Conic Sections: the ellipse, hyperbola and parabola

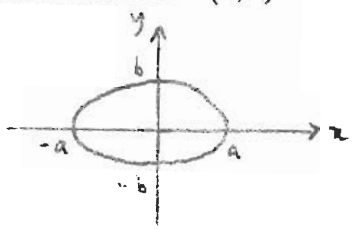
1 Conic sections

1.1 The ellipse:

We normally express the ellipse in the standard form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1}$$

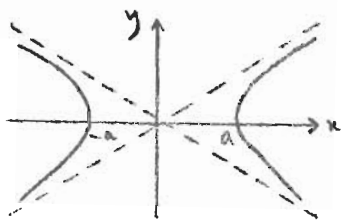
where a and b are numbers. Clearly, if $a = b$ then we have a circle $x^2 + y^2 = a^2$. In the form expressed in (1) the ellipse is centred at $(0, 0)$.



1.2 The hyperbola:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{2}$$

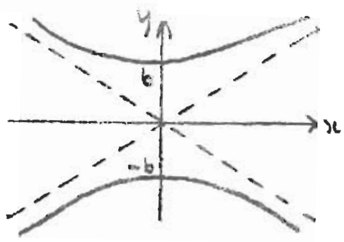
With the minus sign this way round the two branches are depicted in the figure below with asymptotes at $y = \pm\frac{b}{a}x$



If, however, the sign is the opposite way such that

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 \tag{3}$$

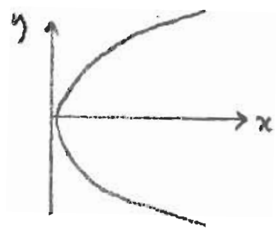
then this looks like



1.3 The parabola:

The standard form for the parabola is

$$y^2 = 4ax.$$



2 The trajectory of a particle in a force field

Consider a particle, such as an asteroid, moving in the force field of a larger object such that its equation of motion in polar co-ordinates is given by

$$r = \frac{L}{(1 + e\cos\theta)}. \tag{4}$$

L has dimensions of length and e, a dimensionless parameter, is the ***eccentricity***. When $e = 0$, then $r = L$ and the asteroid would move in a circle. e is a measure of how much its motion deviates from that of a circle. We now transform (4) into Cartesian co-ordinates; $x = r\cos\theta$, $y = r\sin\theta$ and $r^2 = x^2 + y^2$. Multiplying out and squaring we obtain

$$r^2 = (L - ex)^2 \tag{5}$$

which, on division by $1 - e^2$, becomes

$$x^2 + \frac{2eLx}{(1-e^2)} + \frac{y^2}{(1-e^2)} = \frac{L^2}{(1-e^2)}. \tag{6}$$

2.1 $0 < e < 1$: An elliptic orbit

Completing the square in (6) gives

$$\left(x + \frac{eL}{1-e^2}\right)^2 + \frac{y^2}{1-e^2} = \frac{L^2}{(1-e^2)^2} \tag{7}$$

Dividing by $\frac{L^2}{(1-e^2)^2}$ we get

$$\frac{\left(x + \frac{eL}{1-e^2}\right)^2}{\left(\frac{L}{1-e^2}\right)^2} + \frac{y^2}{\left(\frac{L}{\sqrt{1-e^2}}\right)^2} = 1. \tag{8}$$

This is the standard form given above in (1) for an ellipse centred at $\left(-\frac{eL}{1-e^2}, 0\right)$ with major axis $a = \frac{L}{1-e^2}$ and minor axis $b = \frac{L}{\sqrt{1-e^2}}$.

2.2 $e > 1$: A hyperbolic orbit

When $e > 1$ we have to re-arrange (6) in a different way to take account of the difference in sign on the y^2 term. Then we get

$$\frac{\left(x - \frac{eL}{e^2-1}\right)^2}{\left(\frac{L}{e^2-1}\right)^2} - \frac{y^2}{\left(\frac{L}{\sqrt{e^2-1}}\right)^2} = 1. \tag{9}$$

This is the equation for a hyperbola as in (2) but not centred at $(0,0)$, with $a = \frac{L}{e^2-1}$ and $b = \frac{L}{\sqrt{e^2-1}}$.

2.3 $e = 1$: A parabolic orbit

When $e = 1$, (5) simply reduces to $y^2 = L^2 - 2Lx$, which is the equation for a parabola, again not centred at $(0,0)$.