.

- Complex numbers can be added, subtracted, multiplied, and divided. For complex numbers $z_1 = x_1 + i\,y_1$ and $z_2 = x_2 + i\,y_2$, these operations are carried as follows to obtain a complex number in the standard form $z = x + i\,y$:

$$(x_1 + i\,y_1) \pm (x_2 + i\,y_2) = (x_1 \pm x_2) + i\,(y_1 \pm y_2)\,,$$

$$(x_1 + i\,y_1)(x_2 + i\,y_2) = (x_1 x_2 - y_1 y_2) + i\,(x_1 y_2 + x_2 y_1)\,,$$

$$\frac{x_1 + i\,y_1}{x_2 + i\,y_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\,\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}\,.$$

- For complex numbers expressed in polar form, $z_1 = r_1\,e^{i\,\theta_1}$ and $z_2 = r_2\,e^{i\,\theta_2}$, multiplication and division are given by

$$\left(r_1\,e^{i\,\theta_1}\right)\left(r_2\,e^{i\,\theta_2}\right) = r_1 r_2\,e^{i\,(\theta_1+\theta_2)}\,,$$

$$\frac{r_1\,e^{i\,\theta_1}}{r_2\,e^{i\,\theta_2}} = \frac{r_1}{r_2}\,e^{i(\theta_1-\theta_2)}\,.$$

  Note that each result has the polar form $z = r\,e^{i\,\theta}$.

Summary Sheet 2: Functions of Complex Variables

- The **complex exponential** function e^z is defined by the power series
$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \cdots ,$$
which, like its real counterpart, has an infinite radius of convergence.

- The following properties are immediate consequences of this power series:
$$e^{z_1+z_2} = e^{z_1} e^{z_2} , \qquad e^{-z} = \frac{1}{e^z} ,$$
for complex numbers z, z_1, and z_2.

- The **complex sine** and **cosine** functions are defined by
$$\cos z = \frac{e^{iz} + e^{-iz}}{2} , \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i} ,$$
which imply
$$e^z = \cos z + i \ \sin z .$$

- The **complex hyperbolic sine** and **cosine** functions are defined by
$$\cosh z = \frac{e^z + e^{-z}}{2} , \qquad \sinh z = \frac{e^z - e^{-z}}{2} ,$$
which imply
$$e^z = \cosh z + \sinh z .$$

- The trigonometric and hyperbolic functions are related by
$$\cos z = \cosh iz , \qquad \sin z = -i \sinh iz .$$

- The **complex natural logarithm** of a complex number $z = r\,e^{i\theta}$ is
$$\ln z = \ln r + i\theta ,$$
where $\ln r$ is the natural logarithm of $r > 0$ and $0 \le \theta < 2\pi$.

- A complex number z raised to a complex power w is calculated as
$$z^w = e^{w \ln z} .$$
With $z = r\,e^{i\theta}$ and $w = a + i\,b$,
$$z^w = r^a e^{-b\theta}\left[\cos(b \ln r + a\theta) + i \ \sin(b \ln r + a\theta)\right] .$$

Summary Sheet 3: Ordinary Differential Equations

- An **ordinary differential equation** is an equation involving a function and its derivatives.
- A **solution** to a differential equation is a function which, when substituted into the equation, results in an identity.
- The **order** of a differential equation is the highest-order derivative appearing in the equation.
- A differential equation is said to be **linear** if the function and its derivatives appear only as single powers. Otherwise, the equation is **nonlinear**.
- First-order equations are used in epidemiology, population biology, and other applications in which the functions y are densities of different types of species and the independent variable is the time t. The general form of such an equation is

  $$\frac{dy}{dt} = F(y)\,,$$

  where F, which is determined by the rules of the model, can be a linear or nonlinear function.
- A first-order linear equation can be solved by the method of trial solutions, i.e. where the function $y(t) = e^{mt}$ is substituted into the equation and m is chosen by the requirement that this expression solves the equation.
- Nonlinear first-order equations can sometimes be solved by the method of separation of variables, whereby the equation is rearranged and integrated according to

  $$\int_{y_0}^{y(t)} \frac{dy'}{F(y')} = \int_0^t dt' = t\,.$$

  If F is a simple enough function, this equation can be solved explicitly for $y(t)$.
- Linear second-order equations with constant coefficients occur in applications such as mechanics, electrical circuits, and beam deflection. The general form of such an equation is

  $$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0\,,$$

  in which a, b, and c are constants. Second-order equations must be supplemented by two initial conditions to obtain a unique solution. These are usually imposed at $x = 0$: $y(0) = y_0$ and $y'(0) = y'_0$, where y_0 and y'_0 are specified real numbers.
- The method of trial solutions leads to three types of general solution of second-order equations with constant coefficients:

  $$b^2 - 4ac > 0\,, \qquad y(x) = A\,e^{m_1 x} + B\,e^{m_2 x}\,, \qquad m_1, m_2 = \frac{1}{2a}\left(-b \pm \sqrt{b^2 - 4ac}\right),$$

  $$b^2 - 4ac = 0\,, \qquad y(x) = (A + Bx)\,e^{m_1 x}\,, \qquad m_1 = -\frac{b}{2a}\,,$$

  $$b^2 - 4ac < 0\,, \qquad y(x) = A\,e^{m_1 x} + B\,e^{m_2 x}\,, \qquad m_1, m_2 = \frac{1}{2a}\left(-b \pm i\,\sqrt{4ac - b^2}\right).$$