Mathematics: Linear Algebra

Lecture Notes

Prof. Kim Christensen

Complexity and Networks Programme
Institute for Mathematical Sciences
53 Prince's Gate, Exhibition Road
Imperial College London
SW7 2PG London
United Kingdom

and

Condensed Matter Theory
Blackett Laboratory
Imperial College London
South Kensington
London SW7 2AZ
United Kingdom.

E-mail: k.christensen@imperial.ac.uk

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Introduction

A huge welcome to the mathematical course on *Linear Algebra*. It will be a privilege to introduce you to the world of *Linear Algebra* and I hope that we will have an enjoyable time. *"Oh no"*, you might say. *"I didn't come to the Department of Physics at Imperial College London to learn mathematics. I came here to learn physics!"* Well, that would be the equivalent of a music student at Royal College of Music, London stating: *"Why do I need to learn nodes? I want to be a composer!"*. Mathematics is the universal language for physics and indeed science in general. It is a tool. It is a precision tool that we can use to dissect and investigate phenomena. It is very important to be able to master the tool (mathematics) so you can concentrate on the physics involved in the phenomena you are considering. I will certainly do my utmost to teach you all I know about *Linear Algebra* during the course. The aim is that you should build up a thorough understanding of the material so that you will be able to stand on your own two feet and go on adventures on your own in the world of *Linear Algebra* when the course terminates.

Before we embark on the course, let me give you a brief introduction to the structure and content of the course.

- Lectures [1–17]
- Classworks [1–8]
- Office hours [twice a week - Mondays noon-1pm & Fridays 1pm-2pm, Blackett Lab. 812]
- Assessed problems sheets [3]
- Lecture notes [pp 1–159]
- Problems for lectures [3–15]
- Problems for lectures: Answers [3–15]

All the material is available via the web-site learn.imperial.ac.uk under the item *Mathematics: Linear Algebra (Physics Year 1 Term 1)*.

Regarding the content of the course, then we will be dealing with

- Coordinate systems
- Scalars and vectors
- Vector spaces - a generalisation of algebra with real numbers in one dimension to vectors in n dimensions.

- Geometry - lines and planes
- Linear equations - n equations in n unknowns and n equations in m unknowns
- Matrices and determinants
- Matrices and linear functions (transformations)
- Eigenvalues, eigenvectors and associated concepts.

Finally, I would be very grateful for any suggestions you may have to improve these notes. Maybe you spot some typos or maybe you have a suggestion for how to make the presentation more clear. Do you know of an illustrative example or application that would be useful to include? Also, should you have any questions about the material covered, please do let me know. Then I will produce an FAQs appendix with associated answers. For any such suggestions or questions, please send an e-mail to *k.christensen@imperial.ac.uk* using the subject "Linear Algebra".

Kim Christensen
London, October 2010

Linear Algebra

1 Cartesian Coordinate Systems

The aim is to introduce Cartesian coordinate systems and, in particular, the notion of a right-handed $2D$ Cartesian coordinate system and a right-handed $3D$ Cartesian coordinate system.

The French mathematician and philosopher *René Descartes*[1] (Latin: Cartesius) introduced the notion of a Cartesian coordinate system (also known as a rectangular coordinate system) in 1637 to allow one to specify the position of a point in space.

Definition 1.1. A coordinate system for which the coordinates of a point are its distances from a set of perpendicular directed lines (axes) that intersect in the origin of the system is called a *Cartesian coordinate system* or a *rectangular coordinate system.*

1.1 One-dimensional space, $\mathbb{R}$

Let us consider the one-dimensional space $\mathbb{R}$, that is, the line. Here, the coordinate system consists of a single directed line. One must specify a point on the line to be called the origin, a unit of distance, and a direction on the line to be called positive. The opposite direction is then called negative. Then a positive number $x_0 > 0$ corresponds to the point which is a distance x_0 in the positive direction from the origin. A negative number $x_0 < 0$ corresponds to the point which is a distance $|x_0|$ from the origin in the negative direction.[2] The number zero $x_0 = 0$ corresponds to the origin. In Fig. 1.1, we have identified the origin on the line (axis) with '0' and the unit of distance (scale) by specifying the position of '+1', that is, the positive direction is towards the right as indicated by the arrow on the axis.

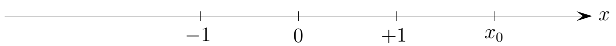

Figure 1.1: The set of real numbers $\mathbb{R}$ is represented geometrically as a straight line. The conventional choice is for the line to be horizontal with the positive direction towards the right.

[1] René Descartes (1596 – 1650) was a French philosopher, mathematician and scientist. His book *Principles of Philosophy (Principia philosophiae)*, published in 1644, contains his famous statement *"Cogito ergo sum" – "I think, therefore I am"*.

[2] $|x|$ denotes the *absolute value* of x, that is, $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$.

1.2 Two-dimensional space, $\mathbb{R}^2$

Let us consider the two dimensional space $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$, that is, the plane. Let us draw two perpendicular directed lines that intersect at the origin. These two lines are often called the x- and y-axis (or the x_1- and x_2-axis), respectively. We have to specify a positive direction for each of the two axis. Given the x-axis, which is normally the horizontal axis with positive to the right, we have two choices for the positive direction of the y-axis, namely upwards or downwards.

1. If the positive direction is upwards, the two axes form a so-called *right-handed coordinate system*, see Fig. 1.2(a).

2. If the positive direction is downwards, the two axes form a so-called *left-handed coordinate system*, see Fig. 1.2(b).

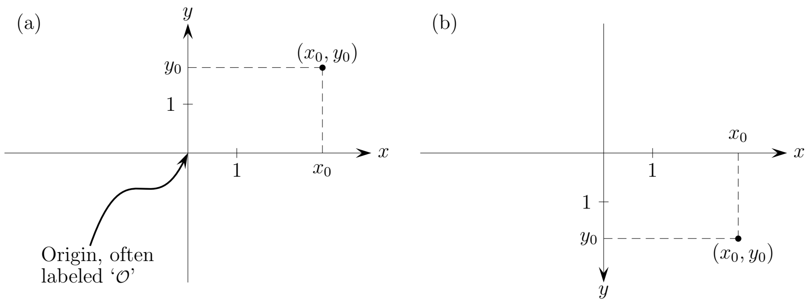

Figure 1.2: The two-dimensional plane $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ is represented geometrically by two perpendicular axis that intersect in the origin. Given a point (x_0, y_0) in the plane. That corresponds to the point whose (perpendicular) projection onto the x-axis is x_0 and whose (perpendicular) projection onto the y-axis is y_0. (a) A right-handed coordinate system. (b) A left-handed coordinate system. Note that the handedness of these coordinate systems is invariant under rotation.

The rule for determining whether a Cartesian coordinate system in two dimensions is right-handed or left-handed is as follows:

Rule: Place your right hand on the plane with the thumb pointing up and consider the smallest angle between the positive x- and y-axis. If the fingers point from the positive x- to the positive y-axis, it is a right-handed coordinate system. Otherwise, it is left handed.

The conventional choice is to take the x-axis horizontal with the positive direction towards the right and the y-axis vertical with the positive direction upwards, that is, a right-handed coordinate system, see Fig. 1.2(a).

1.3 Three-dimensional space, $\mathbb{R}^3$

Let us consider the three-dimensional space $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3$ and draw three perpendicular directed axes through the origin, see Fig. 1.3. These three axes are often called the x-, y- and z-axis (or the x_1-, x_2- and x_3-axis), respectively. We have to specify a positive direction for each of the three axis. Having specified the x- and y-axis, that normally specifies the horizontal plane with positive x-axis pointing outwards and the positive y-axis a horizontal axis with positive to the right, we have two choices for the positive direction of the z-axis.

1. If the positive direction of the z-axis is upwards, the three axes form a so-called *right-handed coordinate system*, see Fig. 1.3(a).
2. If the positive direction of the z-axis is downwards, the three axes form a so-called *left-handed coordinate system*, see Fig. 1.3(b).

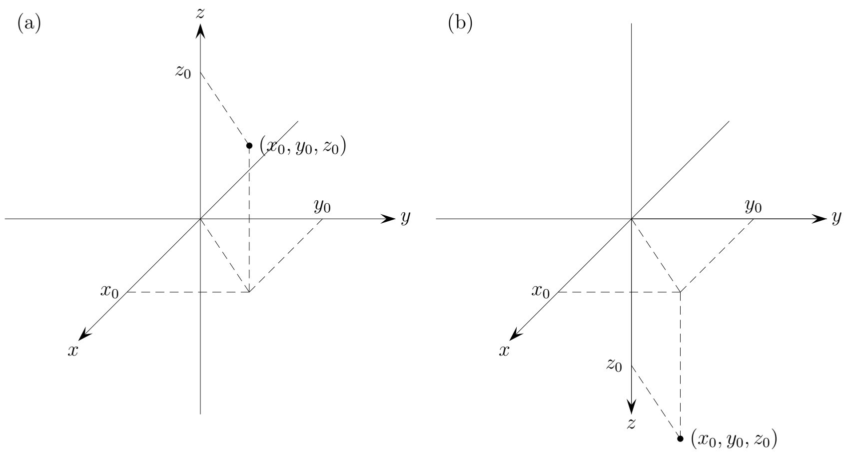

Figure 1.3: The three-dimensional space $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3$ is represented geometrically by three perpendicular axes that intersect in the origin. Given a point (x_0, y_0, z_0) in space. That corresponds to the point whose (perpendicular) projection onto the x-axis is x_0, whose (perpendicular) projection onto the y-axis is y_0, and whose (perpendicular) projection onto the z-axis is z_0. (a) A right-handed coordinate system. (b) A left-handed coordinate system. Note that the handedness of these systems is invariant under rotation.

The rule for determining whether a Cartesian coordinate system in three dimensions is right-handed or left-handed is as follows:

Rule: Using your right hand, let the index finger point along the positive direction of the x-axis and the middle finger along the positive direction of the y-axis. If your thumb points along the positive direction of the z-axis, it is a right-handed Cartesian coordinate system. Otherwise, it is left handed.

1.4 Summary

The notion of right-handedness is not defined for the one-dimensional coordinate system discussed in Sec. 1.1. By convention, we will use right-handed coordinate systems in two dimensions ($\mathbb{R}^2$) and three dimensions ($\mathbb{R}^3$) throughout this course.

After studying Sec. 1, you should know

- what is meant by a right-handed 2D Cartesian coordinate system
- what is meant by a right-handed 3D Cartesian coordinate system.

2 Scalars and vectors

The aim is to introduce the notion of *scalars*, *vectors*, operations (addition and scalar multiplication) with vectors and *linear vector spaces*.

2.1 Scalars

Definition 2.1. A *scalar* is a quantity that is completely specified by a single real number together with its units. We say that scalars are one-dimensional objects.

Example 2.1. Examples of scalars in physics are temperature T = 300 K, time $t = 12.3$ s, density $\rho = 12.25$ kg m^{-3}, mass $m = 69.5$ kg, and charge e $= -1.6 \cdot 10^{-19}$ C to name a few. A scalar can be represented geometrically as a point in the one-dimensional coordinate system representing the real numbers, see Fig. 1.1.

2.2 Vectors

Definition 2.2. A *vector* is a quantity that is specified by a magnitude (length) greater than or equal to zero and a direction (except for the zero-vector). Like scalars, vectors also have units. We say that vectors are 2-*dimensional* objects if the vector has 2 components, and 3-*dimensional* objects if the vector has 3 components.

A vector can be represented geometrically in a coordinate system as directed arrows with a length equal to its magnitude, see Sec. 2.3 below for details.

Example 2.2. Examples of vectors in physics are position $\mathbf{r} = (x, y, z)$, velocity $\mathbf{v} = (v_x, v_y, v_z)$, momentum $\mathbf{p} = (p_x, p_y, p_z)$, acceleration $\mathbf{a} = (a_x, a_y, a_z)$, force $\mathbf{F} = (F_x, F_y, F_z)$, electric field $\mathbf{E} = (E_x, E_y, E_z)$, magnetic field $\mathbf{B} = (B_x, B_y, B_z)$. These are normally 2- or 3-dimensional objects.

Our aim is to introduce the general notion of a so-called vector space and then focus on particular examples of vectors in two dimensions (2D or $\mathbb{R}^2$), three dimensions (3D or $\mathbb{R}^3$), and more generally in n dimensions (nD or $\mathbb{R}^n$). Indeed, it will prove very useful to make this generalisation to n dimensions as we shall see throughout this course.[3] Note also, that when we

[3]You may have heard about theories in Physics and Mathematics that use dimensions larger that 3. For example, the so-called M-theory (a generalisation of string-theory) is formulated in 11 dimensions.

have introduced the notion of a determinant of a matrix, see Sec. 8, we will be able to define the notion of a right-handed rectangular coordinate system generally for any dimension $n \geq 2$, see Sec 8.

Definition 2.3. We denote by $\mathbb{R}^n$ the set of all n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers $x_i \in \mathbb{R}, i = 1, 2, \ldots, n$. We denote the n-tuple by boldface letters $\mathbf{x}$, by underlining $\underline{x}$ or by $\vec{x}$.

It is important to emphasise when you are dealing with a vector. Therefore, use the notation $\underline{x}$ in hand-writing as it is more convenient than the boldface $\mathbf{x}$ commonly used in printed matter.

Definition 2.4. We call the element $\mathbf{x} \in \mathbb{R}^n$ a *vector* (or n-dimensional vector) and the entries $x_i, i = 1, 2, \ldots, n$ are called the *components* or *coordinates* of the vector $\mathbf{x}$.

2.3 Geometrical representation of vectors

Let A and B be two points in $\mathbb{R}^n$. We define the arrow $\overline{AB}$ as the line starting at point A and ending at point B. Two arrows $\overline{AB}$ and $\overline{CD}$ are equivalent, $\overline{AB} \sim \overline{CD}$, if the translation $A \curvearrowright C$ is identical with the translation $B \curvearrowright D$, see Fig. 2.1.

Definition 2.5. The set of all equivalent arrows is a *vector* $\overrightarrow{\mathrm{AB}}$.

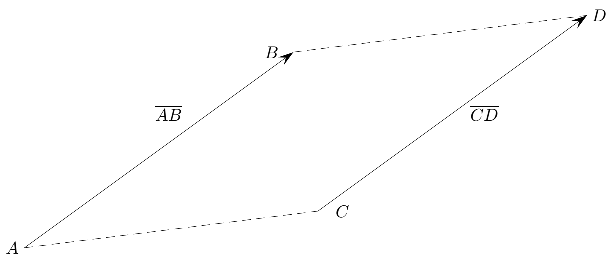

Figure 2.1: Two arrows are displayed: $\overline{AB}$ and $\overline{CD}$. These two arrows are equivalent because the translation that maps A onto C is the same that maps B onto D. We identify the set of all equivalent arrows with the vector $\overrightarrow{\mathrm{AB}}$. Hence, the two arrows shown above are both representations of the vector $\overrightarrow{\mathrm{AB}}$. In general, there are an infinite number of representations of a given vector.

Note that if $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$ then the vector

$$\overrightarrow{\mathrm{AB}} = (b_1 - a_1, b_2 - a_2, \ldots, b_n - a_n)\,, \tag{2.1}$$

that is, the ith coordinates of a vector from point A to point B is found by subtracting a_i from b_i, $i = 1, 2, \ldots, n$.

Example 2.3. In $\mathbb{R}^2$, the vector from point $A = (2, 5)$ to point $B = (1, 3)$ is

$$\overrightarrow{\mathrm{AB}} = (1 - 2, 3 - 5) = (-1, -2)\,. \tag{2.2}$$

For a geometrical representation, see Fig. 2.2.

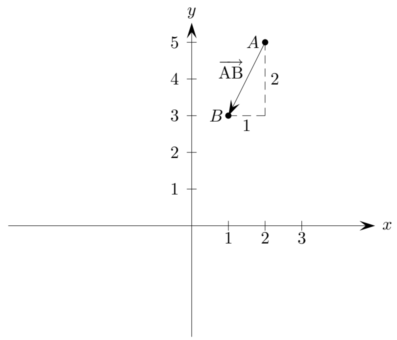

Figure 2.2: Two points $A = (2, 5)$ and $B = (1, 3)$. The vector from point A to point B $\overrightarrow{\mathrm{AB}} = (-1, -2)$. The x-coordinate -1 signifies that the distance along the x-axis between the points A and B is 1 and the negative sign that B is in the negative direction along the x-axis relative to A. The y-coordinate -2 signifies that the distance along the y-axis between the points A and B is 2 and the negative sign that B is in the negative direction along the y-axis relative to A.

Finally, we introduce the concept of a position vector and a displacement vector.

Definition 2.6. A *position vector* is a vector which represents the position of a point or object in relation to the origin of a coordinate system.

Hence, if $A = (a_1, a_2, \ldots, a_n)$ is a point in $\mathbb{R}^n$ then $\mathbf{a} = \overrightarrow{\mathrm{OA}} = (a_1, a_2, \ldots, a_n)$ is the position vector associated with that point, see Fig. 2.3.

Definition 2.7. A *displacement vector* is a vector that specifies the position of a point or object in reference to a previous position.

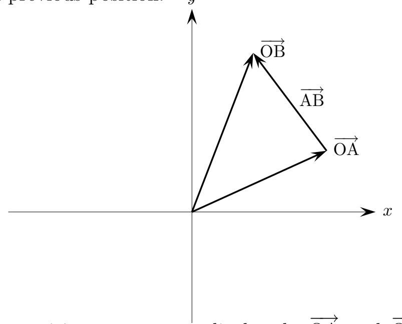

Figure 2.3: Two position vectors are displayed: $\overrightarrow{\mathrm{OA}}$ and $\overrightarrow{\mathrm{OB}}$. One displacement vector is displayed: $\overrightarrow{\mathrm{AB}}$. Note that $\overrightarrow{\mathrm{OA}} + \overrightarrow{\mathrm{AB}} = \overrightarrow{\mathrm{OB}}$ or $\overrightarrow{\mathrm{AB}} = \overrightarrow{\mathrm{OB}} - \overrightarrow{\mathrm{OA}}$.

2.4 Vector operations: Sum and scalar multiplication

Definition 2.8. For any two vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$ we define their sum $\mathbf{x} + \mathbf{y}$ as the n-dimensional vector with components $x_i + y_i, i = 1, 2, \ldots, n$, that is, the sum of the corresponding components in the two vectors (see Fig. 2.4):

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)\,. \tag{2.3}$$

Hence, vectors can only be added if they have the same dimension.

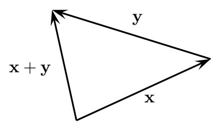

Figure 2.4: Geometrical representation of adding two vectors $\mathbf{x}$ and $\mathbf{y}$. To find $\mathbf{x} + \mathbf{y}$, place the starting point of $\mathbf{y}$ on the ending point of $\mathbf{x}$. The arrow from the starting point of $\mathbf{x}$ to the ending point of $\mathbf{y}$ is then a representative of the vector sum $\mathbf{x} + \mathbf{y}$.

Example 2.4. If $\mathbf{x} = (1, 4, 2)$ and $\mathbf{y} = (5, -2, 6)$ then

$$\mathbf{x} + \mathbf{y} = (1 + 5, 4 + (-2), 2 + 6) = (6, 2, 8)\,. \tag{2.4}$$

Example 2.5. If $\mathbf{F}_1 = (2, 3)$ and $\mathbf{F}_2 = (3, 0)$ are the only two forces acting on a particle, then the total force acting on that particle is given by the vector sum (see Fig. 2.5),

$$\mathbf{F}_{\text{tot}} = \mathbf{F}_1 + \mathbf{F}_2 = (5, 3)\,.$$

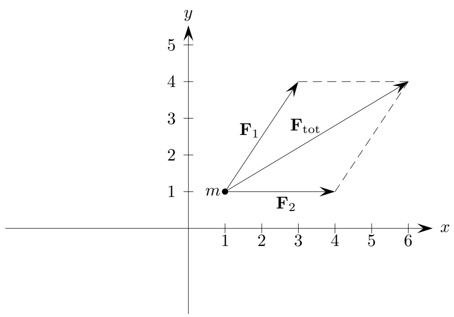

Figure 2.5: Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are acting on a point particle with mass m at position $\mathbf{r} = (1, 1)$. The total force on the particle $\mathbf{F}_{\text{tot}} = \mathbf{F}_1 + \mathbf{F}_2$. The diagram also demonstrate geometrically that the vector sum is commutative because clearly $\mathbf{F}_1 + \mathbf{F}_2 = \mathbf{F}_2 + \mathbf{F}_1$.

Definition 2.9. For any real number $r \in \mathbb{R}$ and any vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ we define the *scalar multiple* $r\mathbf{x}$ as the n-dimensional vector with components $rx_i, i = 1, 2, \ldots, n$, that is, all the components are multiplied with the number r (see Fig. 2.6):

$$r\mathbf{x} = (rx_1, rx_2, \ldots, rx_n)\,. \tag{2.5}$$

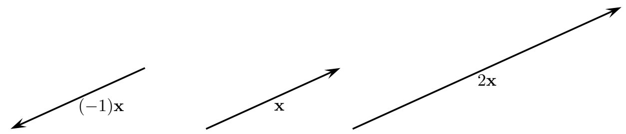

Figure 2.6: Geometrical representation of scalar multiplication of a vector $\mathbf{x}$ with a real number r. When $r > 0$, the direction is preserved and the length is multiplied by a factor of r. When $r < 0$, the direction is reversed and the length is multiplied by a factor of $|r|$. The examples shown are for $r = 2$ and $r = -1$.

Example 2.6. If $\mathbf{x} = (2, -1, 3, 0)$ then

$$3\mathbf{x} = (3 \cdot 2, 3 \cdot (-1), 3 \cdot 3, 3 \cdot 0) = (6, -3, 9, 0)\,. \tag{2.6}$$

Example 2.7. If $\mathbf{y} = (2, -5)$ then

$$(-2)\mathbf{y} = (-4, 10)\,. \tag{2.7}$$

Convention: We write $-\mathbf{x}$ for the (scalar) multiplication by -1, that is,

$$-\mathbf{x} = (-1)\mathbf{x} \tag{2.8}$$

and, similarly,

$$\mathbf{x} - \mathbf{y} = \mathbf{x} + (-1)\mathbf{y} \tag{2.9}$$

and we denote by $\mathbf{0}$ the n-tuple (zero-vector) consisting entirely of zeros, that is,

$$\mathbf{0} = (0, 0, \ldots, 0)\,. \tag{2.10}$$

Example 2.8. If $\mathbf{x} = (2, -1, 3, 0)$ and $\mathbf{y} = (1, 0, -3, 8)$ then

$$\mathbf{x} - \mathbf{y} = (2 - 1, -1 - 0, 3 - (-3), 0 - 8) = (1, -1, 6, -8) \tag{2.11}$$

and

$$\begin{aligned} \mathbf{x} - \mathbf{x} &= (2 - 2, -1 - (-1), 3 - 3, 0 - 0) \\ &= (0, 0, 0, 0) \\ &= \mathbf{0}. \end{aligned} \tag{2.12}$$

2.5 Linear vector space

The following formulas hold for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ and arbitrary $r, s \in \mathbb{R}$:

1. $r\mathbf{x} + s\mathbf{x} = (r+s)\mathbf{x}$ Distributive law.
2. $r\mathbf{x} + r\mathbf{y} = r(\mathbf{x}+\mathbf{y})$ Distributive law.
3. $r(s\mathbf{x}) = (rs)\mathbf{x}$ Associative law for scalar multiplication.
4. $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$ Commutative law for addition.
5. $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x}+(\mathbf{y}+\mathbf{z})$ Associative law for addition.
6. $\mathbf{x}+\mathbf{0} = \mathbf{x}$ Neutral element for addition.
7. $\mathbf{x}+(-\mathbf{x}) = \mathbf{0}$ Inverse element for addition.

These laws for the new operations of addition and scalar multiplication are analogous to the familiar laws for handling real numbers. To show how easy it is prove these formulas, we will explicitly write down the proof of formula 2.

Proof: Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ denote any two vectors in $\mathbb{R}^n$ and $r \in \mathbb{R}$. Then

$$\begin{aligned} r\mathbf{x}+r\mathbf{y} &= (rx_1, rx_2, \ldots, rx_n) + (ry_1, ry_2, \ldots, ry_n) && \text{Def. 2.9} \\ &= (rx_1+ry_1, rx_2+ry_2, \ldots, rx_n+ry_n) && \text{Def. 2.8} \\ &= (r(x_1+y_1), r(x_2+y_2), \ldots, r(x_n+y_n)) && \text{Algebra with real numbers} \\ &= r(x_1+y_1, x_2+y_2, \ldots, x_n+y_n) && \text{Def. 2.9} \\ &= r(\mathbf{x}+\mathbf{y}) && \text{Def. 2.8.} \end{aligned}$$

Q.E.D.

Definition 2.10. Any set with the operations of addition and (scalar) multiplication by a real number defined in such a way that the formulas $1-7$ holds is called a *linear vector space* and its elements are called *vectors*.

Example 2.9. The real numbers $\mathbb{R}$, the plane $\mathbb{R}^2$, the three-dimensional space $\mathbb{R}^3$ are all vector spaces with the definitions of addition and scalar multiplication in 2.8 and 2.9, respectively. Indeed, the n-dimensional space $\mathbb{R}^n$ with the definitions of addition and scalar multiplication in 2.8 and 2.9, respectively is a vector space for $n \geq 1$.

2.6 Natural basis for $\mathbb{R}^n$

For now, we restrict ourselves to consider the vectors that are elements of the n-dimensional vector space $\mathbb{R}^n$. The vectors

$$\begin{aligned} \mathbf{e}_1 &= (1,0,0,\ldots,0,0)\,, \\ \mathbf{e}_2 &= (0,1,0,\ldots,0,0)\,, \\ &\vdots \\ \mathbf{e}_n &= (0,0,0,\ldots,0,1) \end{aligned} \qquad (2.13)$$

where $\mathbf{e}_i$ has entries '0' except for the ith entry which is '1' have the special property that if $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, then

$$\mathbf{x} = (x_1, x_2, \ldots, x_n) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n = \sum_{i=1}^{n} x_i\mathbf{e}_i. \qquad (2.14)$$

Because every vector $\mathbf{x} \in \mathbb{R}^n$ is so simply represented in this way, the set of vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is called the *natural basis* for $\mathbb{R}^n$. The entries x_i in $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ are called the *coordinates* of $\mathbf{x}$ relative to the natural basis.

Clearly, we might formulate addition in terms of the natural basis as

$$\mathbf{x}+\mathbf{y} = \sum_{i=1}^{n} x_i\mathbf{e}_i + \sum_{i=1}^{n} y_i\mathbf{e}_i = \sum_{i=1}^{n}(x_i\mathbf{e}_i + y_i\mathbf{e}_i) = \sum_{i=1}^{n}(x_i+y_i)\,\mathbf{e}_i. \qquad (2.15)$$

and scalar multiplication in terms of the natural basis as

$$r\mathbf{x} = r\sum_{i=1}^{n} x_i\mathbf{e}_i = \sum_{i=1}^{n}(rx_i)\,\mathbf{e}_i. \qquad (2.16)$$

Natural basis for $\mathbb{R}^2$

For $n=2$, that is, $\mathbb{R}^2$ it is common to denote its natural basis with the set $\{\mathbf{i}, \mathbf{j}\}$ where

$$\mathbf{i} = (1,0)\,, \qquad (2.17a)$$
$$\mathbf{j} = (0,1) \qquad (2.17b)$$

rather than with the set $\{\mathbf{e}_1, \mathbf{e}_2\}$. Hence, for example,

$$(4,5) = 4\mathbf{i} + 5\mathbf{j}, \qquad (2.18)$$

or, in general (see Fig. 2.7)

$$(x_0, y_0) = x_0\mathbf{i} + y_0\mathbf{j}. \qquad (2.19)$$

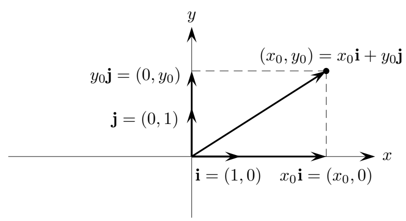

Figure 2.7: A vector $(x_0, y_0) = x_0\mathbf{i} + y_0\mathbf{j}$ in the two-dimensional plane $\mathbb{R}\times\mathbb{R} = \mathbb{R}^2$. The natural basis vectors $\mathbf{i} = (1,0)$ and $\mathbf{j} = (0,1)$ are also shown in the diagram together with the vectors $x_0\mathbf{i} = (x_0, 0)$ and $y_0\mathbf{j} = (0, y_0)$.

Natural basis for $\mathbb{R}^3$

Similarly, for $n = 3$, that is, $\mathbb{R}^3$ it is common to denote its natural basis with the set $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ where

$$\mathbf{i} = (1, 0, 0)\,, \tag{2.20a}$$
$$\mathbf{j} = (0, 1, 0)\,, \tag{2.20b}$$
$$\mathbf{k} = (0, 0, 1) \tag{2.20c}$$

rather than with the set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Hence, for example,

$$(3, -1, 7) = 3\mathbf{i} - \mathbf{j} + 7\mathbf{k}. \tag{2.21}$$

or, in general (see Fig. 2.8)

$$(x_0, y_0, z_0) = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}. \tag{2.22}$$

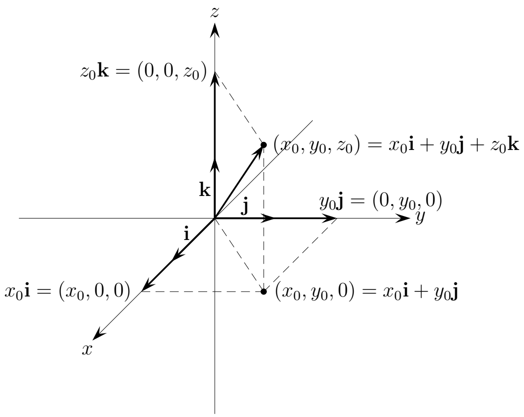

Figure 2.8: A vector $(x_0, y_0, z_0) = x_0\mathbf{i}+y_0\mathbf{j}+z_0\mathbf{k}$ in the three-dimensional space $\mathbb{R}\times\mathbb{R}\times\mathbb{R} = \mathbb{R}^3$. The natural basis vectors $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$ are also shown in the diagram together with the vectors $x_0\mathbf{i} = (x_0, 0, 0)$, $y_0\mathbf{j} = (0, y_0, 0)$ and $z_0\mathbf{k} = (0, 0, z_0)$.

2.7 Magnitude (length) of a vector and unit vectors

Definition 2.11. The magnitude (or length) of a vector $\mathbf{x} \in \mathbb{R}^n$ is defined by

$$|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{\sum_{i=1}^{n} x_i^2}. \tag{2.23}$$

Note that $|\mathbf{x}| \geq 0$ and $|\mathbf{x}| = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

Definition 2.12. A vector with unit magnitude, $|\mathbf{x}| = 1$, is called a *unit vector* or a *normalised vector*.

Note that all the vectors in the natural basis for $\mathbb{R}^n$ are unit vectors because $|\mathbf{e}_i| = 1$.

The unit vector in the direction of any non-zero vector $\mathbf{x}$ is usually denoted $\hat{\mathbf{x}}$ and may be evaluated by

$$\hat{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|}, \quad \text{for } \mathbf{x} \neq \mathbf{0}. \tag{2.24}$$

Example 2.10. The force $\mathbf{F} = (3, 2)$ has magnitude $|\mathbf{F}| = \sqrt{3^3 + 2^2} = \sqrt{13}$. The velocity $\mathbf{v} = (1, 3, -2)$ has magnitude $|\mathbf{v}| = \sqrt{1^2 + 3^2 + (-2)^2} = \sqrt{14}$ also known as the speed.

Example 2.11. If $\mathbf{F} = (3, 2)$, then $|\mathbf{F}| = \sqrt{13}$ and hence a unit vector in the direction of $\mathbf{F}$ is (see Fig. 2.9):

$$\hat{\mathbf{F}} = \frac{\mathbf{F}}{|\mathbf{F}|} = \frac{1}{\sqrt{13}}(3, 2) = \left(\frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}}\right). \tag{2.25}$$

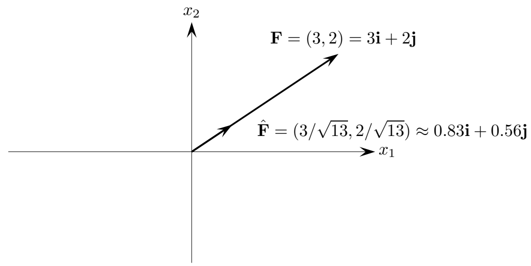

Figure 2.9: The vector $\mathbf{F} = (3, 2)$ and the associated unit vector in the direction of $\mathbf{F}$, namely, $\hat{\mathbf{F}} = \mathbf{F}/|\mathbf{F}| = \left(3/\sqrt{13}, 2/\sqrt{13}\right)$.

2.8 Differentiation of vectors

Consider a vector $\mathbf{r}(t)$ whose coordinates are a function of t.

Definition 2.13. We define the *derivative* of $\mathbf{r}(t)$ with respect to t as

$$\frac{d\mathbf{r}(t)}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \tag{2.26}$$

In general, if

$$\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \tag{2.27}$$

is a vector in $\mathbb{R}^n$ with coordinates $x_i(t), i = 1, 2, \ldots, n$, then the derivative of $\mathbf{x}(t)$ with respect to t is

$$\frac{d\mathbf{x}(t)}{dt} = \left(\frac{dx_1(t)}{dt}, \frac{dx_2(t)}{dt}, \ldots, \frac{dx_n(t)}{dt} \right). \tag{2.28}$$

Example 2.12. Given the position vector of a particle at time t is

$$\mathbf{r}(t) = 2t^2\mathbf{i} + (3t - 2)\,\mathbf{j} + \cos t\,\mathbf{k}. \tag{2.29}$$

Then the associated velocity $\mathbf{v}(t)$ and acceleration $\mathbf{a}(t)$ are

$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = 4t\mathbf{i} + 3\mathbf{j} - \sin t\,\mathbf{k}, \tag{2.30}$$

$$\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \frac{d^2\mathbf{r}(t)}{dt^2} = 4\mathbf{i} - \cos t\,\mathbf{k}. \tag{2.31}$$

2.9 Summary

After studying Sec. 2, you should know

- the meaning of the terms: scalar, vector and vector component (coordinate)
- how to find the vector displacement $\overline{\mathrm{AB}}$ from one point A to another B given the coordinates of A and B
- the meaning of the terms: position vector and displacement vector
- how to add vectors in component form
- how to multiply a vector with a scalar in component form
- how to subtract vectors in component form
- the meaning of the term (linear) vector space
- what is meant by the natural basis for $\mathbb{R}^n$
- how the special unit vectors $\mathbf{i}, \mathbf{j}$ in $\mathbb{R}^2$ are defined
- how the special unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in $\mathbb{R}^3$ are defined
- how to calculate the magnitude (length) of a vector given its components
- what is meant by a unit vector
- how to find a unit vector in the direction of a given vector
- how to differentiate vectors.

3 Dot product of two vectors

The aim is to introduce the operation on vectors, known as the so-called dot (scalar) product $\mathbf{a} \cdot \mathbf{b}$ for vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$.

3.1 Dot-product of vectors in $\mathbb{R}^3$

Consider the three-dimensional space $\mathbb{R}^3$. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ denote two vectors and let θ denote the smallest angle between them $(0 \le \theta \le \pi)$ in the plane defined by $\mathbf{a}$ and $\mathbf{b}$, see Fig. 3.1.

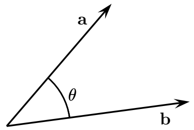

Figure 3.1: The angle θ between two vectors in $\mathbb{R}^3$ is the smallest angle between them $(0 \le \theta \le \pi)$ in the plane defined by $\mathbf{a}$ and $\mathbf{b}$.

Definition 3.1. The dot-product of $\mathbf{a}$ and $\mathbf{b}$ is a scalar equal to the magnitude of $\mathbf{a}$ times the magnitude of $\mathbf{b}$ times the cosine of the angle θ between $\mathbf{a}$ and $\mathbf{b}$, that is,

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta. \tag{3.1}$$

Geometrical interpretation, see Fig. 3.2(a)

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \underbrace{|\mathbf{b}| \cos\theta} = |\mathbf{a}| \text{ (length of projection of } \mathbf{b} \text{ onto } \mathbf{a}) \tag{3.2}$$

or, equivalently, see Fig. 3.2(b)

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{b}| \underbrace{|\mathbf{a}| \cos\theta} = |\mathbf{b}| \text{ (length of projection of } \mathbf{a} \text{ onto } \mathbf{b}). \tag{3.3}$$

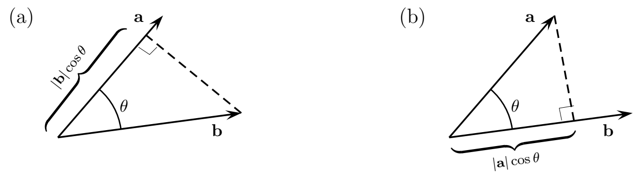

Figure 3.2: The angle between the two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ is denoted θ. (a) The length of the projection of $\mathbf{b}$ onto the direction specified by $\mathbf{a}$ is given by $|\mathbf{b}| \cos\theta$. (b) The length of the projection of $\mathbf{a}$ onto the direction specified by $\mathbf{b}$ is given by $|\mathbf{a}| \cos\theta$.

Relation of dot-product to magnitudes and perpendicularity + formulas

Note that the dot-product of a vector with itself is the magnitude squared, that is, given a vector $\mathbf{a}$ then $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}||\mathbf{a}|\cos 0 = |\mathbf{a}|^2$.

Theorem 3.1. Let $\mathbf{a}, \mathbf{b}$ denote two vectors in $\mathbb{R}^3$ that are not zero vectors, that is, $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$. Then the dot-product is zero if and only if $\mathbf{a}$ is perpendicular to $\mathbf{b}$, that is, $\mathbf{a}\cdot\mathbf{b} = 0 \Leftrightarrow \mathbf{a} \perp \mathbf{b}$.

Proof: ($\Rightarrow$) Assume that $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$. If $\mathbf{a}\cdot\mathbf{b} = 0$ then it follows from the definition Eq.(3.1) that $\cos\theta = 0$ because $|\mathbf{a}| \neq 0$ and $|\mathbf{b}| \neq 0$. Because $0 \leq \theta \leq \pi$, we conclude that $\theta = \pi/2$ and hence $\mathbf{a}$ is perpendicular to $\mathbf{b}$.
($\Leftarrow$) Assume that $\mathbf{a}$ is perpendicular to $\mathbf{b}$. Then $\theta = \pi/2$, implying that $\cos\theta = 0$ and hence $\mathbf{a}\cdot\mathbf{b} = 0$.

Q.E.D.

The following formulas hold for any $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ and arbitrary $r \in \mathbb{R}$:

1. $\mathbf{a}\cdot\mathbf{a} > 0$ except when $\mathbf{a} = \mathbf{0}$.
2. $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$ Commutative law.
3. $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$ Distributive law.
4. $(r\mathbf{a})\cdot\mathbf{b} = r\,(\mathbf{a}\cdot\mathbf{b})$

Dot-product in terms of vector coordinates

For the natural basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ in $\mathbb{R}^3$ (see Fig. 3.3), we find

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1 \quad \text{they are unit vectors} \tag{3.4a}$$
$$\mathbf{i}\cdot\mathbf{j} = \mathbf{i}\cdot\mathbf{k} = \mathbf{j}\cdot\mathbf{k} = 0 \quad \text{they are pairwise perpendicular} \tag{3.4b}$$

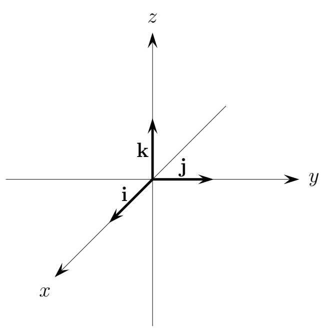

Figure 3.3: The natural basis vectors $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$ and $\mathbf{k} = (0,0,1)$ are unit vectors and they are pairwise perpendicular.

Using the formula and the properties of the natural basis vectors, we can express the dot-product in terms of the coordinates of the vectors. Assume that $\mathbf{a} = (a_x, a_y, a_z)$ and $\mathbf{b} = (b_x, b_y, b_z)$. Then we find

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k})\cdot(b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}) \\ &= a_xb_x + a_yb_y + a_zb_z.\end{aligned} \tag{3.5}$$

Exercise 3.1. Convince yourself about that Sec. 3.1 is also valid in two dimensions, that is, for vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ and Eq. (3.5) would read $\mathbf{a}\cdot\mathbf{b} = a_xb_x + a_yb_y$.

3.2 Dot-product of vectors in $\mathbb{R}^n$

Equation (3.5) suggests a simple generalisation of the definition of the dot-product to vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$:

Definition 3.2. For vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ with $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ we define the dot-product

$$\mathbf{x}\cdot\mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n. \tag{3.6}$$

Hence, from now on, we will take Eq. (3.6) as our definition of the dot-product of two vectors and this will also allows us to define the angle between two non-zero vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$:

Definition 3.3. If $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y} \neq \mathbf{0}$ we define the angle θ between $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ as

$$\cos\theta = \frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}||\mathbf{y}|} \quad 0 \leq \theta \leq \pi. \tag{3.7}$$

Relation of dot-product to magnitudes and perpendicularity + formulas

We notice once again that

$$\mathbf{x}\cdot\mathbf{x} = x_1^2 + x_2^2 + \cdots + x_n^2 = |\mathbf{x}|^2. \tag{3.8}$$

Similarly, we stress that both Theorem 3.1 and the formulas for dot-products listed in Sec. 3.1 generalise to vectors in $\mathbb{R}^n$, that is,

Theorem 3.2. Let $\mathbf{x}, \mathbf{y}$ denote two vectors in $\mathbb{R}^n$ that are not zero vectors, that is, $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y} \neq \mathbf{0}$. Then the dot-product is zero if and only if $\mathbf{x}$ is perpendicular to $\mathbf{y}$, that is, $\mathbf{x}\cdot\mathbf{y} = 0 \Leftrightarrow \mathbf{x} \perp \mathbf{y}$.

The following formulas hold for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ and arbitrary $r \in \mathbb{R}$:

1. $\mathbf{x}\cdot\mathbf{x} > 0$ except when $\mathbf{x} = \mathbf{0}$.
2. $\mathbf{x}\cdot\mathbf{y} = \mathbf{y}\cdot\mathbf{x}$ Commutative law.
3. $\mathbf{x}\cdot(\mathbf{y}+\mathbf{z}) = \mathbf{x}\cdot\mathbf{y} + \mathbf{x}\cdot\mathbf{z}$ Distributive law.
4. $(r\mathbf{x})\cdot\mathbf{y} = r\,(\mathbf{x}\cdot\mathbf{y})$

For the natural basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ in $\mathbb{R}^n$, we find

$$\mathbf{e}_i \cdot \mathbf{e}_i = 1 \quad \text{for all } i = 1, 2, \ldots, n - \text{they are unit vectors} \tag{3.9a}$$
$$\mathbf{e}_i \cdot \mathbf{e}_j = 0 \quad \text{for } i \neq j - \text{they are pairwise perpendicular} \tag{3.9b}$$

and defining the *Kronecker delta* $\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$ we can write this elegantly as

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}. \tag{3.10}$$

Example 3.1. If $\mathbf{x} = (3, 6, 9)$ and $\mathbf{y} = (-2, 3, 1)$, find the angle between $\mathbf{x}$ and $\mathbf{y}$. First, we find the magnitudes

$$|\mathbf{x}| = \sqrt{3^2 + 6^2 + 9^2} = \sqrt{126}$$
$$|\mathbf{y}| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}$$

and the dot-product

$$\mathbf{x} \cdot \mathbf{y} = 3 \cdot (-2) + 6 \cdot 3 + 9 \cdot 1 = 21$$

and hence

$$\cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}||\mathbf{y}|} = \frac{21}{\sqrt{126}\sqrt{14}} \Rightarrow \theta = \pi/3 \quad (0 \leq \theta \leq \pi). \tag{3.11}$$

Example 3.2. Use of dot-product in physics. The work done by a constant force $\mathbf{F} = (F_x, F_y, F_z)$ moving a particle a displacement $\Delta\mathbf{r} = (\Delta x, \Delta y, \Delta z)$ is given by

$$W = \mathbf{F} \cdot \Delta\mathbf{r} = |\mathbf{F}||\Delta\mathbf{r}| \cos\theta = F_x \Delta x + F_y \Delta y + F_z \Delta z.$$

If you know the magnitudes of the force, $|\mathbf{F}|$, and the displacement, $|\Delta\mathbf{r}|$ and the angle in between, use $|\mathbf{F}||\Delta\mathbf{r}| \cos\theta$. If the force and the displacement are given in (Cartesian) coordinates, use $F_x \Delta x + F_y \Delta y + F_z \Delta z$.

3.3 Differentiation of dot product

Consider vectors $\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))\,, \mathbf{y}(t) = (y_1(t), y_2(t), \ldots, y_n(t)) \in \mathbb{R}^n$ whose coordinates are a function of t. From the definition of the dot product and using the product rule for differentiation of functions $(fg)' = f'g + fg'$ we find that (for notational simplicity, we leave out the t-dependence of the coordinates)

$$\begin{aligned} \frac{d}{dt}(\mathbf{x}(t) \cdot \mathbf{y}(t)) &= \frac{d}{dt}(x_1 y_1 + x_2 y_2 + \cdots + x_n y_n) \\ &= \left(\frac{dx_1}{dt} y_1 + \frac{dx_2}{dt} y_2 + \cdots + \frac{dx_n}{dt} y_n\right) + \left(x_1 \frac{dy_1}{dt} + x_2 \frac{dy_2}{dt} + \cdots + x_n \frac{dy_n}{dt}\right) \\ &= \frac{d\mathbf{x}(t)}{dt} \cdot \mathbf{y}(t) + \mathbf{x}(t) \cdot \frac{d\mathbf{y}(t)}{dt}, \end{aligned} \tag{3.12}$$

that is, we have proven that the product rule extends to dot (scalar) products:

$$\frac{d}{dt}(\mathbf{x} \cdot \mathbf{y}) = \frac{d\mathbf{x}}{dt} \cdot \mathbf{y} + \mathbf{x} \cdot \frac{d\mathbf{y}}{dt}. \tag{3.13}$$

3.4 Summary

After studying Sec. 3, you should know

- that the dot-product is a scalar (i.e., not a vector)
- that the dot-product $\mathbf{x} \cdot \mathbf{y}$ is defined for vectors in $\mathbb{R}^n$
- how to calculate the dot-product on component form $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^n x_i y_i$.
- the geometrical interpretation of the dot-product, see Fig. 3.2
- how to use the dot-product to determine the angle $0 \leq \theta \leq \pi$ between two vectors using $\cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}||\mathbf{y}|}$
- that the dot-product of two non-zero vectors is zero if and only if the two vectors are perpendicular, that is, $\mathbf{x} \cdot \mathbf{y} = 0 \Leftrightarrow \mathbf{x} \perp \mathbf{y}$.
- how to use the dot-product to determine the magnitude of a vector, $\mathbf{x} \cdot \mathbf{x} = |\mathbf{x}|^2$.
- how to differentiate a dot-product: $\frac{d}{dt}(\mathbf{x} \cdot \mathbf{y}) = \frac{d\mathbf{x}}{dt} \cdot \mathbf{y} + \mathbf{x} \cdot \frac{d\mathbf{y}}{dt}$.

4 Vector (cross) product of two vectors in $\mathbb{R}^3$

The aim is to introduce the operation on vectors known as the vector (cross) product $\mathbf{a} \times \mathbf{b}$ for vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$.

4.1 Definition of vector product and its geometrical interpretation

Consider the three-dimensional space $\mathbb{R}^3$. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ denote two vectors and let θ denote the smallest angle between them $(0 \leq \theta \leq \pi)$ in the plane defined by $\mathbf{a}$ and $\mathbf{b}$, see Fig. 4.1.

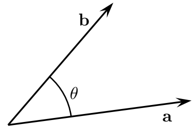

Figure 4.1: The angle θ between two vectors in $\mathbb{R}^3$ is the smallest angle between them $(0 \leq \theta \leq \pi)$ in the plane defined by $\mathbf{a}$ and $\mathbf{b}$.

Definition 4.1. The vector-product (or cross-product) of two vectors $\mathbf{a}$ and $\mathbf{b}$ is a vector $\mathbf{a} \times \mathbf{b}$. The vector $\mathbf{a} \times \mathbf{b}$ has magnitude $|\mathbf{a}||\mathbf{b}| \sin\theta$, that is,

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta, \tag{4.1}$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. The direction of $\mathbf{a} \times \mathbf{b}$ is perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$ such that $\{\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}\}$ form a right-handed coordinate system, see Fig. 4.2(a).

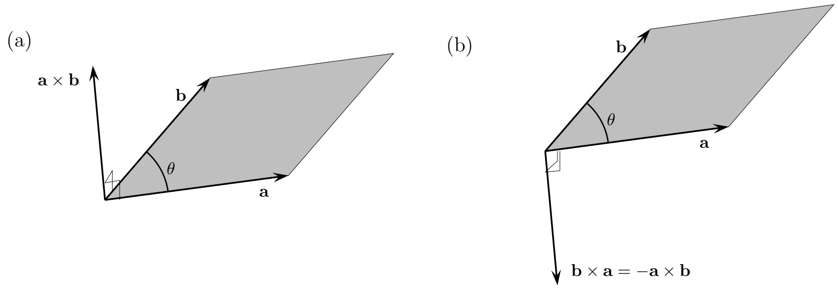

Figure 4.2: (a) The vector product $\mathbf{a} \times \mathbf{b}$ is a vector with magnitude $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$ and direction perpendicular to the plane defined by $\mathbf{a}$ and $\mathbf{b}$ such that the set of vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}\}$ form a right-handed coordinate system. (b) The vector product $\mathbf{b} \times \mathbf{a}$ has same magnitude as $\mathbf{a} \times \mathbf{b}$ but its direction is reversed because $\{\mathbf{b}, \mathbf{a}, \mathbf{b} \times \mathbf{a}\}$ form a right-handed coordinate system, that is, $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$.

Note that the vector product is only defined in $\mathbb{R}^3$, that is, in three dimensions! It cannot be generalised to n-dimensions, $n \neq 3$. Also note that the right-handed coordinate system

$\{\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}\}$ is not necessarily a Cartesian coordinate system because $\mathbf{a}$ need not be perpendicular to $\mathbf{b}$. Only if $\theta = \pi/2$ does $\{\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}\}$ form a right-handed Cartesian coordinate system.

Geometrical interpretation, see Fig. 4.3(a).

$$\begin{aligned} |\mathbf{a} \times \mathbf{b}| &= |\mathbf{a}| \underbrace{|\mathbf{b}| \sin\theta} \\ &= |\mathbf{a}| \text{ (height in parallelogram spanned by the vectors } \mathbf{a} \text{ and } \mathbf{b}) \\ &= \text{area of parallelogram spanned by the vectors } \mathbf{a} \text{ and } \mathbf{b} \end{aligned} \tag{4.2}$$

or, equivalently, see Fig. 4.3(b)

$$\begin{aligned} |\mathbf{a} \times \mathbf{b}| &= |\mathbf{b}| \underbrace{|\mathbf{a}| \sin\theta} \\ &= |\mathbf{b}| \text{ (height in parallelogram spanned by the vectors } \mathbf{a} \text{ and } \mathbf{b}) \\ &= \text{area of parallelogram spanned by the vectors } \mathbf{a} \text{ and } \mathbf{b} \end{aligned} \tag{4.3}$$

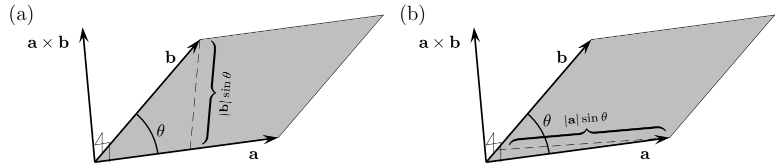

Figure 4.3: The angle between the two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ is denoted θ. (a) Because $|\mathbf{b}| \sin\theta$ is the height in the parallelogram spanned by the two vectors $\mathbf{a}$ and $\mathbf{b}$ and $|\mathbf{a}|$ its base, the magnitude (length) of the vector product $|\mathbf{a}||\mathbf{b}| \sin\theta$ equals the area of the parallelogram. (b) Because $|\mathbf{a}| \sin\theta$ is the height in the parallelogram spanned by the two vectors $\mathbf{a}$ and $\mathbf{b}$ and $|\mathbf{b}|$ its base, the magnitude (length) of the vector product $|\mathbf{a}||\mathbf{b}| \sin\theta$ equals the area of the parallelogram.

Relation of vector product to parallel or anti-parallel vectors

Note that for any vector $\mathbf{a} \neq \mathbf{0}$ we have that $\mathbf{a}$ is parallel with itself, that is, $\theta = 0$ and therefore $\sin\theta = 0$. Hence we can conclude that for all vectors $\mathbf{a} \in \mathbb{R}^3$ we have $\mathbf{a} \times \mathbf{a} = \mathbf{0}$. We can generalise this result to two arbitrary parallel or anti-parallel vectors:

Theorem 4.1. Let $\mathbf{a}, \mathbf{b}$ denote two vectors in $\mathbb{R}^3$ that are not zero vectors, that is, $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$. Then the vector product $\mathbf{a} \times \mathbf{b}$ is zero-vector if and only if $\mathbf{a}$ is parallel or anti-parallel to $\mathbf{b}$, that is, $\mathbf{a} \times \mathbf{b} = \mathbf{0} \Leftrightarrow \mathbf{a}$ and $\mathbf{b}$ are parallel or anti-parallel.

Proof: $(\Rightarrow)$ Assume that $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$. If $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ then if follows from the definition Eq.(4.1) that $\sin\theta = 0$ because $|\mathbf{a}| \neq 0$ and $|\mathbf{b}| \neq 0$. Because $0 \leq \theta \leq \pi$, we conclude that $\theta = 0$ (parallel) or $\theta = \pi$ (anti-parallel).

($\Leftarrow$) Assume that $\mathbf{a}$ and $\mathbf{b}$ are parallel or anti-parallel. Then $\theta = 0$ (parallel) or $\theta = \pi$ (anti-parallel), implying that $\sin\theta = 0$ and hence $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

Q.E.D.

The following formulas hold for any vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ and arbitrary real numbers $r \in \mathbb{R}$:

1. $\mathbf{a} \times \mathbf{a} = \mathbf{0}$,
2. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$, Anti-commutative law.
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$, Distributive law.
4. $(r\mathbf{a}) \times \mathbf{b} = r\,(\mathbf{a} \times \mathbf{b})$.

Vector product in terms of vector coordinates

For the natural basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ in $\mathbb{R}^3$ (see Fig. 4.4(a)), we find

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0} \qquad \text{they are parallel with themselves} \tag{4.4a}$$

$$\mathbf{i} \times \mathbf{j} = +\mathbf{k},\ \mathbf{j} \times \mathbf{k} = +\mathbf{i},\ \mathbf{k} \times \mathbf{i} = +\mathbf{j} \tag{4.4b}$$

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k},\ \mathbf{k} \times \mathbf{j} = -\mathbf{i},\ \mathbf{i} \times \mathbf{k} = -\mathbf{j} \tag{4.4c}$$

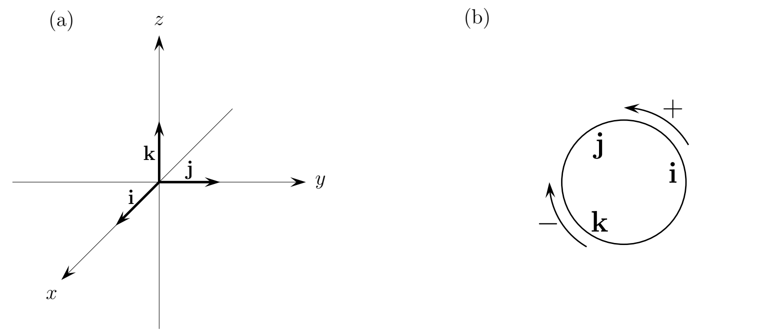

Figure 4.4: (a) The natural basis vectors $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$ are unit vectors and they are pairwise perpendicular. Also, they satisfy $\mathbf{i}\times\mathbf{j} = +\mathbf{k}$; $\mathbf{j}\times\mathbf{k} = +\mathbf{i}$; $\mathbf{k}\times\mathbf{i} = +\mathbf{j}$ and $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$; $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$; $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$. (b) This diagram is designed to show that the sign is positive when the three basis vectors are in cyclic order, that is, ($\mathbf{i}\ \mathbf{j}\ \mathbf{k}\ \mathbf{i}\ \mathbf{j}\ \mathbf{k}\ldots$), see Eq.(4.4b) and negative otherwise, see Eq.(4.4c).

Using the formulas [3-4] and the properties of the natural basis vectors Eqs. (4.4b) and (4.4c), we can express the vector product in terms of the coordinates of the vectors. Assume that $\mathbf{a} = (a_x, a_y, a_z)$ and $\mathbf{b} = (b_x, b_y, b_z)$. Then we find

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}) \times (b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}) \\ &= (a_y b_z - a_z b_y)\,\mathbf{i} - (a_x b_z - a_z b_x)\,\mathbf{j} + (a_x b_y - a_y b_x)\,\mathbf{k}.\end{aligned} \tag{4.5}$$

Vector product expressed by the 3×3 vector-product determinant

Arguably, Eq. (4.5) is a rather complicated formula to remember. Therefore, we will now develop a much more convenient and compact way of writing and indeed evaluating the vector product in terms of the coordinates of the vectors. Consider the following object

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \tag{4.6}$$

known as a 2×2 determinant.

Definition 4.2. Let us call the diagonal which runs from the top left corner to the bottom right corner of a determinant for the *main diagonal*. Then we define a 2×2 determinant as

$$\begin{aligned} \begin{vmatrix} a & b \\ c & d \end{vmatrix} &= \text{product of the two entries in the main diagonal} - \text{product of the two other entries} \\ &= ad - cb. \end{aligned} \tag{4.7}$$

that is, schematically we can illustrate the evaluation of the 2×2 determinant as

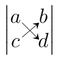

Figure 4.5: The 2×2 determinant is the product of the two entries in the main diagonal, ad, minus the product of the two other entries, cb, that is, $|\cdot| = ad - cb$.

Example 4.1. The 2×2 determinants

$$\begin{vmatrix} 2 & 5 \\ 3 & 4 \end{vmatrix} = 2 \cdot 4 - 3 \cdot 5 = 8 - 15 = -7 \tag{4.8}$$

and

$$\begin{vmatrix} -7 & 6 \\ -2 & 2 \end{vmatrix} = (-7) \cdot 2 - (-2) \cdot 6 = -14 + 12 = -2. \tag{4.9}$$

Using the notation of a 2×2 determinant, we can write

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= (a_y b_z - a_z b_y)\,\mathbf{i} - (a_x b_z - a_z b_x)\,\mathbf{j} + (a_x b_y - a_y b_x)\,\mathbf{k}. \\ &= \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \mathbf{k} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}. \end{aligned} \tag{4.10}$$

In the last equation, we have introduced a so-called 3×3 determinant which is the convenient and compact way of writing and indeed evaluating the vector product we are looking for. We

will later return to the notion of determinants but for now, it suffice to explain how to evaluate the 3×3 determinant.

Let us call the horizontal lines in the determinant for *rows* and the vertical lines for its *columns*. Hence, the three entries in the first row of the 3×3 determinant in Eq. (4.10) are $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$, respectively. We now evaluate (define) the 3×3 determinant in Eq.(4.10) according to

$$\begin{aligned} \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} &= +(\text{determinant of the entries left when deleting the row and column of } \mathbf{i})\,\mathbf{i} \\ &\quad - (\text{determinant of the entries left after deleting the row and column of } \mathbf{j})\,\mathbf{j} \\ &\quad + (\text{determinant of the entries left after deleting the row and column of } \mathbf{k})\,\mathbf{k} \\ &= + \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \mathbf{k}. \end{aligned} \tag{4.11}$$

Note the alternating signs, that is, $+ - +$ of the three terms. Throughout these notes, we will use Eq. (4.11) to evaluate vector (cross) products of vectors.

Example 4.2. Let $\mathbf{a} = (3, 4, -1)$ and $\mathbf{b} = (-2, 5, 6)$. Then the vector product

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & -1 \\ -2 & 5 & 6 \end{vmatrix} \\ &= + \begin{vmatrix} 4 & -1 \\ 5 & 6 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & -1 \\ -2 & 6 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & 4 \\ -2 & 5 \end{vmatrix} \mathbf{k} \\ &= (4 \cdot 6 - 5 \cdot (-1))\,\mathbf{i} - (3 \cdot 6 - (-2) \cdot (-1))\,\mathbf{j} + (3 \cdot 5 - (-2) \cdot 4)\,\mathbf{k} \\ &= 29\mathbf{i} - 16\mathbf{j} + 23\mathbf{k}. \end{aligned} \tag{4.12}$$

Note that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = (29, -16, 23) \cdot (3, 4, -1) = 87 - 64 - 23 = 0$ and similarly, $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = (29, -16, 23) \cdot (-2, 5, 6) = -58 - 80 + 128 = 0$. This should come as no surprise. Indeed, it is a general result.

Because $\mathbf{a} \times \mathbf{b} \perp \mathbf{a}$ and $\mathbf{a} \times \mathbf{b} \perp \mathbf{b}$ for all vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ we have

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = 0, \tag{4.13a}$$

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0. \tag{4.13b}$$

This follows from the definition of the vector product and Theorem 3.1. Also, one can easily prove this by simply writing the left-hand side of Eqs. (4.13) on coordinate form. This is left as an exercise to the reader. Because the dot-product is commutative (see formula 2 on page 17), we also have $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$ and $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0$.

Example 4.3. Find a vector perpendicular to $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$. We know that the vector product $\mathbf{a} \times \mathbf{b} \perp \mathbf{a}$ and $\mathbf{a} \times \mathbf{b} \perp \mathbf{b}$ so we simply evaluate the vector product:

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & 3 & -2 \end{vmatrix} \\
&= + \begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \mathbf{k} \\
&= (1 \cdot (-2) - 3 \cdot (-1))\, \mathbf{i} - (2 \cdot (-2) - 1 \cdot (-1))\, \mathbf{j} + (2 \cdot 3 - 1 \cdot 1)\, \mathbf{k} \\
&= \mathbf{i} + 3\mathbf{j} + 5\mathbf{k}. \qquad (4.14)
\end{aligned}$$

Let us check that we haven't made a simple mistake in evaluating $\mathbf{a} \times \mathbf{b}$: $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 2 \cdot 1 + 1 \cdot 3 + (-1) \cdot 5 = 0$ and $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 1 \cdot 1 + 3 \cdot 3 + (-2) \cdot 5 = 0$, which confirms that $\mathbf{a} \times \mathbf{b} \perp \mathbf{a}$ and $\mathbf{a} \times \mathbf{b} \perp \mathbf{b}$.

Example 4.4. Let $\mathbf{a} = (1, -2, 4)$ and $\mathbf{b} = (3, -6, 12)$. Then

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 4 \\ 3 & -6 & 12 \end{vmatrix} \\
&= + \begin{vmatrix} -2 & 4 \\ -6 & 12 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 4 \\ 3 & 12 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -2 \\ 3 & -6 \end{vmatrix} \mathbf{k} \\
&= ((-2) \cdot 12 - (-6) \cdot 4)\, \mathbf{i} - (1 \cdot 12 - 3 \cdot 4)\, \mathbf{j} + (1 \cdot (-6) - 3 \cdot (-2))\, \mathbf{k} \\
&= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} \\
&= \mathbf{0}. \qquad (4.15)
\end{aligned}$$

This is expected because $\mathbf{b} = 3\mathbf{a}$ and hence, $\mathbf{a} \parallel \mathbf{b}$ so applying Theorem 4.1 yields $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

4.2 Differentiation of vector product

Consider the vectors $\mathbf{a}(t) = (a_x(t), a_y(t), a_z(t))\,, \mathbf{b}(t) = (b_x(t), b_y(t), b_z(t)) \in \mathbb{R}^3$ whose coordinates are a function of t. From Eq. (4.5) and using the product rule for differentiation of functions $(fg)' = f'g + fg'$ we find that (for notational simplicity, we leave out the t-dependence of the coordinates)

$$\begin{aligned}
\frac{d}{dt}(\mathbf{a}(t) \times \mathbf{b}(t)) &= \frac{d}{dt}\left[(a_y b_z - a_z b_y)\, \mathbf{i} - (a_x b_z - a_z b_x)\, \mathbf{j} + (a_x b_y - a_y b_x)\, \mathbf{k}\right] \\
&= \left(\frac{da_y}{dt} b_z - \frac{da_z}{dt} b_y\right) \mathbf{i} - \left(\frac{da_x}{dt} b_z - \frac{da_z}{dt} b_x\right) \mathbf{j} + \left(\frac{da_x}{dt} b_y - \frac{da_y}{dt} b_x\right) \mathbf{k} \\
&+ \left(a_y \frac{db_z}{dt} - a_z \frac{db_y}{dt}\right) \mathbf{i} - \left(a_x \frac{db_z}{dt} - a_z \frac{db_x}{dt}\right) \mathbf{j} + \left(a_x \frac{db_y}{dt} - a_y \frac{db_x}{dt}\right) \mathbf{k} \\
&= \frac{d\mathbf{a}(t)}{dt} \times \mathbf{b}(t) + \mathbf{a}(t) \times \frac{d\mathbf{b}(t)}{dt}, \qquad (4.16)
\end{aligned}$$

that is, we have proven that the product rule extends to vector (cross) products:

$$\frac{d}{dt}(\mathbf{a} \times \mathbf{b}) = \frac{d\mathbf{a}}{dt} \times \mathbf{b} + \mathbf{a} \times \frac{d\mathbf{b}}{dt}. \qquad (4.17)$$

4.3 Examples of vector products in physics

Example 4.5. Force on a particle with charge q moving with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$ is given by the Lorenz force (in zero electrical field $\mathbf{E}$):

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}. \qquad (4.18)$$

Assume that the velocity is in the plane, that is, $\mathbf{v} = (v_x, v_y, 0)$ and that $\mathbf{B}$ is along the z-axis, that is, $\mathbf{B} = (0, 0, B_z)$. Then the force on the moving particle

$$\begin{aligned}
\mathbf{F} &= q\mathbf{v} \times \mathbf{B} \\
&= q \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_x & v_y & 0 \\ 0 & 0 & B_z \end{vmatrix} \\
&= q \left(+ \begin{vmatrix} v_y & 0 \\ 0 & B_z \end{vmatrix} \mathbf{i} - \begin{vmatrix} v_x & 0 \\ 0 & B_z \end{vmatrix} \mathbf{j} + \begin{vmatrix} v_x & v_y \\ 0 & 0 \end{vmatrix} \mathbf{k} \right) \\
&= q\,(v_y B_z \mathbf{i} - v_x B_z \mathbf{j}) \qquad (4.19)
\end{aligned}$$

that is, $\mathbf{F}$ is in the plane as it has no $\mathbf{k}$-component. Furthermore, we note that

$$\begin{aligned}
\mathbf{F} \cdot \mathbf{v} &= q\,(v_y B_z \mathbf{i} - v_x B_z \mathbf{j}) \cdot (v_x \mathbf{i} + v_y \mathbf{j}) \\
&= q\,(v_y B_z v_x - v_x B_z v_y) \\
&= 0. \qquad (4.20)
\end{aligned}$$

Hence, we can conclude that the force $\mathbf{F}$ is perpendicular to the velocity $\mathbf{v}$, that is, $\mathbf{F} \perp \mathbf{v}$. The instantaneous power, or work per time is indeed given by $\mathbf{F} \cdot \mathbf{v}$ so we can conclude that the force does not do any work on the particle. The kinetic energy of the particle is constant and therefore, the particle must move (in a circle) with constant speed $|\mathbf{v}|$.

Example 4.6. What determines how effective a force $\mathbf{F}$ is in causing or changing rotational motion? Magnitude of force $|\mathbf{F}|$, direction of force and the point of action, see Fig. 4.6

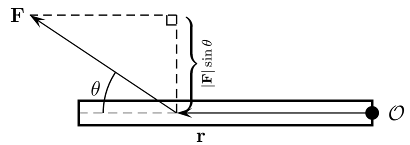

Figure 4.6: The force $\mathbf{F}$ applied at a point described by the position vector $\mathbf{r}$ with respect to the rotational axis $\mathcal{O}$. The angle θ is defined in the diagram - it is the angle between the position vector $\mathbf{r}$ and the applied force $\mathbf{F}$. The magnitude of the torque $|\vec{\tau}| = |\mathbf{r}||\mathbf{F}| \sin\theta$. The magnitude of the force perpendicular to the lever arm is given by $|\mathbf{F}| \sin\theta$. It is this component of the force that causes the change in rotational motion. The torque $\vec{\tau} = \mathbf{r} \times \mathbf{F}$ is directed into the page. Placing your right-hand with the thumb pointing in the direction of the torque, the fingers will point in the direction of rotation that the torque tends to cause.

Definition 4.3. We define the *torque* of the force $\mathbf{F}$ with respect to the point $\mathcal{O}$ as the vector that emerges when taking the vector product between $\mathbf{r}$ and $\mathbf{F}$, that is,

$$\vec{\tau} = \mathbf{r} \times \mathbf{F}. \tag{4.21}$$

The magnitude of the torque

$$|\vec{\tau}| = |\mathbf{r}||\mathbf{F}| \sin\theta \tag{4.22}$$

where θ is the angle between the vector $\mathbf{r}$ and $\mathbf{F}$.

Notice that to maximise the torque, you would let $\theta = \pi/2$ and maximise the distance $|\mathbf{r}|$. That is why, when you want to open a swing door using the minimal force, you should maximize $|\mathbf{r}|$ and attack the door at $\theta = \pi/2$.

Example 4.7.⋆ Maxwell's equations. Define the so-called del-operator $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$. Then two of Maxwell's equations can be elegantly written as

$$\nabla \cdot \mathbf{B} = 0 \tag{4.23a}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{Faraday's law} \tag{4.23b}$$

where the left-hand side of Eq. (4.23a) signifies

$$\begin{aligned} \nabla \cdot \mathbf{B} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (B_x, B_y, B_z) \\ &= \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \end{aligned} \tag{4.24a}$$

and the left-hand side of Eq. (4.23b) signifies

$$\begin{aligned} \nabla \times \mathbf{E} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & E_y & E_z \end{vmatrix} \\ &= + \begin{vmatrix} \partial/\partial_y & \partial/\partial z \\ E_y & E_z \end{vmatrix} \mathbf{i} - \begin{vmatrix} \partial/\partial x & \partial/\partial z \\ E_x & E_z \end{vmatrix} \mathbf{j} + \begin{vmatrix} \partial/\partial x & \partial/\partial_y \\ E_x & E_y \end{vmatrix} \mathbf{k} \\ &= \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) \mathbf{i} - \left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) \mathbf{j} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) \mathbf{i}. \end{aligned} \tag{4.24b}$$

4.4 Summary

After studying Sec. 4, you should know

- that the vector product $\mathbf{a} \times \mathbf{b}$ is defined for vectors in $\mathbb{R}^3$
- that the vector-product is a vector (i.e., not a scalar)
- how to calculate the vector-product of two vectors (magnitude $|\mathbf{a}||\mathbf{b}| \sin\theta$ with direction perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$ such the $\{\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}\}$ form a right-handed coordinate system) and its geometrical interpretation, namely that $|\mathbf{a} \times \mathbf{b}|$ represents the area of the parallelogram spanned by the vectors $\mathbf{a}$ and $\mathbf{b}$
- that the vector-product of two (non-zero) vectors is zero-vector if and only if they are parallel or anti-parallel, that is, $\mathbf{a} \times \mathbf{b} = \mathbf{0} \Leftrightarrow \mathbf{a}$ and $\mathbf{b}$ are parallel or anti-parallel
- how to calculate the vector-product on component form using the notion of a 3×3 vector-product determinant $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$
- how to differentiate a vector product: $\frac{d}{dt}(\mathbf{a} \times \mathbf{b}) = \frac{d\mathbf{a}}{dt} \times \mathbf{b} + \mathbf{a} \times \frac{d\mathbf{b}}{dt}$.

5 Geometry

The aim is to define the notions of *direction ratios* and *direction cosines* and then continue with a discussion of

- one-dimensional lines in $\mathbb{R}^n$
- two-dimensional planes in $\mathbb{R}^3$.

Finally, we briefly mention the generalisation of the latter, that is, an $(n-1)$-dimensional hyper-plan in $\mathbb{R}^n$.

5.1 Direction ratios and direction cosines in $\mathbb{R}^3$

Consider a point (d_x, d_y, d_z) in 3D space with position vector $\mathbf{d} = (d_x, d_y, d_z) = d_x\mathbf{i} + d_y\mathbf{j} + d_z\mathbf{k}$.

Definition 5.1. If the vector $\mathbf{d}$ defines a *direction*, we refer to the coefficients d_x, d_y, d_z as *direction ratios*.

A direction $\mathbf{d}$ in three-dimensional space may also be specified by three angles α, β and γ that the vector $\mathbf{d}$ makes with the x-, y-, and z-axis, respectively, see Fig. 5.1.

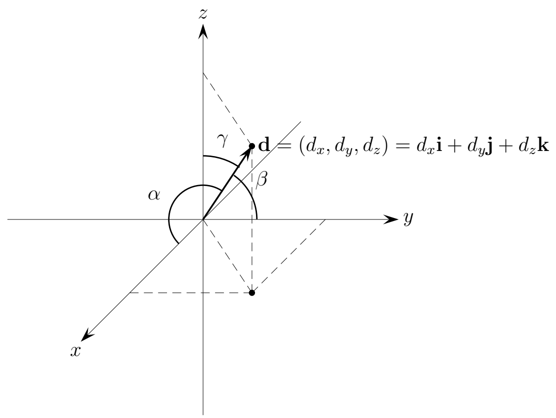

Figure 5.1: The point (d_x, d_y, d_z) in $3D$ space with position vector $\mathbf{d} = (d_x, d_y, d_z)$. The direction $\mathbf{d}$ may be specified by the Cartesian coordinates d_x, d_y and d_z (the so-called direction ratios) or the angles α, β and γ that the vector $\mathbf{d}$ makes with the x-, y-, and z-axis, respectively. Note that these angles are defined in the planes defined by the vector $\mathbf{d}$ and the respective coordinate axes. Hence, the arcs (representing the angles) displayed in the figure are a bit misleading.

What is the relationship between the set of the three Cartesian coordinates $\{d_x, d_y, d_z\}$ and the set of the three angles $\{\alpha, \beta, \gamma\}$? By definition, the Cartesian coordinates d_x, d_y and d_z, are the

(perpendicular) projections onto the x-, y- and z-axis, respectively, that is,

$$d_x = \mathbf{d}\cdot\mathbf{i} = |\mathbf{d}||\mathbf{i}|\cos\alpha = |\mathbf{d}|\cos\alpha, \tag{5.1a}$$
$$d_y = \mathbf{d}\cdot\mathbf{j} = |\mathbf{d}||\mathbf{j}|\cos\beta = |\mathbf{d}|\cos\beta, \tag{5.1b}$$
$$d_z = \mathbf{d}\cdot\mathbf{k} = |\mathbf{d}||\mathbf{k}|\cos\gamma = |\mathbf{d}|\cos\gamma, \tag{5.1c}$$

where $|\mathbf{d}| = \sqrt{d_x^2 + d_y^2 + d_z^2}$ is the magnitude of the direction vector $\mathbf{d}$. Hence, the unit vector $\hat{\mathbf{d}}$ along the direction vector $\mathbf{d}$:

$$\hat{\mathbf{d}} = \frac{\mathbf{d}}{|\mathbf{d}|} = \left(\frac{d_x}{|\mathbf{d}|}, \frac{d_y}{|\mathbf{d}|}, \frac{d_z}{|\mathbf{d}|}\right) = (\cos\alpha, \cos\beta, \cos\gamma)\,. \tag{5.2}$$

Definition 5.2. We define $\cos\alpha, \cos\beta$ and $\cos\gamma$ as *direction cosines* of $\mathbf{d}$, i.e., the direction cosines of a vector are the cosines of the angles between the vector and the 3 coordinate axes.

Clearly, because $\hat{\mathbf{d}}$ is a unit vector, the direction cosines must satisfy

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \tag{5.3}$$

The notion of direction cosines can be generalised to $\mathbb{R}^n$ by defining the direction cosines of a vector $\mathbf{d} \in \mathbb{R}^n$ as the cosines of the angles between the vector and the natural basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ given by $\cos\alpha_i = \mathbf{d}\cdot\mathbf{e}_i/|\mathbf{d}|, i = 1, 2, \ldots, n$.

5.2 Equation of a line

Equation of line in $\mathbb{R}^n$ on vector form

Consider a line with direction $\mathbf{d}$ passing through a point R_0 with position vector $\mathbf{r}_0$, see Fig. 5.2. A position vector $\mathbf{r}$ of any point R on the line can be written on a *parametric vector form*

$$\mathbf{r} = \mathbf{r}_0 + \lambda\mathbf{d}, \quad \lambda \in \mathbb{R}. \tag{5.4}$$

Different values of the (real) parameter λ give different points $\mathbf{r}$ on the line. In particular, $\lambda = 0$ yields $\mathbf{r}_0$, $\lambda > 0$ gives points $\mathbf{r}$ on the side of $\mathbf{r}_0$ towards which $\mathbf{d}$ is pointing while $\lambda < 0$ yields points on the "opposite side" of $\mathbf{r}_0$. Even though Fig. 5.2 demonstrating the parametric equation, Eq. (5.4), is drawn in $\mathbb{R}^3$, then Eq. (5.4) is valid in all dimensions $n \geq 2$. For a line in $\mathbb{R}^n$, clearly the vectors $\mathbf{r}, \mathbf{r}_0, \mathbf{d} \in \mathbb{R}^n$ while the parameter $\lambda \in \mathbb{R}$.

Equation of line in $\mathbb{R}^n$ on component form

Using the notation $\mathbf{r} = (x_1, x_2, \ldots, x_n)$, $\mathbf{r}_0 = (x_{01}, x_{02}, \ldots, x_{0n})$ and $\mathbf{d} = (d_1, d_2, \ldots, d_n)$, then Eq. (5.4) yields n equations

$$x_1 = x_{01} + \lambda d_1 \tag{5.5a}$$
$$x_2 = x_{02} + \lambda d_2 \tag{5.5b}$$
$$\vdots \quad \vdots$$
$$x_n = x_{0n} + \lambda d_n \tag{5.5c}$$

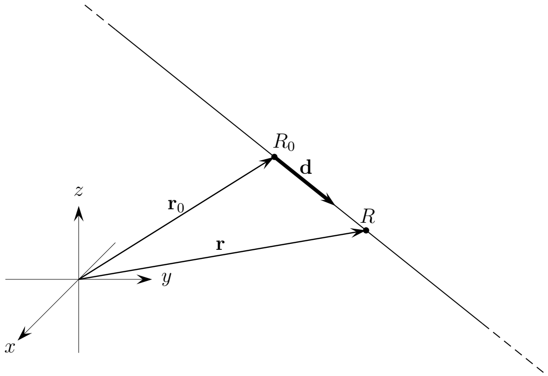

Figure 5.2: The line passing through the point R_0 with position vector $\mathbf{r}_0$ along the direction specified by the direction vector $\mathbf{d}$. Any point R on the line with position vector $\mathbf{r}$ can be written as $\mathbf{r} = \mathbf{r}_0 + \lambda\mathbf{d}$ where λ is a real number.

or, equivalently

$$x_i = x_{0i} + \lambda d_i \quad i = 1, 2, \ldots, n, \ \lambda \in \mathbb{R}. \tag{5.6}$$

These are the equations for a line in n dimensions on component form. Isolating λ, we find

$$\lambda = \frac{x_1 - x_{01}}{d_1} = \frac{x_2 - x_{02}}{d_2} = \cdots = \frac{x_n - x_{0n}}{d_n}. \tag{5.7}$$

We see from Eq. (5.7) that given say x_1, that determines (fixes) $x_2, x_3, \ldots, x_n$. In this sense, there is only one degree of freedom – the line is a one-dimensional object.

Equation of line in $\mathbb{R}^2$ on component form

Consider $n = 2$, that is, $\mathbb{R}^2$. Let $\mathbf{r} = (x, y)$, $\mathbf{r}_0 = (x_0, y_0)$ and $\mathbf{d} = (d_x, d_y) \neq \mathbf{0}$. Then the parametric vector equation for a line in two dimensions is

$$\mathbf{r} = \mathbf{r}_0 + \lambda\mathbf{d}, \quad \lambda \in \mathbb{R}. \tag{5.8}$$

The aim is to show that this general form for a line is, in two dimensions, equivalent to $y - y_0 = a\,(x - x_0)$ for a line passing through (x_0, y_0) with slope a. First, let us write Eq. (5.8) on component form:

$$x = x_0 + \lambda d_x, \tag{5.9a}$$
$$y = y_0 + \lambda d_y. \tag{5.9b}$$

Isolating λ and assuming that $d_x \neq 0$ and $d_y \neq 0$ we find

$$\lambda = \frac{x - x_0}{d_x} = \frac{y - y_0}{d_y}. \tag{5.10}$$

Once again, note that if for example x is given, then that fixes y so Eq. (5.10) has only one degree of freedom. Rearranging Eq. (5.10) we find

$$y - y_0 = \frac{d_y}{d_x}\,(x - x_0) \tag{5.11}$$

and we identify $a = d_y/d_x$ as the slope of the line. Hence, we recover the equation for a line passing through (x_0, y_0) with direction $\mathbf{d} = (d_x, d_y)$ or slope $a = d_y/d_x$ as

$$y - y_0 = a\,(x - x_0)\,. \tag{5.12}$$

In particular, the equation for the line passing through $(0, y_0)$ with slope a is given by

$$y - y_0 = ax. \tag{5.13}$$

Equation of line in $\mathbb{R}^n$ passing through two points R_0 and R_1

What is the equation of a line passing through points R_0 and R_1 ($R_0 \neq R_1$) with position vectors $\mathbf{r_0}$ and $\mathbf{r}_1$, respectively? Note that this question and the following argument is valid in any dimension $n \geq 2$. Clearly, the vector $\mathbf{r}_1 - \mathbf{r_0}$ specifies the direction of the line, so a parametric vector equation for the line is

$$\mathbf{r} = \mathbf{r_0} + \lambda\,(\mathbf{r}_1 - \mathbf{r}_0)\,, \quad \lambda \in \mathbb{R}. \tag{5.14}$$

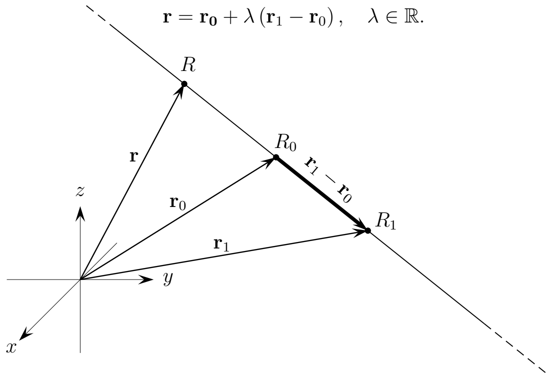

Figure 5.3: A line passing through two points R_0 and R_1 with position vectors $\mathbf{r}_0$ and $\mathbf{r}_1$, respectively. The vector $\mathbf{r}_1 - \mathbf{r}_0$ define a direction vector for the line. Any point R on the line with position vector $\mathbf{r}$ can be written as $\mathbf{r} = \mathbf{r}_0 + \lambda\,(\mathbf{r}_1 - \mathbf{r}_0)$ where λ is a real number.

Example 5.1. Find the equation of a line in $\mathbb{R}^3$ passing through the two points with position vectors $\mathbf{r}_0 = (1, 2, 0)$ and $\mathbf{r}_1 = (2, -1, -1)$. The direction of the line is given by $\mathbf{r}_1 - \mathbf{r}_0 = (2 - 1, -1 - 2, -1 - 0) = (1, -3, -1)$ so a parametric vector equation for this line is

$$\begin{aligned} \mathbf{r} &= \mathbf{r}_0 + \lambda\,(\mathbf{r}_1 - \mathbf{r}_0) \quad \lambda \in \mathbb{R} \\ &= (1, 2, 0) + \lambda\,(1, -3, -1)\,. \end{aligned} \tag{5.15}$$

Denoting $\mathbf{r} = (x, y, z)$, then on component form, this equations separates into three equations

$$x = 1 + \lambda \tag{5.16a}$$
$$y = 2 - 3\lambda \tag{5.16b}$$
$$z = -\lambda \tag{5.16c}$$

or, if we isolate λ:

$$\lambda = \frac{x-1}{1} = \frac{y-2}{-3} = \frac{z-0}{-1} \tag{5.17}$$

from which we also can read off that the line passes through the point $(1, 2, 0)$ and has direction vector $\mathbf{d} = (1, -3, -1)$. Note again that there is only one degree of freedom. Given say x, that fixes both y and z. Indeed, the line is a one-dimensional object.

5.3 Equation of a plane in $\mathbb{R}^3$

A plane is a two-dimensional object in three-dimensional space. We will now derive the equation of a plane through a point R_0 with position vector $\mathbf{r}_0$ and *perpendicular* to a given vector $\mathbf{n}$, the so-called normal vector to the plane.

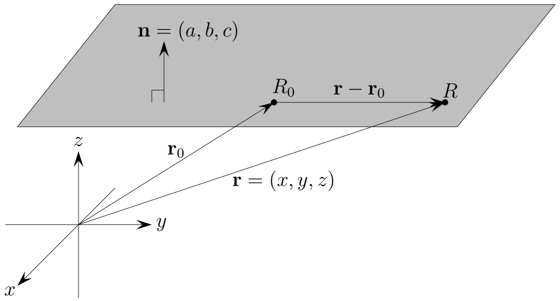

Figure 5.4: Part of a two-dimensional plane in 3D. The plane passes through the point R_0 with position vector $\mathbf{r}_0$ and is perpendicular to the normal vector $\mathbf{n}$ of the plane. Given any point R on the plane with position vector $\mathbf{r}$. Then the vector $\mathbf{r} - \mathbf{r}_0$ is in the plane, implying that $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$.

Consider a position vector $\mathbf{r} = (x, y, z)$ of any point in the plane. The vector $\mathbf{r} - \mathbf{r}_0$ is in the plane and therefore we must have $(\mathbf{r} - \mathbf{r}_0) \perp \mathbf{n}$ for all $\mathbf{r}$ in the plane, see Fig. 5.4. Hence, the equation for the plane can be elegantly written using the dot-product:

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0 \tag{5.18}$$

because the requirement in Eq. (5.18) guarantees that $(\mathbf{r} - \mathbf{r}_0) \perp \mathbf{n}$, see Theorem 3.1.

Let us investigate further Eq. (5.18). Simple re-arrangement yields

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{r}_0 \cdot \mathbf{n}. \tag{5.19}$$

We note that $\mathbf{r}_0$ is a fixed vector and so is $\mathbf{n}$. Hence, the scalar product on the right-hand side of Eq. (5.19) is a constant. Let us denote $\mathbf{r}_0 \cdot \mathbf{n} = k$. Let $\mathbf{r} = (x, y, x)$ and $\mathbf{n} = (a, b, c)$ where a, b, c are the direction ratios of the normal vector. Then from Eq. (5.19) we find

$$(x, y, z) \cdot (a, b, c) = k \Leftrightarrow ax + by + cz = k \tag{5.20}$$

Hence, we arrive at the equation of a plane perpendicular to $\mathbf{n} = (a, b, c)$

$$ax + by + cz = k. \tag{5.21}$$

Knowing a point $\mathbf{r}_0 = (x_0, y_0, z_0)$ that the plane has to pass through will determine the constant $k = \mathbf{r}_0 \cdot \mathbf{n} = ax_0 + by_0 + cz_0$.

Notice that given say x there are now two degrees of freedom in Eq. (5.21), namely y and z. The plane is indeed a two-dimensional object.

Example 5.2. Given the plane $2x + 3y - z = 10$ in $\mathbb{R}^3$. Then we know that the normal vector to this plane is $\mathbf{n} = (2, 3, -1)$. Denoting $\mathbf{r} = (x, y, z)$ we may write the equation of the plane on vector form: $\mathbf{r} \cdot \mathbf{n} = 10$.

If you were keen to find a point $\mathbf{r}_0$ on the plane, you make arbitrarily set for example $x_0 = 1$ and $y_0 = 1$ and then determine z_0 from the equation: $2 + 3 - z_0 = 10$ so $z_0 = 5 - 10 = -5$, that is, the plane passes through the point $\mathbf{r}_0 = (x_0, y_0, z_0) = (1, 1, -5)$. As a check of consistency, we can evaluate $\mathbf{n} \cdot \mathbf{r}_0 = 2 \cdot 1 + 3 \cdot 1 + (-1) \cdot (-5) = 10$ as it must be.

Equation of plane passing through three points R_0, R_1 and R_2

Find the equation of the plane passing through the points R_0, R_1 and R_2 with position vectors $\mathbf{r}_0, \mathbf{r}_1$ and $\mathbf{r}_2$, respectively. We already have a point on the plane (any of the three points will do). Hence, the aim is to find a normal vector to the plane. We notice that the vectors $\mathbf{r}_1 - \mathbf{r}_0$ and $\mathbf{r}_2 - \mathbf{r}_0$ are both vectors in the plane, see Fig. 5.5.

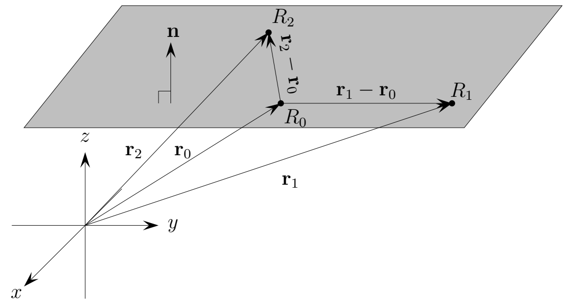

Figure 5.5: Part of a two-dimensional plane in 3D. The plane passes through three points R_0, R_1 and R_2 with position vectors $\mathbf{r}_0, \mathbf{r}_1$ and $\mathbf{r}_2$, respectively. The vector $\mathbf{n} = (\mathbf{r}_1 - \mathbf{r}_0) \times (\mathbf{r}_2 - \mathbf{r}_0)$ is a normal vector to the plane. Hence, given any point R on the plane with position vector $\mathbf{r}$ then the vector $\mathbf{r} - \mathbf{r}_0$ is in the plane, implying that $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$.

Therefore, if we define

$$\mathbf{n} = (\mathbf{r}_1 - \mathbf{r}_0) \times (\mathbf{r}_2 - \mathbf{r}_0) \tag{5.22}$$

then that would be a normal vector to the plane and so the equation for the plane would be

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0. \tag{5.23}$$

Perpendicular distance of plane from the Origin

Let us investigate the right-hand side of Eq. (5.18) further (see Fig. 5.4):

$$\begin{aligned}\mathbf{r}\cdot\mathbf{n} &= \mathbf{r}_0\cdot\mathbf{n}\\ &= |\mathbf{n}|\ \text{(length of the projection of } \mathbf{r}_0 \text{ onto } \mathbf{n})\\ &= |\mathbf{n}|\ \text{(perpendicular distance of plane to origin)}\end{aligned} \tag{5.24}$$

Hence, the constant $k = \mathbf{r}_0 \cdot \mathbf{n}$ is the magnitude of the normal vector multiplied with the perpendicular distance of the plane to the origin. If the normal vector to the plane is a unit normal vector $\hat{\mathbf{n}} = \mathbf{n}/|\mathbf{n}|$, that is, $|\hat{\mathbf{n}}| = 1$, we have

$$\mathbf{r}\cdot\hat{\mathbf{n}} = \text{perpendicular distance of plane to origin} = p \tag{5.25}$$

Example 5.3. In example 5.2 we considered the plane $2x+3y-z = 10$ with normal vector $\mathbf{n} = (2, 3, -1)$. Because $|\mathbf{n}| = \sqrt{2^2 + 3^3 + (-1)^2} = \sqrt{14}$, we can also conclude that the perpendicular distance from the plane to the origin is $p = k/|\mathbf{n}| = 10/\sqrt{14} \approx 2.67$.

Equation of an $n-1$ dimensional hyperplane in $\mathbb{R}^n$

What would happen if we apply Eq. (5.19) in $\mathbb{R}^2$ you may eagerly ask! Well, let's do it.

$$\begin{aligned}\mathbf{r}\cdot\mathbf{n} &= \mathbf{r}_0\cdot\mathbf{n},\\ ax+by &= k.\end{aligned} \tag{5.26}$$

We recognise this as the equation of a line. Indeed, it is the line that is perpendicular to the normal vector $\mathbf{n}$ passing through the point $\mathbf{r}_0$. In general, the equation $\mathbf{r}\cdot\mathbf{n} = \mathbf{r}_0\cdot\mathbf{n}$ or equivalently $\mathbf{r}\cdot\mathbf{n} = k$ is the equation of an $n-1$ dimensional hyperplane in $\mathbb{R}^n$. For example, the plane is a two-dimensional surface in three-dimensional space while the line is a one-dimensional line in two-dimensional space.

5.4 Summary

After studying Sec. 5, you should know

- the meaning of *direction ratios* and *direction cosines*
- the equation of a line on a (parametric) vector form: $\mathbf{r} = \mathbf{r}_0 + \lambda\mathbf{d}, \lambda \in \mathbb{R}$
- the equation of a line on component form: Isolating λ yields $\frac{x-x_0}{d_x} = \frac{y-y_0}{d_y} = \frac{z-z_0}{d_z}$
- the equation of a two-dimensional plane on vector form: $(\mathbf{r}-\mathbf{r}_0)\cdot\mathbf{n} = 0$
- the equation of a two-dimensional plane on component form: $ax+by+cz = k = \mathbf{r}_0\cdot\mathbf{n}$
- how to determine a normal vector to a plane from $ax+by+cz = k$
- how to determine the equation to a plane passing through three points
- how to determine the perpendicular distance of a plane from the origin: $p = k/|\mathbf{n}|$

6 Geometry in $\mathbb{R}^3$

The aim is to discuss practical applications of vectors, demonstrating the power of vectors in simplifying geometrical problems.

Let us remind ourselves of the following facts regarding lines and planes.
Lines in $\mathbb{R}^n$: Let R_0 denote a point with position vector $\mathbf{r}_0$. Then the equation of a line in $\mathbb{R}^n$ through a point R_0 and direction $\mathbf{d}$ is given by the (parametric) vector equation (see Sec. 5.2)

$$\mathbf{r} = \mathbf{r}_0 + \lambda\mathbf{d}, \quad \lambda \in \mathbb{R}. \tag{6.1}$$

Planes in $\mathbb{R}^3$: The equation of a plane in $\mathbb{R}^3$ through a point R_0 with position vector $\mathbf{r}_0$ and a normal vector $\mathbf{n} = (a, b, c)$ is on the form (see Sec. 5.3)

$$(\mathbf{r}-\mathbf{r}_0)\cdot\mathbf{n} = 0 \Leftrightarrow \mathbf{r}\cdot\mathbf{n} = \mathbf{r}_0\cdot\mathbf{n} \Leftrightarrow ax+by+cz = k, \tag{6.2}$$

where the constant $k = \mathbf{r}_0\cdot\mathbf{n} = |\mathbf{n}|p$, p being the perpendicular distance of the plane to the origin.

6.1 The equation of the line of intersection of two planes in $\mathbb{R}^3$

Consider two planes in $\mathbb{R}^3$:

$$a_1x + b_1y + c_1z = k_1, \tag{6.3a}$$
$$a_2x + b_2y + c_2z = k_2, \tag{6.3b}$$

with normal vectors

$$\mathbf{n}_1 = (a_1, b_1, c_1)\,, \tag{6.4a}$$
$$\mathbf{n}_2 = (a_2, b_2, c_2)\,, \tag{6.4b}$$

respectively.

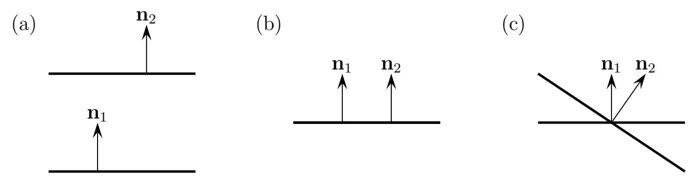

Figure 6.1: (a) Two parallel planes where the intersection is empty. The system of linear equations (6.3) does not have any solutions. The equations are incompatible. (b) Two parallel planes where the intersection is the plane itself. The system of linear equations (6.3) has infinitely many solutions. The two equations are proportional. (c) Two planes that are not parallel intersect in a line. The system of linear equations (6.3) has infinitely many solutions that form a straight line.

When will the two planes defined in Eqs. (6.3) intersect in a line? Two planes will intersect in a line if and only if they are not parallel, see Fig. 6.1. The two planes are parallel if and only if there normal vectors are parallel. Hence, if $\mathbf{n}_1$ and $\mathbf{n}_2$ are not parallel, the planes intersect in a line.

Using Theorem 4.1, we then have that the two planes intersect in a line if and only if

$$\mathbf{n}_1 \times \mathbf{n}_2 \neq \mathbf{0}. \tag{6.5}$$

Now, if two planes do intersect in a line, how do we find the equation of the line of intersection of two planes? We need a point $\mathbf{r}_0 = (x_0, y_0, z_0)$ on the line and a direction vector $\mathbf{d}$ for the line. To determine $\mathbf{r}_0$, we notice that x_0, y_0, z_0 must satisfy both equations (6.3). The simplest procedure to find a point that belongs to both planes is to choose (freely) one of the coordinates, say x_0 and then solve the resulting two equations in the two unknowns y_0 and z_0.

To determine the direction vector for the line, we notice that the normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ must both be perpendicular to $\mathbf{d}$, see Fig. 6.1(c) and Fig. 6.2, and therefore we may choose $\mathbf{d}$ as the vector product of $\mathbf{n}_1$ and $\mathbf{n}_2$:

$$\mathbf{d} = \mathbf{n}_1 \times \mathbf{n}_2. \tag{6.6}$$

The parametric vector equation for the line of intersection of the two planes is then given by $\mathbf{r} = \mathbf{r}_0 + \lambda\mathbf{d}, \lambda \in \mathbb{R}$.

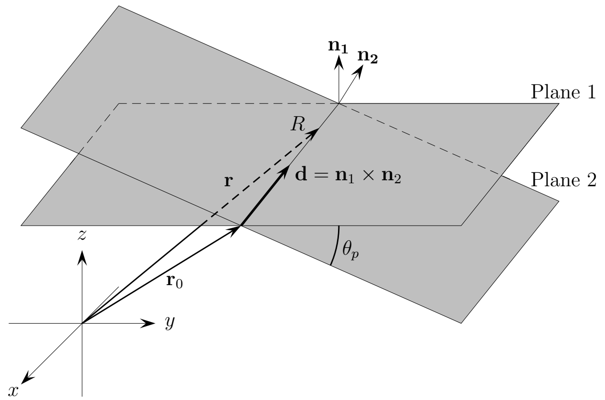

Figure 6.2: Two non-parallel planes intersect in a line. A directional vector for the line of intersection is given by $\mathbf{d} = \mathbf{n}_1 \times \mathbf{n}_2$ because it has to be perpendicular to both normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$. Then any point R with position vector $\mathbf{r}$ on the the line of intersections can be written on the vector form $\mathbf{r} = \mathbf{r}_0 + \lambda\mathbf{d}, \lambda \in \mathbb{R}$, where $\mathbf{r}_0$ is a point on the line.

We summarise the procedure to determine whether two planes intersect and if affirmative, how to find the equation of the line of intersection:

Procedure to determine the equation of the line of intersection of two planes in $\mathbb{R}^3$

1. Two planes intersect in a line if and only if they are not parallel, that is, if the vector product of their respective normal vectors is not zero-vector: $\mathbf{n}_1 \times \mathbf{n}_2 \neq \mathbf{0}$.
2. Find *any* point $\mathbf{r}_0$ on the line of intersection. Do this by choosing an arbitrary value for one of the variables (say $x_0 = 0$) and solving the two equations for the other two variables (i.e., y_0 and z_0 in this case).
3. The direction vector of the line of intersection $\mathbf{d} \perp \mathbf{n}_1$ and $\mathbf{d} \perp \mathbf{n}_2$ and $\mathbf{d}$ is (proportional to) the vector product of the two normal vectors, that is, we may choose $\mathbf{d} = \mathbf{n}_1 \times \mathbf{n}_2$.
4. The equation for the line of intersection in vector form is then $\mathbf{r} = \mathbf{r}_0 + \lambda\mathbf{d}, \lambda \in \mathbb{R}$.

Example 6.1. Consider the two planes

$$5x - 4y - 3z = 10, \tag{6.7a}$$
$$-2x + y + z = 2. \tag{6.7b}$$

The associated normal vectors are

$$\mathbf{n}_1 = (5, -4, -3), \tag{6.8a}$$
$$\mathbf{n}_2 = (-2, 1, 1), \tag{6.8b}$$

respectively. The normal vectors are not parallel ($\mathbf{n}_1$ is not a numerical multiple of $\mathbf{n}_2$) and hence the two planes do intersect in a line.

Let us fist determine a point on this line of intersection by choosing $x_0 = 0$ and substitute into the Eqs. (6.7) to yield:

$$-4y_0 - 3z_0 = 10 \tag{6.9a}$$
$$y_0 + z_0 = 2. \tag{6.9b}$$

Adding 4 times the second equation to the first yields $z_0 = 18$ and substituting that back into the second equation, we find $y_0 = -16$. Hence, $\mathbf{r}_0 = (x_0, y_0, z_0) = (0, -16, 18)$ is a point on the line of intersection.

To determine the direction vector $\mathbf{d}$ of the line, we calculate the vector product between the two normal vectors:

$$\begin{aligned}\mathbf{d} &= \mathbf{n}_1 \times \mathbf{n}_2 \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -4 & -3 \\ -2 & 1 & 1 \end{vmatrix} \\ &= \begin{vmatrix} -4 & -3 \\ 1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 5 & -3 \\ -2 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 5 & -4 \\ -2 & 1 \end{vmatrix}\mathbf{k} \\ &= -\mathbf{i} + \mathbf{j} - 3\mathbf{k}, \end{aligned} \tag{6.10}$$

and therefore, the parametric vector equation for the line of intersection is

$$\mathbf{r} = \mathbf{r}_0 + \lambda\mathbf{d}, \quad \lambda \in \mathbb{R} \Leftrightarrow \tag{6.11a}$$
$$(x, y, z) = (0, -16, 18) + \lambda(-1, 1, -3), \quad \lambda \in \mathbb{R} \tag{6.11b}$$

so isolating the parameter λ yields the equation for the line on component form:

$$\frac{x-0}{-1} = \frac{y+16}{1} = \frac{z-18}{-3}. \tag{6.12}$$

6.2 How to determine the (acute) angle θ_p between two planes?

We define the angle θ_p between two planes as the acute angle between them, that is, $0 \leq \theta_p \leq \pi/2$, see Fig. 6.2. Hence, the angle between two planes can be inferred by knowing the angle θ between the two normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, see Fig. 6.3.

Figure 6.3: (a) The (acute) angle between two planes θ_p equals the angle θ between the two normal vectors when $\theta \in [0, \pi/2]$. (b) The (acute) angle between two planes θ_p equals $\pi - \theta$ when the angle between the two normal vectors $\theta \in [\pi/2, \pi]$. Note that the normal vector $\mathbf{n}_2$ in (b) is the reverse of that displayed in (a).

To determine the angle θ between the two normal vectors, we apply definition 3.3:

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2 \quad 0 \leq \theta \leq \pi, \tag{6.13}$$

where $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$ are the unit normal vectors to the planes, respectively. Then we identify

$$\theta_p = \begin{cases} \theta & \text{if } 0 \leq \theta \leq \pi/2, \\ \pi - \theta & \text{if } \pi/2 \leq \theta \leq \pi. \end{cases} \tag{6.14}$$

Note that when the angle $\theta \in [\pi/2, \pi]$ between the two normal vectors, the (acute) angle between the planes $\theta_p = \pi - \theta$, see Fig. 6.3(b). Equivalently, if you find $\theta \in [\pi/2, \pi]$, you may reverse the direction of one on the normal vectors (say $\mathbf{n}_2 \mapsto -\mathbf{n}_2$) as that would yield a (new) $\theta \in [0, \pi/2]$ which is identical to the angle θ_p of the planes, see Fig. 6.3(b) $\mapsto$ Fig. 6.3(a).

Procedure to determine the (acute) angle θ_p between two planes in $\mathbb{R}^3$

1. Find the angle $\theta \in [0, \pi]$ between the two normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ of the planes by solving the equation $\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}$, $0 \leq \theta \leq \pi$.

2. The acute angle $\theta_p \in [0, \pi/2]$ between the planes is related to θ by

$$\theta_p = \begin{cases} \theta & \text{if } 0 \leq \theta \leq \pi/2, \\ \pi - \theta & \text{if } \pi/2 \leq \theta \leq \pi. \end{cases} \tag{6.15}$$

Example 6.2. Consider the two planes from Ex. 6.1. The magnitude of the normal vectors are

$$\mathbf{n}_1 = (5, -4, -3) \Rightarrow |\mathbf{n}_1| = \sqrt{5^2 + (-4)^2 + (-3)^2} = \sqrt{50}, \tag{6.16a}$$
$$\mathbf{n}_2 = (-2, 1, 1) \Rightarrow |\mathbf{n}_2| = \sqrt{(-2)^2 + 1^2 + 1^2} = \sqrt{6}. \tag{6.16b}$$

The dot-product

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = (5, -4, -3) \cdot (-2, 1, 1) = -10 - 4 - 3 = -17. \tag{6.17}$$

Hence the angle θ between the two normal vectors:

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{-17}{\sqrt{50}\sqrt{6}} \Rightarrow \theta = 2.95 \text{ rad} = 169.0°. \tag{6.18}$$

Hence, the (acute) angle between the two planes is

$$\theta_p = \pi - \theta \approx 0.193 \text{ rad} = 11.0°. \tag{6.19}$$

6.3 The minimum distance from a point to a plane

Consider a plane in $\mathbb{R}^3$:

$$ax + by + cz = k \tag{6.20}$$

with normal vector

$$\mathbf{n} = (a, b, c)\,. \tag{6.21}$$

Our aim is to find the minimum distance of a point P in $\mathbb{R}^3$ to this plane, see Fig. 6.4. Let A denote *any* point in the plane and consider the vector from the point A to the point P, that is, $\overrightarrow{AP}$, see Fig. 6.4. By inspection, we see from Fig. 6.4 that the minimum distance from the point to the plane:

$$\begin{aligned} \text{dist}\,(P, \text{plane}) &= \text{length of projection of } \overrightarrow{AP} \text{ onto } \mathbf{n} \\ &= |\overrightarrow{AP} \cdot \hat{\mathbf{n}}| \end{aligned} \tag{6.22}$$

where $\hat{\mathbf{n}}$ is a unit normal vector to the plane, that is,

$$\hat{\mathbf{n}} = \pm\frac{\mathbf{n}}{|\mathbf{n}|} = \pm\frac{a\mathbf{i} + b\mathbf{j} + c\mathbf{k}}{\sqrt{a^2 + b^2 + c^2}} \tag{6.23}$$

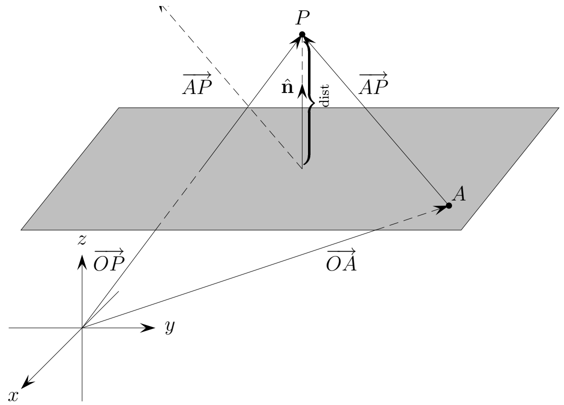

Figure 6.4: The minimum distance, dist, from a point P to the plane is length of the projection of the vector $\overrightarrow{AP}$ onto the direction of the normal vector $\mathbf{n}$ of the plane, that is, dist $(P, \text{plane}) = |\overrightarrow{AP} \cdot \hat{\mathbf{n}}|$, where A is any point in the plane and $\hat{\mathbf{n}}$ a unit normal vector to the plane.

and $|\cdot|$ signifies the absolute value of the dot-product. The need for taking the absolute value is as follows: The plane "splits" $\mathbb{R}^3$ into two halves. If the normal vector to the plane happens to point into the half where P is situated, the dot-product is positive because the angle between the normal vector $\mathbf{n}$ and the vector $\overrightarrow{AP}$ will be in the range $[0, \pi/2[$. However, if the normal vector to the plane happens to point into the opposite half, the angle between the normal vector $\mathbf{n}$ and the vector $\overrightarrow{AP}$ will be in the range $]\pi/2, \pi]$, leaving the dot-product negative. (If the point P is in the plane, the angle between the normal vector $\mathbf{n}$ and the vector $\overrightarrow{AP}$ will be $\pi/2$.) Obviously, the distance is a non-negative quantity and hence, we need to take the absolute value.

Procedure to find minimum distance from a point to a plane

1. Find a unit normal vector to the plane $\hat{\mathbf{n}} = \pm\frac{\mathbf{n}}{|\mathbf{n}|}$. Any of the two will do.
2. Find any point A on the plane by choosing arbitrary values for two of the coordinates, say x_0 and y_0, and using the equation of the plane Eq. (6.20) to find the value of the third coordinate, z_0.
3. Find the vector $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$ from point A to point P.
4. The minimum distance from the point P to the plane is given by

$$\text{dist}\,(P, \text{plane}) = |\overrightarrow{AP} \cdot \hat{\mathbf{n}}| = \frac{\left|\overrightarrow{AP} \cdot \mathbf{n}\right|}{|\mathbf{n}|}. \tag{6.24}$$

Example 6.3. Find the minimum distance between the point $P = (1, 1, 1)$ and the plane define by

$$3x - 2y + z = 5. \tag{6.25}$$

We note that $\mathbf{n} = (3, -2, 1)$ is a normal vector to the plane with magnitude $|\mathbf{n}| = \sqrt{3^2 + (-2)^2 + 1^1} = \sqrt{14}$. To find an arbitrary point A on the plane, we choose $x_0 = y_0 = 0$ and substitute into Eq. (6.25) to find $z_0 = 5$, that is, $A = (0, 0, 5)$ is a point on the plane. We can now determine the vector $\overrightarrow{AP}$:

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = (1, 1, 1) - (0, 0, 5) = (1, 1, -4)\,. \tag{6.26}$$

Hence, the minimum distance from the point $P = (1, 1, 1)$ to the plane:

$$\begin{aligned} \text{dist}\,(P, \text{plane}) &= \frac{\left|\overrightarrow{AP} \cdot \mathbf{n}\right|}{|\mathbf{n}|} \\ &= \frac{|(1, 1, -4) \cdot (3, -2, 1)|}{\sqrt{14}} \\ &= \frac{3}{\sqrt{14}} \\ &\approx 0.802. \end{aligned} \tag{6.27}$$

6.4 The minimum distance from a point to a line

Given a point P and a line with direction vector $\mathbf{d}$ in $\mathbb{R}^3$. Let A be *any* point on the line and consider the vector $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$. Let θ denote the angle between the vectors $\overrightarrow{AP}$ and $\mathbf{d}$. By inspection, we see from Fig. 6.5 that

$$\begin{aligned} \text{dist}\,(P, \text{line}) &= \left|\overrightarrow{AP}\right| \sin\theta \\ &= \left|\overrightarrow{AP}\right| \left|\hat{\mathbf{d}}\right| \sin\theta \\ &= |\overrightarrow{AP} \times \hat{\mathbf{d}}| \end{aligned} \tag{6.28}$$

where $\hat{\mathbf{d}}$ is a unit directional vector of the line and $|\cdot|$ signifies the magnitude of the vector-product.

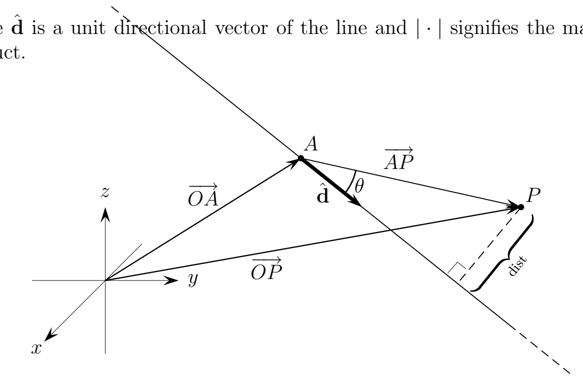

Figure 6.5: The minimum distance, dist, from a point P to a line. By inspection we see that dist $(P, \text{line}) = |\overrightarrow{AP} \times \hat{\mathbf{d}}|$ where A is any point on the line and $\hat{\mathbf{d}}$ a unit directional vector.

Procedure to determine the minimum distance from a point P to a line in $\mathbb{R}^3$

1. Find a direction vector for the line $\mathbf{d}$.
2. Find any point A on the line by choosing an arbitrary value for one of the coordinates, say x_0, and using the equation of the line to find the value of the two remaining coordinates, here y_0 and z_0.
3. Find the vector $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$ from point A to point P.
4. The minimum distance from the point P to the line is given by

$$\text{dist}\,(P, \text{line}) = |\overrightarrow{AP} \times \hat{\mathbf{d}}| = \frac{\left|\overrightarrow{AP} \times \mathbf{d}\right|}{|\mathbf{d}|}. \tag{6.29}$$

Example 6.4. Find the minimum distance between the point $P = (-2, 0, 1)$ and the line define by

$$\mathbf{r} = (1, 0, 1) + \lambda\,(1, 2, 3)\,, \quad \lambda \in \mathbb{R}. \tag{6.30}$$

Clearly, $A = (1, 0, 1)$ is a point on the line. Hence, the vector

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = (-2, 0, 1) - (1, 0, 1) = (-3, 0, 0)\,. \tag{6.31}$$

The directional vector $\mathbf{d} = (1, 2, 3)$ with magnitude $|\mathbf{d}| = \sqrt{1^2 + 2^2 + 3^3} = \sqrt{14}$ Therefore,

$$\begin{aligned} \overrightarrow{AP} \times \mathbf{d} = &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 0 & 0 \\ 1 & 2 & 3 \end{vmatrix} \\ &= 0\mathbf{i} + 9\mathbf{j} - 6\mathbf{k} \end{aligned}$$

from which

$$\text{dist}\,(P, \text{line}) = \frac{\left|\overrightarrow{AP} \times \mathbf{d}\right|}{|\mathbf{d}|} = \frac{\sqrt{117}}{\sqrt{14}} \approx 2.89. \tag{6.32}$$

6.5 The minimum distance between two skew lines in $\mathbb{R}^3$

Consider two lines in $\mathbb{R}^3$. We want to determine the minimum distance between these two lines. Let A_1 be an arbitrary point on line 1 and A_2 an arbitrary point on line 2 and consider the vector $\overrightarrow{A_1A_2}$, see Fig. 6.6.

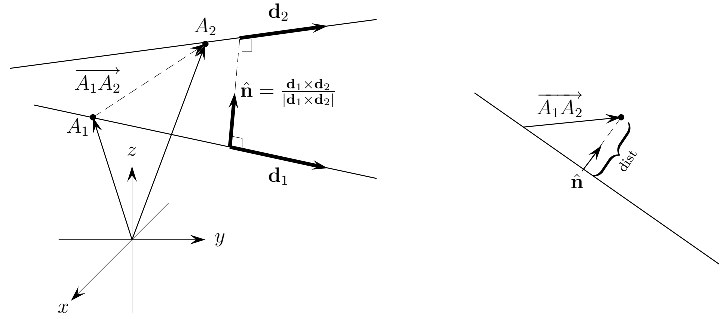

Figure 6.6: The minimum distance between two lines, dist, is the projection of the vector $\overrightarrow{A_1A_2}$ onto the direction of $\hat{\mathbf{n}}$, a unit vector perpendicular to both the directional vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ for the two lines, that is, $\hat{\mathbf{n}} = \mathbf{d}_1 \times \mathbf{d}_2 / |\mathbf{d}_1 \times \mathbf{d}_2|$. Hence dist $= \left|\overrightarrow{A_1A_2} \cdot \hat{\mathbf{n}}\right|$.

6.6 The condition for three planes in $\mathbb{R}^3$ to intersect in a point

Consider three planes in $\mathbb{R}^3$:

$$a_1x + b_1y + c_1z = k_1, \tag{6.33a}$$
$$a_2x + b_2y + c_2z = k_2, \tag{6.33b}$$
$$a_3x + b_3y + c_3z = k_3, \tag{6.33c}$$

with normal vectors

$$\mathbf{n}_1 = (a_1, b_1, c_1), \tag{6.34a}$$
$$\mathbf{n}_2 = (a_2, b_2, c_2), \tag{6.34b}$$
$$\mathbf{n}_3 = (a_3, b_3, c_3), \tag{6.34c}$$

respectively.

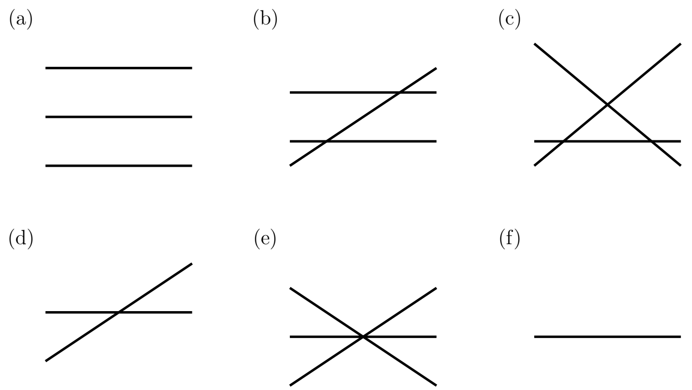

Figure 6.7: Intersection of three planes in $\mathbb{R}^3$. The intersection is empty in (a) where the three planes are parallel but non-overlapping, in (b) where two planes are parallel and non-overlapping and in (c) where the three planes form a "toblerone". In these three cases, the system of linear Eqs. (6.33) has no solutions. The intersection is a line in (d) where two planes are coinciding and the third plane not parallel to those two and in (e) where all planes cross in a line. In these two cases, the system of linear Eqs. (6.33) has infinitely many solutions that form a straight line. (f) The intersection is a plane when all the planes are identical. In this case, the system of linear Eqs. (6.33) has infinitely many solutions that form a plane.

What is the condition that these three planes intersect in a point? We must have that the first two planes intersect in a line. That happens if they are not parallel, see the discussion in Sec. 6.1, which we can express elegantly via

$$\mathbf{n}_1 \times \mathbf{n}_2 \neq \mathbf{0}. \tag{6.35}$$

The third plane cannot be parallel with either of the planes 1 and 2. If $\mathbf{n}_3$ is parallel with either $\mathbf{n}_1$ or $\mathbf{n}_2$ it would be perpendicular to $\mathbf{n}_1 \times \mathbf{n}_2$. This condition excludes the cases (a), (b), (d) and (f) sketched in Fig. 6.7.
In the cases (c) and (e) in Fig. 6.7, the three normal vectors $\mathbf{n}_1, \mathbf{n}_2$ and $\mathbf{n}_3$ are all in the same plane. This again would imply that say $\mathbf{n}_3 \perp \mathbf{n}_1 \times \mathbf{n}_2$. Hence, for all the six cases, we have $\mathbf{n}_3 \cdot (\mathbf{n}_1 \times \mathbf{n}_2) = 0$.
To avoid these situation, that is, to guarantee that the three planes intersect in a point, the normal vectors of the three planes must fulfill the condition $\mathbf{n}_3 \cdot (\mathbf{n}_1 \times \mathbf{n}_2) \neq 0$.

Theorem 6.1. Three planes in $\mathbb{R}^3$ with normal vectors $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$, respectively will intersect in a point if and only if

$$\mathbf{n}_3 \cdot (\mathbf{n}_1 \times \mathbf{n}_2) \neq 0. \tag{6.36}$$

6.7 Summary

After studying Sec. 6, you should know how to

- find the equation of the line of intersection between two planes in $\mathbb{R}^3$
- find the (acute) angle θ_p between two planes in $\mathbb{R}^3$
- find the minimum distance from a point to a plane in $\mathbb{R}^3$
- find the minimum distance from a point to a line in $\mathbb{R}^3$
- find the minimal distance between two skew lines in $\mathbb{R}^3$
- determine whether three planes in $\mathbb{R}^3$ intersect in a point

7 System of linear equations: I

The aim is to discuss a system of 2 linear equations in 2 unknowns (x_1, x_2) and a system of 3 linear equations in 3 unknowns (x_1, x_2, x_3). Depending on the given set of equations for the unknowns, the system may have either a unique solution, no solution or infinitely many solutions. The objective is to find a systematic procedure to determine whether such systems have a *unique* solution and, if so, how we can evaluate the unique solution.

7.1 One linear equation in one unknown

Consider one linear equation in one unknown $ax = b$, where $a, b \in \mathbb{R}$ are constant real numbers. We call a the coefficient of the unknown x. There are two different situations we have to consider, namely when the coefficient $a \neq 0$ and when the coefficient $a = 0$.

Theorem 7.1. The linear equation in one unknown

$$ax = b, \tag{7.1}$$

has a unique solution x if and only if $a \neq 0$ and the unique solution to Eq. (7.1) is given by

$$x = \frac{b}{a}. \tag{7.2}$$

If $a = 0$, Eq. (7.1) reads

$$0 \cdot x = b. \tag{7.3}$$

Equation (7.3) has no solutions if $b \neq 0$. We say that the equation is incompatible and that the set of solutions is an empty set. Equation (7.3) has infinitely many solutions if $b = 0$ because every $x \in \mathbb{R}$ is a solution to the equation $0 \cdot x = 0$.

This is a generic finding. A set of linear equations might have a unique solution, no solutions or infinitely many solutions. The aim of Sec. 7 is to develop a systematic procedure to determine whether a system of linear equations (two equations in two unknowns, three equations in three unknowns and generally n equations in n unknowns) has a unique solution and, if so, how we can evaluate this unique solution.

7.2 System of two linear equations in two unknowns

Consider a system of 2 linear equations in two unknowns

$$a_{11}x_1 + a_{12}x_2 = b_1 \tag{7.4a}$$
$$a_{21}x_1 + a_{22}x_2 = b_2, \tag{7.4b}$$

where the coefficients $a_{ij}, i = 1, 2; j = 1, 2$ of the two unknowns x_1 and x_2 are (known) constant real numbers as are the numbers $b_i, i = 1, 2$ on the right-hand side of Eqs. (7.4).

Let us elaborate on the notation with double indices (suffices) used. We have two equations which we label equation 1 (i.e., Eq. (7.4a)) and equation 2 (i.e., Eq. (7.4b)) and we have two

unknowns which we label x_1 and x_2, respectively. The first index (suffix) of the coefficients a_{ij} refer to the label of the equation: a_{11} and a_{12} are coefficients in the first equation while a_{21} and a_{22} are coefficients in the second equation. The second index (suffix) of the coefficients a_{ij} refer to which unknown it belongs: a_{11} and a_{21} are coefficients of the unknown x_1 in equation 1 and equation 2, respectively while a_{12} and a_{22} are coefficients of the unknown x_2 in equation 1 and equation 2, respectively. In general, a_{ij} is the coefficient in the ith equation of the unknown x_j.

Example 7.1. An example of a system of 2 linear equations in two unknowns would be

$$4x_1 - 3x_2 = 1, \tag{7.5a}$$
$$-2x_1 + x_2 = \sqrt{\pi}, \tag{7.5b}$$

and we identify the coefficients

$$a_{11} = 4, \quad a_{12} = -3, \quad a_{21} = -2, \quad a_{22} = 1 \tag{7.6}$$

and the numbers of the right-hand side of Eq. (7.5) are $b_1 = 1$ and $b_2 = \sqrt{\pi}$, respectively.

Geometrical interpretations

We may think of a system of 2 linear equations in 2 unknowns Eq. (7.4) geometrically as representing two straight lines in $\mathbb{R}^2$. If the two lines are parallel but have no point in common, then the two equations are incompatible and the set of solutions is an empty set, see Fig. 7.1(a). If the two lines are parallel and they do have a point in common, the lines are identical and there are infinitely many solutions. In this case, the two equations are proportional, that is, one equation is a numerical multiple of the other, see Fig. 7.1(b). If the two lines are not parallel, they will intersect in a unique point and hence the system of the two equations has a unique solution, see Fig. 7.1(c).

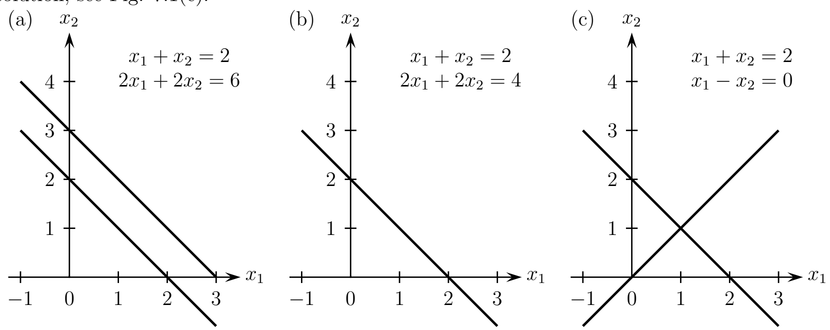

Figure 7.1: (a) The two lines are parallel but have no point in common. The two equations $x_1 + x_2 = 2$ and $2x_1 + 2x_2 = 6$ are incompatible. (b) The two lines are parallel and coincide. The two equations $x_1 + x_2 = 2$ and $2x_1 + 2x_2 = 4$ are proportional. (c) If the two lines are not parallel, they will intersect in a unique point. The two equations $x_1 + x_2 = 2$ and $x_1 - x_2 = 0$ has the unique solution $(x_0, y_0) = (1, 1)$.

Quantitatively, how can we express the condition that the two lines specified by the equations (7.4) are not parallel? Well,

$$\mathbf{n}_1 = (a_{11}, a_{12}), \tag{7.7a}$$
$$\mathbf{n}_2 = (a_{21}, a_{22}), \tag{7.7b}$$

are normal vector to the two lines, respectively (see Eq. (5.26)). Now, rotate $\mathbf{n}_2$ through $-\pi/2$ to obtain a direction vector $\mathbf{d}_2$ for line 2, see Fig. 7.2. The is done by exchanging the first and second coordinates and then negate the new second coordinate: $(a_{21}, a_{22}) \mapsto (a_{22}, -a_{21})$, that is, $\mathbf{d}_2 = (a_{22}, -a_{21})$. For example, the vector $(4, 1)$ rotated by an angle of $-\pi/2$ yields $(1, -4)$. One way of proving this would be to use the notion of complex numbers. Take a complex number $z = x + iy$ or (x, y). Multiplying z by the complex number $-i$ that has unit length and an argument of $-\pi/2$ would rotate the complex number z by $-\pi/2$ (because in general $z_1 z_2 = |z_1| \exp(i\phi_1)|z_2| \exp(i\phi_2) = |z_1||z_2| \exp(i(\phi_1 + \phi_2))$ and we find $-iz = y - ix$, that is $(y, -x)$.

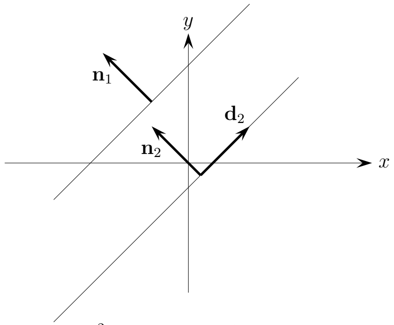

Figure 7.2: Two lines in $\mathbb{R}^2$ are parallel if their two normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are parallel. Rotating the normal vector $\mathbf{n}_2$ by an angel of $-\pi/2$, it turns into a direction vector $\mathbf{d}_2$. If the normal vector $\mathbf{n}_1$ is perpendicular to $\mathbf{d}_2$, that is, $\mathbf{n}_1 \cdot \mathbf{d}_2 = 0$, the lines are parallel.

Hence we have

$$\begin{aligned}\text{Lines are parallel} &\Leftrightarrow \mathbf{n}_1 \parallel \mathbf{n}_2 \\ &\Leftrightarrow \mathbf{n}_1 \perp \mathbf{d_2} \\ &\Leftrightarrow \mathbf{n}_1 \cdot \mathbf{d_2} = 0 \\ &\Leftrightarrow (a_{11}, a_{12}) \cdot (a_{22}, -a_{21}) = 0 \\ &\Leftrightarrow a_{11}a_{22} - a_{12}a_{21} = 0. \end{aligned} \tag{7.8}$$

Because non-parallel lines intersect in a unique point, we find by negating the above statement that the system of two equations in two unknowns (7.4) has a unique solution if and only if $a_{11}a_{22} - a_{12}a_{21} \neq 0$. Clearly, the quantity $a_{11}a_{22} - a_{12}a_{21}$ plays a very special and crucial rôle and it will shortly be revealed in which way.

Solving the equations by brute force when a unique solution exists

Let us solve the system of linear equations (7.4) by brute force. Multiplying Eq. (7.4a) by a_{22} and Eq. (7.4b) by a_{12} we obtain

$$a_{11}a_{22}x_1 + a_{12}a_{22}x_2 = b_1 a_{22}, \tag{7.9a}$$
$$a_{21}a_{12}x_1 + a_{22}a_{12}x_2 = b_2 a_{12}. \tag{7.9b}$$

Subtracting Eq. (7.9b) from Eq. (7.9a) we obtain

$$(a_{11}a_{22} - a_{21}a_{12})\, x_1 = b_1 a_{22} - b_2 a_{12}, \tag{7.10}$$

so if $(a_{11}a_{22} - a_{21}a_{12}) \neq 0$ we find that

$$x_1 = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{21}a_{12}} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}. \tag{7.11a}$$

Similarly, multiplying Eq. (7.4a) by a_{21} and Eq. (7.4b) by a_{11} and subtracting the "new" Eq. (7.4a) from the "new" Eq. (7.4b) we obtain $(a_{11}a_{22} - a_{21}a_{12})\, x_2 = a_{11}b_2 - a_{21}b_1$ and hence

$$x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{21}a_{12}} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}. \tag{7.11b}$$

We recognise that the denominator in Eqs. (7.11) is the 2×2 determinant of the coefficients of the unknowns in the system of linear equations (7.4) while the numerator for x_1 is the 2×2 determinant obtained when replacing the 1st column in the 2×2 determinant of the coefficients with the right-hand side of (7.4) and the numerator for x_2 is the 2×2 determinant obtained when replacing the 2nd column in the 2×2 determinant of the coefficients with the right-hand side of (7.4).

Theorem 7.2. The system of two equations in two unknowns

$$a_{11}x_1 + a_{12}x_2 = b_1 \tag{7.12a}$$
$$a_{21}x_1 + a_{22}x_2 = b_2, \tag{7.12b}$$

has a unique solution (x_1, x_2) if and only if the 2×2 determinant of the coefficients

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0. \tag{7.13}$$

and the unique solution to Eq. (7.12) is given by

$$(x_1, x_2) = \left(\frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \right). \tag{7.14}$$

Example 7.2. Consider the system of two linear equations in two unknowns

$$5x_1 - 2x_2 = 1, \tag{7.15a}$$
$$-3x_1 + 4x_2 = 5. \tag{7.15b}$$

The determinant of the coefficients of the system

$$\begin{vmatrix} 5 & -2 \\ -3 & 4 \end{vmatrix} = 5 \cdot 4 - (-3) \cdot (-2) = 14 \neq 0 \tag{7.16}$$

so we can conclude that a unique solution exists. Applying Theorem 7.2 we find that the unique solution is

$$(x_1, x_2) = \left(\frac{\begin{vmatrix} 1 & -2 \\ 5 & 4 \end{vmatrix}}{\begin{vmatrix} 5 & -2 \\ -3 & 4 \end{vmatrix}}, \frac{\begin{vmatrix} 5 & 1 \\ -3 & 5 \end{vmatrix}}{\begin{vmatrix} 5 & -2 \\ -3 & 4 \end{vmatrix}} \right) = \left(\frac{14}{14}, \frac{28}{14} \right) = (1, 2)\,. \tag{7.17}$$

We can always check whether our solution is correct by substituting these values into the original Eqs. (7.15): $5 \cdot 1 - 2 \cdot 2 = 1$ and $-3 \cdot 1 + 4 \cdot 2 = 5$.

7.3 System of three linear equations in three unknowns

Consider a system of 3 linear equations in three unknowns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \tag{7.18a}$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2, \tag{7.18b}$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3, \tag{7.18c}$$

where the coefficients $a_{ij}, i = 1, 2, 3; j = 1, 2, 3$ of the three unknowns x_1, x_2 and x_3 are (known) constant real numbers as are the numbers $b_i, i = 1, 2, 3$ on the right-hand side of Eqs. (7.18).

Geometrical interpretations

We may think of a system of 3 linear equations in 3 unknowns Eq. (7.18) geometrically as representing three planes in $\mathbb{R}^3$. The three equations are incompatible and hence the set of solutions is an empty set when the three planes are parallel but non-overlapping, see Fig. 7.3(a), two planes are parallel and non-overlapping, see Fig. 7.3(b) and when the three planes form a "toblerone", Fig. 7.3(c). The three equations have infinitely many solutions when two planes are coinciding and the third plane not parallel to those two, see Fig. 7.3(d) and when all planes cross in a line, see Fig. 7.3(e). In these two cases, the set with infinitely many solutions forms a straight line. The three equations have infinitely many solutions when the three planes are identical, Fig. 7.3(f). In this case, the set with infinitely many solutions forms a plane. The latter case happens if and only if all equations are proportional.
The normal vectors to the three planes given by the system of linear Eqs. (7.18) are

$$\mathbf{n}_1 = (a_{11}, a_{12}, a_{13})\,, \tag{7.19a}$$
$$\mathbf{n}_2 = (a_{21}, a_{22}, a_{23})\,, \tag{7.19b}$$
$$\mathbf{n}_3 = (a_{31}, a_{32}, a_{33})\,. \tag{7.19c}$$

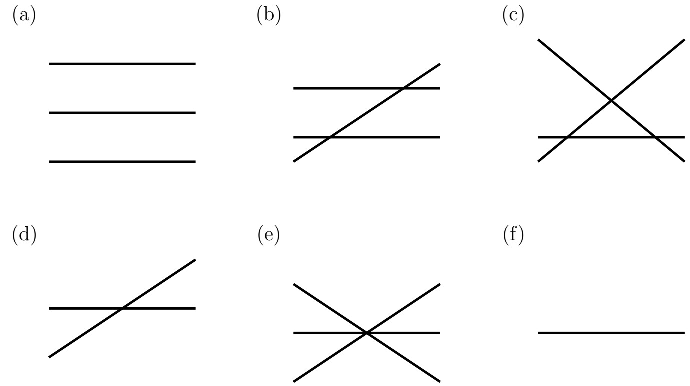

Figure 7.3: Intersection of three planes in $\mathbb{R}^3$ when there is no unique solution to the system of linear equations (7.18).

From Theorem 6.1 in Sec. 6.6 we have that three planes in $\mathbb{R}^3$ intersect in a point if and only if their normal vectors are not co-planar. With other words, the system of three linear equations in three unknowns (7.18) has a unique solution if and only if

$$\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3) \neq 0. \tag{7.20}$$

This quantity (real number) in Eq. (7.20) plays a very special and crucial rôle so we are urged to investigate it more closely.

$$\begin{aligned} \mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3) &= (a_{11}\mathbf{i} + a_{12}\mathbf{j} + a_{13}\mathbf{k}) \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= (a_{11}\mathbf{i} + a_{12}\mathbf{j} + a_{13}\mathbf{k}) \cdot \left(\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \mathbf{k} \right) \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &\stackrel{\text{def}}{=} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}. \end{aligned} \tag{7.21}$$

We recognise this quantity as the 3×3 determinant of the coefficients of the system of linear equations and, in the definition above, we say that the determinant has been expanded after the 1st row using the now familiar alternating sign pattern $+ - +$.

When a unique solution solution exists, we could, with some patience and a large piece of paper, solve the system of 3 linear equations in three unknowns. If we so did, we would arrive at the following conclusion:

Theorem 7.3. The system of three equations in three unknowns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \tag{7.22a}$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2, \tag{7.22b}$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3, \tag{7.22c}$$

has a unique solution (x_1, x_2, x_3) if and only if the 3×3 determinant of the coefficients

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0 \tag{7.23}$$

and the unique solution to Eq. (7.22) it is given by

$$(x_1, x_2, x_3) = \left(\frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}, \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}, \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} \right). \tag{7.24}$$

We recognise that the denominator in Eqs. (7.24) is the 3×3 determinant of the coefficients in the system of linear equations (7.22). The numerator for x_1 is the 3×3 determinant obtained when replacing the 1st column in the 3×3 determinant of the coefficients with the right-hand side of (7.22). The numerator for x_2 is the 3×3 determinant obtained when replacing the 2nd column in the 3×3 determinant of the coefficients with the right-hand side of (7.22). The numerator for x_3 is the 3×3 determinant obtained when replacing the 3rd column in the 3×3 determinant of the coefficients with the right-hand side of (7.22).

Example 7.3. Consider the system of three linear equations in three unknowns

$$2x_1 + 4x_2 + 3x_3 = 4, \tag{7.25a}$$
$$x_1 - 2x_2 - 2x_3 = 0, \tag{7.25b}$$
$$-3x_1 + 3x_2 + 2x_3 = -7. \tag{7.25c}$$

The determinant of the coefficients of the system

$$\begin{aligned} \begin{vmatrix} 2 & 4 & 3 \\ 1 & -2 & -2 \\ -3 & 3 & 2 \end{vmatrix} &= 2\begin{vmatrix} -2 & -2 \\ 3 & 2 \end{vmatrix} - 4\begin{vmatrix} 1 & -2 \\ -3 & 2 \end{vmatrix} + 3\begin{vmatrix} 1 & -2 \\ -3 & 3 \end{vmatrix} \\ &= 2 \cdot 2 - 4 \cdot (-4) + 3 \cdot (-3) \\ &= 11 \neq 0, \end{aligned} \tag{7.26}$$

so we can conclude that a unique solution exists. Applying Theorem 7.3 we find that the unique solution is

$$\begin{aligned} (x_1, x_2, x_3) &= \left(\frac{\begin{vmatrix} 4 & 4 & 3 \\ 0 & -2 & -2 \\ -7 & 3 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 4 & 3 \\ 1 & -2 & -2 \\ -3 & 3 & 2 \end{vmatrix}}, \frac{\begin{vmatrix} 2 & 4 & 3 \\ 1 & 0 & -2 \\ -3 & -7 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 4 & 3 \\ 1 & -2 & -2 \\ -3 & 3 & 2 \end{vmatrix}}, \frac{\begin{vmatrix} 2 & 4 & 4 \\ 1 & -2 & 0 \\ -3 & 3 & -7 \end{vmatrix}}{\begin{vmatrix} 2 & 4 & 3 \\ 1 & -2 & -2 \\ -3 & 3 & 2 \end{vmatrix}} \right) \\ &= \left(\frac{22}{11}, \frac{-33}{11}, \frac{44}{11} \right) \\ &= (2, -3, 4)\,. \end{aligned} \tag{7.27}$$

As usual, we can always check whether our solution is correct by substituting these values into the original Eqs. (7.25): $4 - 12 + 12 = 4, 2 + 6 - 8 = 0$ and $-6 - 9 + 8 = -7$.

7.4 System of n linear equations in n unknowns

Consider a system of n linear equations in n unknowns

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1j}x_j + \cdots + a_{1n}x_n = b_1 \tag{7.28a}$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2j}x_j + \cdots + a_{2n}x_n = b_2, \tag{7.28b}$$
$$\vdots \quad \vdots \quad \vdots$$
$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n = b_i, \tag{7.28c}$$
$$\vdots \quad \vdots \quad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nj}x_j + \cdots + a_{nn}x_n = b_n, \tag{7.28d}$$

where the coefficients $a_{ij}, i = 1, 2, \ldots, n; j = 1, 2, \ldots, n$ of the n unknowns $x_1, x_2, \ldots, x_n$ are (known) constant real numbers as are the numbers $b_i, i = 1, 2, \ldots, n$ on the right-hand side of Eqs. (7.28).

The left-hand side of the system of linear equations (7.28) is completely described by the array of the coefficients:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix} \tag{7.29}$$

The array is known as the $n \times n$ matrix of coefficients a_{ij} of the system of linear equations. We call the horizontal lines of numbers in the matrix $\mathbf{A}$ for its *rows* (value of the first index i) and the vertical lines its *columns* (value of the second index j). Hence, the real number a_{ij} is placed in the ith row and jth column and it is called the ijth entry or element of the matrix

A. The matrix **A** has n rows ($i = 1, 2, \ldots, n$) and n columns ($j = 1, 2, \ldots, n$) and the shape of the matrix is specified by (no. rows) $\times$ (no. columns), here an $n \times n$ (pronounced n by n) matrix.

We also defined the *column vectors* with n entries

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{pmatrix} \tag{7.30}$$

which you may also consider as $n \times 1$ matrices. Hence, we can write the system of n linear equations in n unknowns as

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1j}x_j + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2j}x_j + \cdots + a_{2n}x_n = b_2, \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n = b_i, \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nj}x_j + \cdots + a_{nn}x_n = b_n, \end{cases} \Leftrightarrow \tag{7.31a}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{pmatrix} \Leftrightarrow \tag{7.31b}$$

$$\mathbf{Ax} = \mathbf{b}. \tag{7.31c}$$

Example 7.4. In Ex. 7.2 we have a system of two equations in two unknowns. The associated 2×2 matrix of coefficients is

$$\mathbf{A} = \begin{pmatrix} 5 & -2 \\ -3 & 4 \end{pmatrix} \tag{7.32}$$

and the two column vectors (2×1 matrices) are

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \tag{7.33}$$

and we can write the system of linear equations as

$$\begin{cases} 5x_1 - 2x_2 = 1 \\ -3x_1 + 4x_2 = 5 \end{cases} \Leftrightarrow \begin{pmatrix} 5 & -2 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \Leftrightarrow \mathbf{Ax} = \mathbf{b}. \tag{7.34}$$

Example 7.5. In Ex. 7.3 we have a system of 3 equations in 3 unknowns. The associated 3×3 matrix of coefficients is

$$\mathbf{A} = \begin{pmatrix} 2 & 4 & 3 \\ 1 & -2 & -2 \\ -3 & 3 & 2 \end{pmatrix} \tag{7.35}$$

and the two column vectors (3×1 matrices) are

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 4 \\ 0 \\ 7 \end{pmatrix} \tag{7.36}$$

and we can write the system of linear equations as

$$\begin{cases} 2x_1 + 4x_2 + 3x_3 = 4, \\ x_1 - 2x_2 - 2x_3 = 0, \\ -3x_1 + 3x_2 + 2x_3 = -7. \end{cases} \Leftrightarrow \begin{pmatrix} 2 & 4 & 3 \\ 1 & -2 & -2 \\ -3 & 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 7 \end{pmatrix} \Leftrightarrow \mathbf{Ax} = \mathbf{b}. \tag{7.37}$$

We can now generalise Theorems 7.1, 7.2 and 7.3 to a system of n linear equations in n unknowns:

Theorem 7.4. Cramer's[4] rule: A system of n linear equations in n unknowns $x_1, x_2, \ldots, x_n$ written on the matrix form

$$\mathbf{Ax} = \mathbf{b} \tag{7.38}$$

where **A** is the $n \times n$ matrix of coefficients of the system has a unique solution if and only if $\det\mathbf{A} \neq 0$. If a unique solution $(x_1, x_2, \ldots, x_n)$ exists, it is given by

$$x_j = \frac{\det\mathbf{B}^j}{\det\mathbf{A}} \quad \text{for } j = 1, 2, \ldots, n \tag{7.39}$$

where the matrix $\mathbf{B}^j$ is obtained from the matrix **A** by replacing its jth column with column vector **b** making up the right-hand side of the system of equations.

Cramer's rule is only applicable for a system of n linear equations in n unknowns. It cannot be applied if the number of equations does not equal the number of unknowns. Moreover, Cramer's rule determines whether or not a unique solution exists. If $\det\mathbf{A} \neq 0$, a unique solution does exists and the Cramer's rule can be applied to evaluate the solution. If $\det\mathbf{A} = 0$, you can conclude that no unique solution exists. However, Cramer's rule does not yield any more information about the system of linear equations when $\det\mathbf{A} = 0$. It may have no solutions at all (the equations are incompatible) or the system may have infinitely many solutions (a line (1d), plane (2d) or hyperplane (higher-dimensional).

[4] 1750, Gabriel Cramer, Swiss mathematician, 1704 – 1752.

7.5 Summary

After studying Sec. 7, you should know

- that a system of two equations in two unknowns has a unique solution if and only if the 2×2 determinant of the coefficients $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$ and it is given by

$$(x_1, x_2) = \left(\frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \right).$$

- that a system of 3 equations in 3 unknowns has a unique solution if and only if the 3×3 determinant of the coefficients $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$ and it is given by

$$(x_1, x_2, x_3) = \left(\frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}, \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}, \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} \right).$$

- what is the $n \times n$ matrix of coefficients $\mathbf{A}$ of a system of n linear equations in n unknowns
- what are the $n \times 1$ column vectors $\mathbf{x}$ and $\mathbf{b}$ of a system of n linear equations in n unknowns
- how to write a system of n linear equations in n unknowns on matrix form: $\mathbf{Ax} = \mathbf{b}$
- **Cramer's rule:** A system of n linear equations in n unknowns $x_1, x_2, \ldots, x_n$ written on the matrix form

$$\mathbf{Ax} = \mathbf{b} \tag{7.40}$$

where $\mathbf{A}$ is the $n \times n$ matrix of coefficients of the system has a unique solution if and only if $\det\mathbf{A} \neq 0$. If a unique solution $(x_1, x_2, \ldots, x_n)$ exists, it is given by

$$x_j = \frac{\det\mathbf{B}^j}{\det\mathbf{A}} \quad \text{for } j = 1, 2, \ldots, n \tag{7.41}$$

where the matrix $\mathbf{B}^j$ is obtained from the matrix $\mathbf{A}$ by replacing its jth column with column vector $\mathbf{b}$ making up the right-hand side of the system of equations.

8 Determinant of an $n \times n$ matrix

The aim is to define the determinant, $\det\mathbf{A}$, of an $n \times n$ matrix $\mathbf{A}$ inductively, using the notion of a *minor* of a matrix. We will review the general properties of the determinant without proofs. In particular, we will discover that the determinant, $\det\mathbf{A}$

- can be evaluated by expanding by <u>any</u> row or column,
- is <u>invariant</u> under the operation of adding a numerical multiple of one row (column) to another row (column).

In the following, let $\mathbf{A}$ be an $n \times n$ matrix where we denote the ijth entry (element) of the matrix with a_{ij}. As usual, the first index i denotes the row number and the second index j the column number. The determinant of a square matrix $\mathbf{A}$ is denoted by $\det\mathbf{A}$ or $|\mathbf{A}|$.
Determinants are only defined for square matrices, that is, matrices where the number of rows equals the number of columns. We will define the determinant of an $n \times n$ matrix inductively, that is, we will define it for 1×1 and 2×2 matrices and then give the definition of an $n \times n$ determinant in terms of determinants for $(n-1) \times (n-1)$ matrices. In this way, you can "work your way down", expressing the $(n-1) \times (n-1)$ determinants in terms of determinants for $(n-2) \times (n-2)$ matrices and so on, eventually ending up with an expression only containing determinants of 2×2 matrices.

8.1 Definition of a minor $\mathbf{A}_{ij}$ of a matrix $\mathbf{A}$

First, we need to define the notion of a minor of a matrix $\mathbf{A}$. Therefore, consider an $n \times n$ matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1\,j-1} & a_{1j} & a_{1\,j+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2\,j-1} & a_{2j} & a_{2\,j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{i-1\,1} & a_{i-1\,2} & \cdots & a_{i-1\,j-1} & a_{i-1\,j} & a_{i-1\,j+1} & \cdots & a_{i-1\,n} \\ a_{i\,1} & a_{i\,2} & \cdots & a_{i\,j-1} & a_{i\,j} & a_{i\,j+1} & \cdots & a_{i\,n} \\ a_{i+1\,1} & a_{i+1\,2} & \cdots & a_{i+1\,j-1} & a_{i+1\,j} & a_{i+1\,j+1} & \cdots & a_{i+1\,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n\,j-1} & a_{nj} & a_{n\,j+1} & \cdots & a_{nn} \end{pmatrix} \tag{8.1}$$

where we have explicitly included the rows $i-1, i$ and $i+1$ and the columns $j-1, j$ and $j+1$.

Definition 8.1. We define the ijth *minor of* **A** as the $(n-1)\times(n-1)$ matrix $\mathbf{A}_{ij}$ obtained from **A** by removing its ith row and jth column, that is,

$$\mathbf{A}_{ij} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1\,j-1} & a_{1\,j+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2\,j-1} & a_{2\,j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ a_{i-1\,1} & a_{i-1\,2} & \cdots & a_{i-1\,j-1} & a_{i-1\,j+1} & \cdots & a_{i-1\,n} \\ a_{i+1\,1} & a_{i+1\,2} & \cdots & a_{i+1\,j-1} & a_{i+1\,j+1} & \cdots & a_{i+1\,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n\,j-1} & a_{n\,j+1} & \cdots & a_{nn} \end{pmatrix} \tag{8.2}$$

8.2 Definition of an $n \times n$ determinant

Definition 8.2.
For a 1×1 matrix **A**, we define the 1×1 determinant

$$\det\mathbf{A} = |a_{11}| = a_{11}. \tag{8.3}$$

For a 2×2 matrix **A**, we define the 2×2 determinant

$$\begin{aligned} \det\mathbf{A} &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \\ &= \text{sum with alternating signs } + \; - \text{ of the elements in the first row of } \mathbf{A} \\ &\quad \text{multiplied with the determinant of the corresponding minors} \\ &= a_{11}\det\mathbf{A}_{11} - a_{12}\det\mathbf{A}_{12} \\ &= a_{11}a_{22} - a_{12}a_{21}. \end{aligned} \tag{8.4}$$

For a 3×3 matrix **A**, we define the 3×3 determinant

$$\begin{aligned} \det\mathbf{A} &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= \text{sum with alternating signs } + \; - \; + \text{ of the elements in the first row of } \mathbf{A} \\ &\quad \text{multiplied with the determinant of the corresponding minors} \\ &= a_{11}\det\mathbf{A}_{11} - a_{12}\det\mathbf{A}_{12} + a_{13}\det\mathbf{A}_{13} \\ &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}. \end{aligned} \tag{8.5}$$

For an $n\times n$ matrix **A**, we define the $n\times n$ determinant

$$\begin{aligned} \det\mathbf{A} &= \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & \cdots & a_{nn} \end{vmatrix} \\ &= \text{sum with alternating signs } + \; - \; + \; - \; + \; \cdots \; (-1)^{1+n} \text{ of the elements in the first} \\ &\quad \text{row of } \mathbf{A} \text{ multiplied with the determinant of the corresponding minors} \\ &= a_{11}\det\mathbf{A}_{11} - a_{12}\det\mathbf{A}_{12} + a_{13}\det\mathbf{A}_{13} - a_{14}\det\mathbf{A}_{14} + \cdots + (-1)^{1+n}a_{1n}\det\mathbf{A}_{1n} \\ &= \sum_{j=1}^{n}(-1)^{1+j}a_{1j}\det\mathbf{A}_{1j}. \end{aligned} \tag{8.6}$$

We say that the determinant has been *expanded by the first row.* Note that the term $(-1)^{1+j}$, $j = 1, 2, \ldots, n$ is responsible for the pattern of alternating signs $+ \; - \; + \; - \; + \; \cdots \; (-1)^{1+n}$.

Example 8.1. Consider the 1×1 matrix

$$\mathbf{A} = (-4) \tag{8.7}$$

then we have that the 1×1 determinant

$$\det\mathbf{A} = |-4| = -4. \tag{8.8}$$

Example 8.2. Consider the 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ -2 & 3 \end{pmatrix} \tag{8.9}$$

then we have

$$\text{the entry} \quad a_{11} = 1 \quad \text{and the minor } (1\times 1 \text{ matrix}) \quad \mathbf{A}_{11} = \big(3\big)\,, \tag{8.10a}$$

$$\text{the entry} \quad a_{12} = 4 \quad \text{and the minor } (1\times 1 \text{ matrix}) \quad \mathbf{A}_{12} = \big(-2\big)\,. \tag{8.10b}$$

The 2×2 determinant of the matrix **A**:

$$\begin{aligned} \det\mathbf{A} &= \begin{vmatrix} 1 & 4 \\ -2 & 3 \end{vmatrix} \\ &= 1\cdot 3 - 4\cdot(-2) \\ &= 11. \end{aligned} \tag{8.11}$$

Example 8.3. Consider the 3×3 matrix

$$\mathbf{A} = \begin{pmatrix} -5 & -6 & 7 \\ 8 & -9 & 0 \\ -3 & 4 & 2 \end{pmatrix} \tag{8.12}$$

then we have

$$\text{the entry} \quad a_{11} = -5 \quad \text{and the minor } (2\times 2 \text{ matrix}) \quad \mathbf{A}_{11} = \begin{pmatrix} -9 & 0 \\ 4 & 2 \end{pmatrix}, \tag{8.13a}$$

$$\text{the entry} \quad a_{12} = -6 \quad \text{and the minor } (2\times 2 \text{ matrix}) \quad \mathbf{A}_{12} = \begin{pmatrix} 8 & 0 \\ -3 & 2 \end{pmatrix}, \tag{8.13b}$$

$$\text{the entry} \quad a_{13} = 7 \quad \text{and the minor } (2\times 2 \text{ matrix}) \quad \mathbf{A}_{13} = \begin{pmatrix} 8 & -9 \\ -3 & 4 \end{pmatrix}. \tag{8.13c}$$

The 3×3 determinant of the matrix $\mathbf{A}$:

$$\begin{aligned} \det\mathbf{A} &= \begin{vmatrix} -5 & -6 & 7 \\ 8 & -9 & 0 \\ -3 & 4 & 2 \end{vmatrix} \\ &= (-5)\cdot\begin{vmatrix} -9 & 0 \\ 4 & 2 \end{vmatrix} - (-6)\cdot\begin{vmatrix} 8 & 0 \\ -3 & 2 \end{vmatrix} + 7\cdot\begin{vmatrix} 8 & -9 \\ -3 & 4 \end{vmatrix} \\ &= (-5)\cdot(-18) + 6\cdot 16 + 7\cdot 5 \\ &= 221. \end{aligned} \tag{8.14}$$

Example 8.4. Consider the 4×4 matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}. \tag{8.15}$$

The 4×4 determinant of the matrix $\mathbf{A}$:

$$\begin{aligned} \det\mathbf{A} = &\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{vmatrix} \\ &= 1\cdot\begin{vmatrix} 6 & 7 & 8 \\ 10 & 11 & 12 \\ 14 & 15 & 16 \end{vmatrix} - 2\cdot\begin{vmatrix} 5 & 7 & 8 \\ 9 & 11 & 12 \\ 13 & 15 & 16 \end{vmatrix} + 3\cdot\begin{vmatrix} 5 & 6 & 8 \\ 9 & 10 & 12 \\ 13 & 14 & 16 \end{vmatrix} - 4\cdot\begin{vmatrix} 5 & 6 & 7 \\ 9 & 10 & 11 \\ 13 & 14 & 15 \end{vmatrix} \\ &= 1\cdot\left(6\cdot\begin{vmatrix} 11 & 12 \\ 15 & 16 \end{vmatrix} - 7\cdot\begin{vmatrix} 10 & 12 \\ 14 & 16 \end{vmatrix} + 8\cdot\begin{vmatrix} 10 & 11 \\ 14 & 15 \end{vmatrix}\right) - 2\cdot\left(5\cdot\begin{vmatrix} 11 & 12 \\ 15 & 16 \end{vmatrix} - 7\cdot\begin{vmatrix} 9 & 12 \\ 13 & 16 \end{vmatrix} + 8\cdot\begin{vmatrix} 9 & 11 \\ 13 & 15 \end{vmatrix}\right) + \\ &\quad 3\cdot\left(5\cdot\begin{vmatrix} 10 & 12 \\ 14 & 16 \end{vmatrix} - 6\cdot\begin{vmatrix} 9 & 12 \\ 13 & 16 \end{vmatrix} + 8\cdot\begin{vmatrix} 9 & 10 \\ 13 & 14 \end{vmatrix}\right) - 4\cdot\left(5\cdot\begin{vmatrix} 10 & 11 \\ 14 & 15 \end{vmatrix} - 6\cdot\begin{vmatrix} 9 & 11 \\ 13 & 15 \end{vmatrix} + 7\cdot\begin{vmatrix} 9 & 10 \\ 13 & 14 \end{vmatrix}\right) \\ &= 1\cdot(-24+56-32) - 2\cdot(-20+84-64) + 3\,(-40+72-32) - 4\,(-20+48-28) \\ &= 0. \end{aligned} \tag{8.16}$$

This example demonstrates that it becomes increasingly tedious to calculate determinants of matrices as soon as $n > 3$.

Example 8.5.* Let us evaluate the number of operations T_n required in an elementary (brute force) calculate of an $n\times n$ determinant. We remind ourselves that according to the definition of an $n\times n$ determinant

$$\det\mathbf{A} = \underbrace{a_{11}\det\mathbf{A}_{11} - a_{12}\det\mathbf{A}_{12} + a_{13}\det\mathbf{A}_{13} - \cdots + (-1)^{1+n}a_{1n}\det\mathbf{A}_{1n}}_{n \text{ terms}} \tag{8.17}$$

This sum contains n terms, that is, we have to perform $(n-1)$ additions. Each of these n terms contains one multiplication operation, that is, we have to perform an additional $n\cdot 1$ operations. Each of the n terms also contains the evaluation of an $(n-1)\times(n-1)$ determinant. The number of operations required to evaluate each of those is, by definition, T_{n-1}, that is, we have to perform an additional nT_{n-1} operations. Therefore, we find that the total number of operations required to calculate an $n\times n$ determinant is

$$T_n = nT_{n-1} + n + n - 1 = nT_{n-1} + 2n - 1, \quad \text{where } T_1 = 0 \tag{8.18}$$

which is a recursion relation.

Using the initial condition that for a 1×1 matrix, no operations are needed, that is, $T_1 = 0$, we may easily solve Eq. (8.18) numerically, see Table 8.1. Applying analytical methods, one may show that for $n \gg 1, T_n \approx en!$, where $e = 2.71828\ldots$ is the basis of the natural logarithm, see third column in Table 8.1.

n	T_n	$T_n/(en!)$
1	0	0.00000
2	3	0.55182
3	14	0.85839
4	63	0.96568
5	324	0.99327
6	1,955	0.99889
7	13,698	0.99984
8	109,599	0.99998
9	986,408	1.00000
10	9,864,099	1.00000
25	$4.216\cdot 10^{25}$	1.00000

Table 8.1: The total number of elementary operations T_n when calculating an $n\times n$ determinant. The third column is the ratio between T_n and the number $en! \approx 2.71828n!$.

The fastest computer in the world can carry out about 3 Pentaflops $= 3\cdot 10^{15}$ operations per second. Hence, to calculate a 25×25 determinant via brute force using the world's fastest computer would take $4.216\cdot 10^{25}/3\cdot 10^{15}\ \mathrm{s}^{-1} = 1.405\cdot 10^{10}\ \mathrm{s} \approx 445$ years. Therefore, we cannot rely on brute force calculation. Luckily, we don't have to. ⌣̈

8.3 Properties of the determinant

The evaluation of the determinant can be simplified greatly by applying the following Theorem 8.1 and one or more of the following general properties $1-8$ of the determinant.

Expansion by any row or column

Theorem 8.1. The determinant of an $n \times n$ matrix may be expanded by any row or any column, that is,

$$\det\mathbf{A} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det\mathbf{A}_{ij} \quad \text{expansion by row } i \tag{8.19a}$$

where the $n \times n$ determinant has been expanded by the ith row or

$$\det\mathbf{A} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det\mathbf{A}_{ij} \quad \text{expansion by column } j \tag{8.19b}$$

where the $n \times n$ determinant has been expanded by the jth column. The sign $(-1)^{i+j}$ associated with the term $a_{ij}\det\mathbf{A}_{ij}$ follows the checker-board pattern

$$\begin{pmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ldots \end{pmatrix}. \tag{8.20}$$

Classwork 3, exercise 6 demonstrates explicitly Theorem 8.1 for 3×3 matrices.

Properties of $n \times n$ determinants

1. The sign of a determinant is reversed if two rows (columns) are interchanged.
2. The value of a determinant is zero if any row (column) is made up exclusively of zero.
3. The value of a determinant is zero if two rows (columns) are identical.
4. The value of a determinant is zero if two rows (columns) are proportional.
5. If all elements in any row (column) are multiplied by a common factor $r \in \mathbb{R}$, the value of the determinant is multiplied by r.
6. The value of a determinant is unchanged if rows and columns are interchanged.
7. It the elements of any row (column) are the sums of two terms, the determinant can be written as the sum of two determinants.
8. The value of a determinant is unchanged if equal multiples of the elements of any row (column) are added to the corresponding elements of any other row (column).

The following examples with 2×2 determinants demonstrate the generic properties $1-8$ of $n \times n$ determinants.

Example 8.6. Property 1: The sign of a determinant is reversed if two rows (columns) are interchanged.

$$\begin{vmatrix} 1 & 3 \\ 5 & 2 \end{vmatrix} = 2 - 15 = -13, \tag{8.21a}$$

$$\begin{vmatrix} 5 & 2 \\ 1 & 3 \end{vmatrix} = 15 - 2 = 13. \tag{8.21b}$$

In general, if row i and row k are interchanged

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \ldots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \vdots & \vdots & \ldots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \tag{8.22}$$

and similar if two columns are interchanged.

Example 8.7. Property 2: The value of a determinant is zero if any row (column) is made up exclusively of zero.

$$\begin{vmatrix} 1 & 3 \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0, \tag{8.23a}$$

$$\begin{vmatrix} 5 & 0 \\ 1 & 0 \end{vmatrix} = 0 - 0 = 0. \tag{8.23b}$$

In general, if a row is made up exclusively of zeros

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = 0, \tag{8.24}$$

and similar for a matrix with a zero column.

Example 8.8. Property 3: The value of a determinant is zero if two rows (columns) are identical.

$$\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 3 - 3 = 0, \tag{8.25a}$$

$$\begin{vmatrix} 5 & 5 \\ 1 & 1 \end{vmatrix} = 5 - 5 = 0. \tag{8.25b}$$

In general, if two rows are identical

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = 0, \tag{8.26}$$

and similar for a matrix with two identical columns.

Example 8.9. Property 4: The value of a determinant is zero if two rows (columns) are proportional.

$$\begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix} = 9 - 9 = 0, \tag{8.27a}$$

$$\begin{vmatrix} 5 & 2 \\ 10 & 4 \end{vmatrix} = 20 - 20 = 0. \tag{8.27b}$$

In general, if $r \in \mathbb{R}$, and another row is $r\times$ row i then

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ ra_{i1} & ra_{i2} & \cdots & ra_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = 0, \tag{8.28}$$

and similar for a matrix with two proportional columns.

Example 8.10. Property 5: If all elements in any row (column) are multiplied by a common factor $r \in \mathbb{R}$, the value of the determinant is multiplied by r. Here, we multiply the first row by a factor 5 and the first column by a factor 3:

$$\begin{vmatrix} 1 & 3 \\ 5 & 2 \end{vmatrix} = 2 - 15 = -13, \tag{8.29a}$$

$$\begin{vmatrix} 5 & 15 \\ 5 & 2 \end{vmatrix} = 10 - 75 = -65 = 5 \cdot (-13). \tag{8.29b}$$

$$\begin{vmatrix} 3 & 3 \\ 15 & 2 \end{vmatrix} = 6 - 45 = -39 = 3 \cdot (-13). \tag{8.29c}$$

In general, if row i is multiplied by the common factor $r \in \mathbb{R}$, then

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ ra_{i1} & ra_{i2} & \cdots & ra_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = r \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \tag{8.30}$$

and similar for a matrix where a column j has been multiplied by a common factor r.

Example 8.11. Property 6: The value of a determinant is unchanged if rows and columns are interchanged. Here, the first row becomes the first column and the second row becomes the second column. We say that the matrix has been *transposed*.

$$\begin{vmatrix} 1 & 3 \\ 5 & 2 \end{vmatrix} = 2 - 15 = -13, \tag{8.31a}$$

$$\begin{vmatrix} 1 & 5 \\ 3 & 2 \end{vmatrix} = 2 - 15 = -13. \tag{8.31b}$$

In general, is rows and columns are interchanged

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{vmatrix}. \tag{8.32}$$

Example 8.12. Property 7: If the elements of any row (column) are the sums of two terms, the determinant can be written as the sum of two determinants. Here, the first row is the sum of two terms.

$$\begin{vmatrix} 1+6 & 3+4 \\ 5 & 2 \end{vmatrix} = 14 - 35 = -21, \tag{8.33a}$$

$$\begin{vmatrix} 1 & 3 \\ 5 & 2 \end{vmatrix} + \begin{vmatrix} 6 & 4 \\ 5 & 2 \end{vmatrix} = -13 + (-8) = -21. \tag{8.33b}$$

In general,

$$\begin{vmatrix} a_{11}+w_{11} & a_{12}+w_{12} & \cdots & a_{1n}+w_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \tag{8.34}$$

and similar for columns.

Example 8.13. Property 8: The value of a determinant is unchanged if equal multiples of the elements of any row (column) are added to the corresponding elements of any other row (column). Here, we add 2× the first row to the second row and 3× the first column to the second column:

$$\begin{vmatrix} 1 & 3 \\ 5 & 2 \end{vmatrix} = 2 - 15 = -13, \tag{8.35a}$$

$$\begin{vmatrix} 1 & 3 \\ 7 & 8 \end{vmatrix} = 8 - 21 = -13. \tag{8.35b}$$

$$\begin{vmatrix} 1 & 6 \\ 5 & 17 \end{vmatrix} = 17 - 30 = -13. \tag{8.35c}$$

In general, if $r \in \mathbb{R}$, then adding a numerical multiple of row i to row j leaves the determinant unchanged, that is,

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{j1} + ra_{i1} & a_{j2} + ra_{i2} & \cdots & a_{jn} + ra_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \tag{8.36}$$

and similar for a matrix where equal multiples of the elements from any column are added to the corresponding element of any other column.

Applying property 8 and Theorem 8.1

In particular property 8 together with Theorem 8.1 is frequently utilised to simplify the evaluation of a determinant by creating as many zeros as possible in a given row (column) and then expand the determinant by that row (column). Here are two examples:

Example 8.14.

$$\begin{vmatrix} 1 & 3 & 1 \\ 5 & 2 & 3 \\ -2 & -6 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 1 \\ 5 & 2 & 3 \\ 0 & 0 & 5 \end{vmatrix} = 5 \begin{vmatrix} 1 & 3 \\ 5 & 2 \end{vmatrix} = 5 \cdot (-13) = -65, \tag{8.37}$$

where we have added 2× the 1st row to the 3rd row to generate two zeros and then expanded the determinant by the 3rd row.

Example 8.15. Let us evaluate the 4×4 determinant from Ex. 8.4. Adding $(-1)\times$ the first row to the second row and $(-1)\times$ the first row to the third row yields

$$|\mathbf{A}| = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \\ 8 & 8 & 8 & 8 \\ 13 & 14 & 15 & 16 \end{vmatrix} = 0, \tag{8.38}$$

where we apply property 4 to conclude that the determinant is zero. Alternatively, you may add $(-2)\times$ the new second row to the new third row to create a row with zeros only and then expand by that row.

8.4 Summary

After studying Sec. 8, you should know

- the notion of the ijth minor $\mathbf{A}_{ij}$ of an $n \times n$ matrix $\mathbf{A}$
- how the determinant of an $n \times n$ matrix $\mathbf{A}$ is defined in terms of the determinants of its minors: $\det \mathbf{A} = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det \mathbf{A}_{1j}$.
- how to evaluate the determinant of an $n \times n$ matrix by expanding by any row i: $\det \mathbf{A} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det \mathbf{A}_{ij}$
- how to evaluate the determinant of an $n \times n$ matrix by expanding by any column j $\det \mathbf{A} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det \mathbf{A}_{ij}$
- the general properties $1 - 8$ of determinants, emphasising property 8.

9 System of linear equations: II

The aim is to discuss a system of n linear equations in n unknowns $(x_1, x_2, \ldots, x_n)$. When the associated matrix of coefficients has $\det\mathbf{A} \neq 0$, the system has a unique solution and we may apply **Cramer's rule** to identify the solution. What happens if $\det \mathbf{A} = 0$? From Cramer's rule, we know that no unique solution exists in such cases and therefore, we have no solutions or infinitely many solutions. This leads us to discuss the systematic algorithm of **Gaussian elimination** that can be applied to solve a system of linear equations even when $\det\mathbf{A} = 0$. Applying **Gaussian elimination** will reveal the nature of the solution (no solutions, unique solution, line solution, plane solution, or hyperplane solution).

We recall that a system of n linear equations in n unknowns can be written on matrix form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1j}x_j + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2j}x_j + \cdots + a_{2n}x_n = b_2, \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n = b_i, \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nj}x_j + \cdots + a_{nn}x_n = b_n, \end{cases} \Leftrightarrow \tag{9.1a}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{pmatrix} \Leftrightarrow \tag{9.1b}$$

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \tag{9.1c}$$

where $\mathbf{A}$ is an $n \times n$ matrix, $\mathbf{x}, \mathbf{b} \in \mathbb{R}^n$ are column vectors with n coordinates, that is, they are elements of the vector space $\mathbb{R}^n$. The 1st entry in the column vector $\mathbf{A}\mathbf{x}$ on the left-hand-side of Eq. (9.1b) is $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1j}x_j + \cdots + a_{1n}x_n$, the 2nd entry in the column vector $\mathbf{A}\mathbf{x}$ on the left-hand-side of Eq. (9.1b) is $a_{21}x_1 + a_{22}x_2 + \cdots + a_{2j}x_j + \cdots + a_{2n}x_n$ and, in general, the ith entry in the column vector $\mathbf{A}\mathbf{x}$ on the left-hand-side of Eq. (9.1b) is $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n, i = 1, 2, \ldots, n$.

9.1 Viewing the matrix A as a mapping (function) from $\mathbb{R}^n$ into $\mathbb{R}^n$

We note that the vector of unknowns $\mathbf{x} \in \mathbb{R}^n$ and the vector of constant real numbers on the right-hand side of Eq. (9.1) $\mathbf{b} \in \mathbb{R}^n$ so we may view a set of n linear equations in n unknowns, Eqs. (9.1), as asking which vector(s) $\mathbf{x} \in \mathbb{R}^n$ the $n \times n$ matrix $\mathbf{A}$ maps onto the vector $\mathbf{b} \in \mathbb{R}^n$, see Fig. 9.1.[5]

[5]We will later investigate in much more detail the equivalence of a matrix and a linear mapping (function).

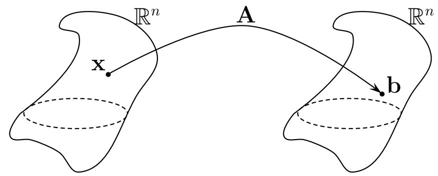

Figure 9.1: Solving a system of n equations in n unknowns can be viewed as searching for the vector(s) $\mathbf{x} \in \mathbb{R}^n$ that the $n \times n$ matrix $\mathbf{A}$ maps onto the vector $\mathbf{b} \in \mathbb{R}^n$.

When $\det\mathbf{A} \neq 0$: Applying Cramer's rule Theorem 7.4 we know that when $\det\mathbf{A} \neq 0$, a unique solution $\mathbf{x}_0$ exists with coordinates $x_j = \frac{\det\mathbf{B}^j}{\det\mathbf{A}}, j = 1, 2, \ldots, n$ where the matrix $\mathbf{B}^j$ is obtained from the matrix $\mathbf{A}$ by replacing its jth column with column vector $\mathbf{b}$ making up the right-hand side of the system of linear equations (9.1), see Fig. 9.2.

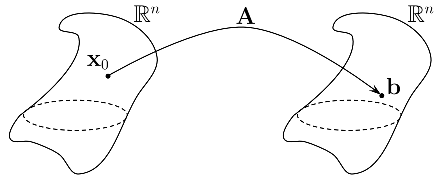

Figure 9.2: When $\det\mathbf{A} \neq 0$, the system of n equations in n unknowns has a unique solution $\mathbf{x}_0 \in \mathbb{R}^n$, that is, $\mathbf{A}\mathbf{x}_0 = \mathbf{b}$.

When $\det\mathbf{A} = 0$: Applying Cramer's rule Theorem 7.4 we know that when $\det\mathbf{A} = 0$, no unique solution exists. The system of n equations in n unknowns has either no solutions or infinitely many solutions. When infinitely many solutions exist, they may form a line in $\mathbb{R}^n$, see Fig. 9.3(a), a plane in $\mathbb{R}^n$, see Fig. 9.3(b), or hyper-planes in $\mathbb{R}^n$ (not shown).

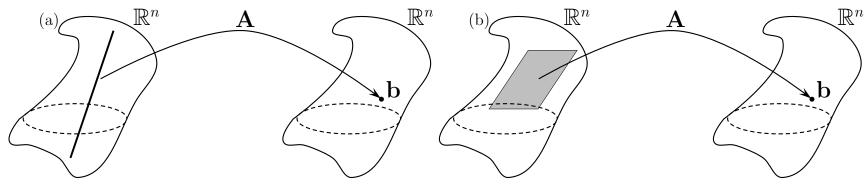

Figure 9.3: When $\det\mathbf{A} = 0$, the system of n equations in n unknowns has no solutions (not shown) or infinitely many solutions. (a) Line solution: All the points $\mathbf{x} \in \mathbb{R}^n$ on a straight line is mapped onto the vector $\mathbf{b} \in \mathbb{R}^n$, that is, $\forall\mathbf{x} \in$ line : $\mathbf{A}\mathbf{x} = \mathbf{b}$. (b) Plane solution: All the points $\mathbf{x} \in \mathbb{R}^n$ on a plane is mapped onto the vector $\mathbf{b} \in \mathbb{R}^n$, that is, $\forall\mathbf{x} \in$ plane : $\mathbf{A}\mathbf{x} = \mathbf{b}$. Note that the solutions $\mathbf{x} \in \mathbb{R}^n$ may also form hyperplanes of dimension $d > 2$ (not shown).

When no unique solution exists, what do we do to identify whether there are no solutions (the set of equations are incompatible) or infinitely many solutions and in the latter case, whether we have a line solution, a plane solution or a hyperplane solution?

9.2 Gaussian elimination

Gaussian elimination[6] is an efficient algorithm for solving a system of linear equations. Gaussian elimination is applicable to any system of linear equations. It will reveal whether there is no solutions, a unique solution or infinitely many solutions (a line, a plane, or a hyperplane).

Gaussian elimination algorithm

None of the following operations change the solution of a system of linear equations and hence may be applied to simplify the system of linear equations:

1. Changing the order of the equations.
2. Multiplying all terms in an equation by the same <u>non-zero</u> constant.
3. Adding a multiple $r \in \mathbb{R}$ of any equation to any other equation. Because r may be negative, addition includes subtraction.

The strategy applied is to simplify the set of equations as much as possible. Our aim is to end up with a system of linear equations where:

- All non-trivial equations that contain unknowns (i.e., where not all coefficients are zero) are above any trivial equations that do not (i.e., where all coefficients are zero).
- The leading coefficient (the first non-zero coefficient from the left) of all non-trivial equations is always strictly to the right of the leading coefficient of the equations above it.
- The leading coefficient of each non-trivial equation is 1.

9.3 Examples with 2 linear equations in 2 unknowns

Example 9.1. Consider the set of 2 linear equations in two unknowns x, y:

$$\begin{cases} 2x + 3y &= 5 \\ -x + 7y &= 1 \end{cases} \Leftrightarrow \begin{pmatrix} 2 & 3 \\ -1 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}. \tag{9.2}$$

The determinant of the matrix of coefficients

$$\begin{vmatrix} 2 & 3 \\ -1 & 7 \end{vmatrix} = 2 \cdot 7 - (-1) \cdot 3 = 17 \neq 0 \tag{9.3}$$

so the system has a unique solution according to Cramer's Rule, see Theorem 7.4. Using Cramer's rule to determine the unique solution, we find

$$(x, y) = \left(\frac{\begin{vmatrix} 5 & 3 \\ 1 & 7 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ -1 & 7 \end{vmatrix}}, \frac{\begin{vmatrix} 2 & 5 \\ -1 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ -1 & 7 \end{vmatrix}} \right) = \left(\frac{32}{17}, \frac{7}{17} \right). \tag{9.4}$$

[6] 1809, Carl Friedrich Gauss, German mathematician, 1777 – 1855. However, the method of Gaussian elimination appears in Chap. 8, *Rectangular Arrays*, of the Chinese mathematical text *Jiŭzhāng suànshù - The Nine Chapters on the Mathematical Art*, from around 200 BC.

Applying Gaussian elimination on the same set of equations, we find:

$$\text{Exchange the two equations} \begin{cases} 2x + 3y &= 5 \\ -x + 7y &= 1 \end{cases} \qquad \left(\begin{array}{cc|c} 2 & 3 & 5 \\ -1 & 7 & 1 \end{array} \right) \quad \Leftrightarrow \tag{9.5a}$$

$$2\ \text{Eq.}(1) + \text{Eq. }(2) \begin{cases} -x + 7y &= 1 \\ 2x + 3y &= 5 \end{cases} \qquad \left(\begin{array}{cc|c} -1 & 7 & 1 \\ 2 & 3 & 5 \end{array} \right) \quad \Leftrightarrow \tag{9.5b}$$

$$\frac{1}{17}\ \text{Eq.}(2) \begin{cases} -x + 7y &= 1 \\ 0 + 17y &= 7 \end{cases} \qquad \left(\begin{array}{cc|c} -1 & 7 & 1 \\ 0 & 17 & 7 \end{array} \right) \quad \Leftrightarrow \tag{9.5c}$$

$$-7\ \text{Eq.}(2) + \text{Eq.}(1) \begin{cases} -x + 7y &= 1 \\ y &= \frac{7}{17} \end{cases} \qquad \left(\begin{array}{cc|c} -1 & 7 & 1 \\ 0 & 1 & \frac{7}{17} \end{array} \right) \quad \Leftrightarrow \tag{9.5d}$$

$$-\ \text{Eq.}(1) \begin{cases} -x + 0 &= 1 - \frac{49}{17} \\ y &= \frac{7}{17} \end{cases} \qquad \left(\begin{array}{cc|c} -1 & 0 & 1 - \frac{49}{17} \\ 0 & 1 & \frac{7}{17} \end{array} \right) \Leftrightarrow \tag{9.5e}$$

$$\begin{cases} x &= \frac{32}{17} \\ y &= \frac{7}{17} \end{cases} \qquad \left(\begin{array}{cc|c} 1 & 0 & \frac{32}{17} \\ 0 & 1 & \frac{7}{17} \end{array} \right) \tag{9.5f}$$

and we find that there is a unique solution $(x, y) = \left(\frac{32}{17}, \frac{7}{17}\right)$.

The augmented matrix associated with a system of linear equations

Note that alongside the set of linear equations, we have written the associated *augmented matrix* of the system, that is, the $n \times (n+1)$ matrix obtained from the matrix of coefficients by adding an extra column (indicated by the vertical line), namely the right-hand-side of the system of equations. We see that the Gaussian elimination algorithm is equivalent to the following elementary <u>row operations</u> on the augmented matrix:

1. Interchanging two rows.
2. Multiplying all terms in a row equation by the same <u>non-zero</u> constant.
3. Adding a multiple $r \in \mathbb{R}$ of any row to any other row. Because r may be negative, addition includes subtraction.

It is worthwhile stressing that <u>only</u> row operations are allowed. No column operations may be performed at all when using elementary operations on the augmented matrix.

Throughout the remaining of these lecture notes, when applying Gaussian elimination, we will start with the original set of equations and write the associated augmented matrix of the system. We will then perform the elementary row operations on the augmented matrix until it cannot be reduced any more. The strategy applied is to simplify the set of equations as much as possible. Our aim is to end up with an augmented matrix where:

- All nonzero rows (rows with at least one nonzero element) are above any rows of all zeroes.
- The leading coefficient (the first non-zero number from the left) of a nonzero row is always strictly to the right of the leading coefficient of the row above it.

- The leading coefficient of each nonzero row is 1.

When this reduced form of the augmented matrix has been reached, we convert this information back into a set of linear equations that is then equivalent to the original set of linear equations. From this new simplified set of linear equations, we can readily see whether there is no solutions, a unique solution, or infinitely many solutions and, in the latter case, whether we have a line solution, a plane solution, or a hyperplane solution.

Example 9.2. Consider the set of 2 linear equations in 2 unknowns x, y:

$$\begin{cases} 2x + 5y &= 2 \\ -4x - 10y &= 3 \end{cases} \Leftrightarrow \begin{pmatrix} 2 & 5 \\ -4 & -10 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}. \tag{9.6}$$

The determinant of the matrix of coefficients

$$\begin{vmatrix} 2 & 5 \\ -4 & -10 \end{vmatrix} = 2 \cdot (-10) - (-4) \cdot 5 = 0, \tag{9.7}$$

so the system has no unique solution according to Cramer's Rule, see Theorem 7.4. Therefore, we must use Gaussian elimination to solve the system.

$$\begin{cases} 2x + 5y &= 2 \\ -4x - 10y &= 3 \end{cases} \Leftrightarrow \tag{9.8a}$$

$$\left(\begin{array}{cc|c} 2 & 5 & 2 \\ -4 & -10 & 3 \end{array}\right) \Leftrightarrow \left(\begin{array}{cc|c} 2 & 5 & 2 \\ 0 & 0 & 7 \end{array}\right) \Leftrightarrow \left(\begin{array}{cc|c} 1 & 2.5 & 1 \\ 0 & 0 & 1 \end{array}\right) \Leftrightarrow \tag{9.8b}$$

$$\begin{cases} x + 2.5y &= 1 \\ 0 &= 1 \end{cases} \tag{9.8c}$$

where we have added $2\times$ the first row to the second row and multiplied the first row by the factor $1/2$ and the second row by a factor $1/7$. The second equation $0 = 1$ is clearly inconsistent. Hence, we can conclude that the system of linear equations (9.6) has no solutions. Geometrically, the two lines are parallel but do not have a point in common.

Example 9.3. Consider the set of 2 linear equations in 2 unknowns x, y:

$$\begin{cases} 2x + 4y &= -2 \\ x + 2y &= -1 \end{cases} \Leftrightarrow \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}. \tag{9.9}$$

The determinant of the matrix of coefficients

$$\begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} = 2 \cdot 2 - 1 \cdot 4 = 0, \tag{9.10}$$

so the system has no unique solution according to Cramer's Rule, see Theorem 7.4.

Therefore, we must use Gaussian elimination to solve the system.

$$\begin{cases} 2x + 4y &= -2 \\ x + 2y &= -1 \end{cases} \Leftrightarrow \tag{9.11a}$$

$$\left(\begin{array}{cc|c} 2 & 4 & -2 \\ 1 & 2 & -1 \end{array}\right) \Leftrightarrow \left(\begin{array}{cc|c} 1 & 2 & -1 \\ 1 & 2 & -1 \end{array}\right) \Leftrightarrow \left(\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & 0 & 0 \end{array}\right) \Leftrightarrow \tag{9.11b}$$

$$\begin{cases} x + 2y &= -1 \\ 0 &= 0 \end{cases} \tag{9.11c}$$

where we have multiplied the first row by the factor $1/2$, added $(-1)\times$ the first row to the second row. The system of two linear equations has reduced to one linear equation in two unknowns because the equation $0 = 0$ is always true and therefore redundant. Hence we have one degree of freedom. Clearly, the solution is a line. It will prove insightful to express the line solution on a parametric vector form as this is the generic form of a straight line in $\mathbb{R}^n$. From the remaining equation, we have

$$x = -1 - 2y \Leftrightarrow \begin{cases} x = -1 - 2\lambda \\ y = \lambda, \qquad \text{where } \lambda \in \mathbb{R}, \end{cases} \tag{9.12}$$

where we have assigned an arbitrary value $\lambda \in \mathbb{R}$ to y. The latter form for the solution, we can write on a parametric vector form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad \text{where } \lambda \in \mathbb{R} \Leftrightarrow \tag{9.13a}$$

$$\mathbf{r} = \mathbf{x}_0 + \lambda \mathbf{x}_1 \qquad \text{where } \lambda \in \mathbb{R} \tag{9.13b}$$

which represents a straight line passing through the point $\mathbf{x}_0 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ with direction vector $\mathbf{x}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, see Fig. 9.4

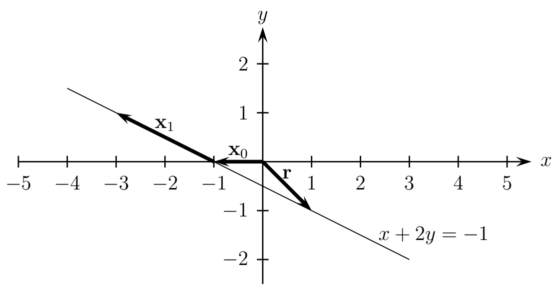

Figure 9.4: The line solution to Eq. (9.9). When the parameter λ runs through the real numbers, the points $\mathbf{r} = \mathbf{x}_0 + \lambda \mathbf{x}_1$ outline the line.

9.4 Examples with 3 linear equations in 3 unknown

Example 9.4. Consider the system of 3 linear equations in 3 unknowns

$$\begin{cases} x_1 - 2x_2 + 4x_3 &= 1 \\ -x_1 + x_2 - x_3 &= 2 \\ 2x_1 + 3x_2 - x_3 &= 3 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & -2 & 4 \\ -1 & 1 & -1 \\ 2 & 3 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}. \tag{9.14}$$

The determinant of the matrix of coefficients

$$\begin{vmatrix} 1 & -2 & 4 \\ -1 & 1 & -1 \\ 2 & 3 & -1 \end{vmatrix} = \begin{vmatrix} -1 & -2 & 2 \\ 0 & 1 & 0 \\ 5 & 3 & 2 \end{vmatrix} = 1 \cdot \begin{vmatrix} -1 & 2 \\ 5 & 2 \end{vmatrix} = -12 \neq 0, \tag{9.15}$$

so the system has a unique solution according to Cramer's Rule, see Theorem 7.4. Using Cramer's rule to determine the unique solution, we find

$$\begin{aligned}(x_1, x_2, x_3) &= \left(\frac{\begin{vmatrix} 1 & -2 & 4 \\ 2 & 1 & -1 \\ 3 & 3 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & -2 & 4 \\ -1 & 1 & -1 \\ 2 & 3 & -1 \end{vmatrix}}, \frac{\begin{vmatrix} 1 & 1 & 4 \\ -1 & 2 & -1 \\ 2 & 3 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & -2 & 4 \\ -1 & 1 & -1 \\ 2 & 3 & -1 \end{vmatrix}}, \frac{\begin{vmatrix} 1 & -2 & 1 \\ -1 & 1 & 2 \\ 2 & 3 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & -2 & 4 \\ -1 & 1 & -1 \\ 2 & 3 & -1 \end{vmatrix}} \right) \\ &= \left(\frac{16}{-12}, \frac{-30}{-12}, \frac{-22}{-12} \right) = \left(-\frac{4}{3}, \frac{5}{2}, \frac{11}{6} \right). \end{aligned} \tag{9.16}$$

Applying Gaussian elimination on the same set of equations, we find:

$$\text{R1 + R2 and } -2\text{R1 + R3} \begin{pmatrix} 1 & -2 & 4 & | & 1 \\ -1 & 1 & -1 & | & 2 \\ 2 & 3 & -1 & | & 3 \end{pmatrix} \Leftrightarrow \tag{9.17a}$$

$$-2\text{R2 + R1 and } 7\text{R2 + R3} \begin{pmatrix} 1 & -2 & 4 & | & 1 \\ 0 & -1 & 3 & | & 3 \\ 0 & 7 & -9 & | & 1 \end{pmatrix} \Leftrightarrow \tag{9.17b}$$

$$1/12 \text{ R3}, -3\text{R3 + R2 and } 2\text{R3 + R1} \begin{pmatrix} 1 & 0 & -2 & | & -5 \\ 0 & -1 & 3 & | & 3 \\ 0 & 0 & 12 & | & 22 \end{pmatrix} \Leftrightarrow \tag{9.17c}$$

$$-\text{R2} \begin{pmatrix} 1 & 0 & 0 & | & -5 + 22/6 \\ 0 & -1 & 0 & | & 3 - 33/6 \\ 0 & 0 & 1 & | & 11/6 \end{pmatrix} \Leftrightarrow \tag{9.17d}$$

$$\begin{pmatrix} 1 & 0 & 0 & | & -4/3 \\ 0 & 1 & 0 & | & 15/6 \\ 0 & 0 & 1 & | & 11/6 \end{pmatrix} \tag{9.17e}$$

and we find that there is a unique solution $(x_1, x_2, x_3) = \left(-\frac{4}{3}, \frac{5}{2}, \frac{11}{6}\right)$.

Example 9.5. Consider the system of 3 linear equations in 3 unknowns

$$\begin{cases} x - 2y - 3z &= 2 \\ x - 4y - 13z &= 14 \\ -3x + 5y + 4z &= 0 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & -2 & -3 \\ 1 & -4 & -13 \\ -3 & 5 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 14 \\ 0 \end{pmatrix}. \tag{9.18}$$

The determinant of the matrix of coefficients

$$\begin{vmatrix} 1 & -2 & -3 \\ 1 & -4 & -13 \\ -3 & 5 & 4 \end{vmatrix} = \begin{vmatrix} 1 & -2 & -3 \\ 0 & -2 & -10 \\ 0 & -1 & -5 \end{vmatrix} = \begin{vmatrix} 1 & -2 & -3 \\ 0 & 0 & 0 \\ 0 & -1 & -5 \end{vmatrix} = 0, \tag{9.19}$$

so the system has no unique solution according to Cramer's Rule, see Theorem 7.4. Therefore, we must use Gaussian elimination to solve the system of linear equations. We form the associated augmented matrix and perform elementary row operations:

$$-1\text{R1 + R2 and } 3\text{R1 + R3} \begin{pmatrix} 1 & -2 & -3 & | & 2 \\ 1 & -4 & -13 & | & 14 \\ -3 & 5 & 4 & | & 0 \end{pmatrix} \Leftrightarrow \tag{9.20a}$$

$$-2\text{R3 + R2 and } -2\text{R3 + R1} \begin{pmatrix} 1 & -2 & -3 & | & 2 \\ 0 & -2 & -10 & | & 12 \\ 0 & -1 & -5 & | & 6 \end{pmatrix} \Leftrightarrow \tag{9.20b}$$

$$-\text{R3 and exchange R2 and R3} \begin{pmatrix} 1 & 0 & 7 & | & -10 \\ 0 & 0 & 0 & | & 0 \\ 0 & -1 & -5 & | & 6 \end{pmatrix} \Leftrightarrow \tag{9.20c}$$

$$\begin{pmatrix} 1 & 0 & 7 & | & -10 \\ 0 & 1 & 5 & | & -6 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \tag{9.20d}$$

that is,

$$\begin{cases} x + 7z &= -10, \\ y + 5z &= -6 \end{cases} \tag{9.21a}$$

as the last equation is redundant. Hence we have 2 equations in 3 unknowns, that is, one degree of freedom. The solution is a line. We assign an arbitrary value $\lambda \in \mathbb{R}$ to z and we find

$$\begin{cases} x &= -10 - 7\lambda, \\ y &= -6 - 5\lambda, \\ z &= \lambda, \end{cases} \qquad \text{where } \lambda \in \mathbb{R} \tag{9.22a}$$

or on parametric vector form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -10 \\ -6 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -7 \\ -5 \\ 1 \end{pmatrix} \quad \text{where } \lambda \in \mathbb{R}, \tag{9.23}$$

that is, a line in $\mathbb{R}^3$ passing through the point $\mathbf{x}_0 = \begin{pmatrix} -10 \\ -6 \\ 0 \end{pmatrix}$ along the direction of $\mathbf{x}_1 = \begin{pmatrix} -7 \\ -5 \\ 1 \end{pmatrix}$.

Example 9.6. Consider the system of 3 linear equations in 3 unknowns

$$\begin{cases} 2x + 4y - 6z &= 6 \\ 3x + 6y - 9z &= 9 \\ -x - 2y + 3z &= -3 \end{cases} \Leftrightarrow \begin{pmatrix} 2 & 4 & -6 \\ 3 & 6 & -9 \\ -1 & -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \\ -3 \end{pmatrix}. \tag{9.24}$$

The determinant of the matrix of coefficients

$$\begin{vmatrix} 2 & 4 & -6 \\ 3 & 6 & -9 \\ -1 & -2 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 3 & 6 & -9 \\ -1 & -2 & 3 \end{vmatrix} = 0, \tag{9.25}$$

so the system has no unique solution according to Cramer's Rule, see Theorem 7.4. Therefore, we must use Gaussian elimination to solve the system of linear equations. We form the associated augmented matrix and perform elementary row operations:

$$2\text{R3} + \text{R1 and } 3\text{R3} + \text{R2} \left(\begin{array}{ccc|c} 2 & 4 & -6 & 6 \\ 3 & 6 & -9 & 9 \\ -1 & -2 & 3 & -3 \end{array}\right) \Leftrightarrow \tag{9.26a}$$

$$-\text{R3, exchange R1 and R3} \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -2 & 3 & -3 \end{array}\right) \Leftrightarrow \tag{9.26b}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \tag{9.26c}$$

that is,

$$x + 2y - 3z = 3, \tag{9.27}$$

as the two last equations $0 = 0$ are redundant. Hence, we have one equation in 3 unknowns, that is, two degrees of freedom. Indeed, Eq. (9.27) is the equation for a two-dimensional plane in $\mathbb{R}^3$ with normal vector $\mathbf{n} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$.

We may assign arbitrary values $\mu \in \mathbb{R}$ to y and $\lambda \in \mathbb{R}$ to z to find

$$\begin{cases} x &= 3 - 2\mu + 3\lambda, \\ y &= \mu, \\ z &= \lambda, \end{cases} \quad \text{where } \mu, \lambda \in \mathbb{R} \tag{9.28a}$$

so the solution on a parametric vector form reads

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \quad \text{where } \mu, \lambda \in \mathbb{R}, \Leftrightarrow \tag{9.29a}$$

$$\mathbf{r} = \mathbf{x}_0 + \mu \mathbf{x}_1 + \lambda \mathbf{x}_2, \quad \text{where } \mu, \lambda \in \mathbb{R}, \tag{9.29b}$$

that is, a plane in $\mathbb{R}^3$ passing through the point $\mathbf{x}_0 = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ and spanned by the two vectors $\mathbf{x}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$, see Fig. 9.5.

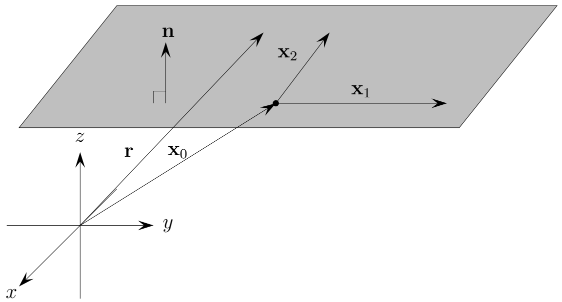

Figure 9.5: Schematic representation of the plane solution to Eq. (9.24). The plane passes through the point $\mathbf{x}_0$ and is normal to the vector $\mathbf{n}$. When the parameters μ and λ runs through the real numbers, the points $\mathbf{r} = \mathbf{x}_0 + \mu\mathbf{x}_1 + \lambda\mathbf{x}_2$ outline the plane solution. We say that the vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ "span the plane".

Indeed, the vector product of these two vectors must be proportional the normal vector of the plane. Let us check:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 0 \\ 3 & 0 & 1 \end{vmatrix} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}, \tag{9.30}$$

which is exactly the normal vector we identified previously from the equation of the plane on component form Eq. (9.27).

9.5 Summary

After studying Sec. 9, you should know

- how to view the system of linear equations as searching for $\mathbf{x} \in \mathbb{R}^n$ that the $n \times n$ matrix $\mathbf{A}$ maps onto $\mathbf{b} \in \mathbb{R}^n$
- how to apply Cramer's rule to determine whether or not a system of linear equations has a unique solution
- how to apply Gaussian elimination to solve a system on linear equations using elementary operations
  1. Changing the order of the equations.
  2. Multiplying all terms in an equation by the same non-zero constant.
  3. Adding a multiple $r \in \mathbb{R}$ of any equation to any other equation. Because r may be negative, addition includes subtraction.
- how to perform elementary operations on the augmented matrix to solve a system of linear equations
  1. Interchanging two rows.
  2. Multiplying all terms in a row equation by the same non-zero constant.
  3. Adding a multiple $r \in \mathbb{R}$ of any row to any other row. Because r may be negative, addition includes subtraction.

  The strategy applied is to end up with a reduced augmented matrix where:
  - All nonzero rows (rows with at least one nonzero element) are above any rows of all zeroes.
  - The leading coefficient (the first non-zero number from the left) of a nonzero row is always strictly to the right of the leading coefficient of the row above it.
  - The leading coefficient of each nonzero row is 1.
- how to write line and plane solutions for a system of linear equations on a parametric vector form.

10 Homogeneous equations and independent vectors

The aim is to discuss the homogeneous equation $\mathbf{Ax} = \mathbf{0}$. The homogeneous equation has the trivial solution $\mathbf{x} = \mathbf{0}$. If the determinant of the matrix of coefficients is non-zero, $\det\mathbf{A} \neq 0$, this is the only (unique) solution of the homogeneous equation. However, if the determinant of the matrix of coefficients is zero, $\det\mathbf{A} = 0$, then the homogeneous equation has non-trivial solutions $\mathbf{x} \neq \mathbf{0}$. We discuss the relationship between the solutions for the homogeneous equations and those for an inhomogeneous equation $\mathbf{Ax} = \mathbf{b}, \mathbf{b} \neq \mathbf{0}$ that has infinitely many solutions.

10.1 Introduction

When solving a system of n linear equations in n unknowns, Cramer's rule Theorem 7.4 states that a unique solution exists when the determinant of the matrix of coefficients is non-zero, $\det\mathbf{A} \neq 0$. When $\det\mathbf{A} = 0$, either there are no solutions or there are infinitely many solutions to the system of linear equations. We will investigate further the situation when $\det\mathbf{A} = 0$. We are interested in the situation when infinitely many solutions exists, that is, in this Sec. we are not interested in the situation when no solutions exist. Therefore, let us review the examples where we have $\det\mathbf{A} = 0$ with infinitely many solutions.

Example 10.1. In Ex. 9.3 on page 75 we solved a system of 2 linear equations in 2 unknowns with a line solution:

$$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \Leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad \text{where } \lambda \in \mathbb{R}. \tag{10.1a}$$

Notice that if we apply the matrix of coefficients on the point on the line and the directional vector, respectively we find

$$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \tag{10.1b}$$

$$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{10.1c}$$

We notice that applying the matrix of coefficients on the point on the line yields the right-hand side of the original equation. Hence, this point on the line is a particular solution to the original equation. Applying the matrix of coefficients on the directional vector yields zero-vector.

Example 10.2. In Ex. 9.5 on page 78 we solved a system of 3 linear equations in 3 unknowns with a line solution:

$$\begin{pmatrix} 1 & -2 & -3 \\ 1 & -4 & -13 \\ -3 & 5 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 14 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -10 \\ -6 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -7 \\ -5 \\ 1 \end{pmatrix}, \quad \text{where } \lambda \in \mathbb{R}. \tag{10.2a}$$

Notice that if we apply the matrix of coefficients on the point on the line and the directional vector, respectively, we find

$$\begin{pmatrix} 1 & -2 & -3 \\ 1 & -4 & -13 \\ -3 & 5 & 4 \end{pmatrix} \begin{pmatrix} -10 \\ -6 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 14 \\ 0 \end{pmatrix} \tag{10.2b}$$

$$\begin{pmatrix} 1 & -2 & -3 \\ 1 & -4 & -13 \\ -3 & 5 & 4 \end{pmatrix} \begin{pmatrix} -7 \\ -5 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{10.2c}$$

Once again, we notice that applying the matrix of coefficients on the point on the line yields the right-hand side of the original equation, that is, it is a particular solution to the original equation. Applying the matrix of coefficients on the directional vector yields zero-vector.

Example 10.3. Finally, In Ex. 9.6 on page 79 we solved a system of 3 linear equations in 3 unknowns with a plane solution:

$$\begin{pmatrix} 2 & 4 & -6 \\ 3 & 6 & -9 \\ -1 & -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \\ -3 \end{pmatrix} \Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \quad \text{where } \mu, \lambda \in \mathbb{R}. \tag{10.3a}$$

Notice that if we apply the matrix of coefficients on the point on the plane and the two vectors spanning the plane, we find

$$\begin{pmatrix} 2 & 4 & -6 \\ 3 & 6 & -9 \\ -1 & -2 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \\ -3 \end{pmatrix} \tag{10.3b}$$

$$\begin{pmatrix} 2 & 4 & -6 \\ 3 & 6 & -9 \\ -1 & -2 & 3 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{10.3c}$$

$$\begin{pmatrix} 2 & 4 & -6 \\ 3 & 6 & -9 \\ -1 & -2 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{10.3d}$$

Also here we notice that applying the matrix of coefficients on the point on the plane yields the right-hand side of the original equation, that is, it is a particular solution to the original equation. Applying the matrix of coefficients on the two vectors that span the plane yields zero-vector.

In these three examples, the system of linear equations has infinitely many solutions. Let us consider a general case where a system of linear equations $\mathbf{Ax} = \mathbf{b}$ has infinitely many solutions:

$$\mathbf{Ax} = \mathbf{b} \Leftrightarrow \mathbf{x} = \mathbf{x}_0 + \sum_{i=1}^{k} \lambda_i \mathbf{x}_i \quad \text{where } \lambda_i \in \mathbb{R}. \tag{10.4a}$$

If $k = 1$ we have a line solution and we identify $\mathbf{x} = \mathbf{x}_0$ as a point on the line and $\mathbf{x}_1$ as the directional vector for the line. If $k = 2$, we have a plane solution and we identify $\mathbf{x} = \mathbf{x}_0$ as a point on the plane and $\mathbf{x}_1$ and $\mathbf{x}_2$ as two vectors in the plane that span that plane. If $k > 2$,

we have a hyper-plane solution and we identify $\mathbf{x} = \mathbf{x}_0$ as a point on the hyper-plane and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ as the vectors that span that hyper-plane when the parameters λ_i runs through the real numbers. In Ex. 9.3 and Ex. 9.5 we have $k = 1$ while in Ex. 9.6 we have $k = 2$.

Assuming that the system of linear equations has infinitely many solutions, is it generally true that

$$\mathbf{Ax}_0 = \mathbf{b}, \tag{10.4b}$$

$$\mathbf{Ax}_i = \mathbf{0} \quad \text{for } i = 1, 2, \ldots, k. \tag{10.4c}$$

The answer is a resounding: “Yes it is!” To reach this conclusion, let us look at the homogeneous equation.

10.2 Homogeneous equation $\mathbf{Ax} = \mathbf{0}$

Consider a system of n linear equations in n unknowns written on matrix form:

$$\mathbf{Ax} = \mathbf{b}, \tag{10.5}$$

where $\mathbf{A}$ is the $n \times n$ matrix of coefficients, $\mathbf{x} \in \mathbb{R}^n$ the n-dimensional vector of unknowns and $\mathbf{b} \in \mathbb{R}^n$ the n-dimensional vector of constant real numbers. We are searching for the vector(s) $\mathbf{x} \in \mathbb{R}^n$ that $\mathbf{A}$ maps onto $\mathbf{b}$, see Fig. 10.1

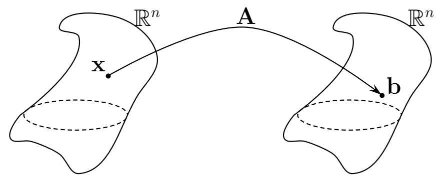

Figure 10.1: Solving a system of n equations in n unknowns can be viewed as searching for the vector(s) $\mathbf{x} \in \mathbb{R}^n$ that the $n \times n$ matrix $\mathbf{A}$ maps onto the vector $\mathbf{b} \in \mathbb{R}^n$. If $\det\mathbf{A} \neq 0$, a unique solution exists. If $\det\mathbf{A} = 0$, there is no solutions or infinitely many solutions. We are interested in the latter situation.

Definition 10.1. The equation

$$\mathbf{Ax} = \mathbf{0} \tag{10.6}$$

is called the *homogeneous equation* associated with the *inhomogeneous equation* $\mathbf{Ax} = \mathbf{b}$.

Because $\mathbf{A0} = \mathbf{0}$, the zero-vector $\mathbf{x} = \mathbf{0}$ is always a solution to the homogeneous equation $\mathbf{Ax} = \mathbf{0}$.

Definition 10.2. We call $\mathbf{x} = \mathbf{0}$ a *trivial solution* to the homogeneous equation (10.6). If there exists $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{Ax} = \mathbf{0}$, we call $\mathbf{x} \neq \mathbf{0}$ a *non-trivial solution* to the homogeneous equation, see Fig. 10.2 for a schematic illustration.

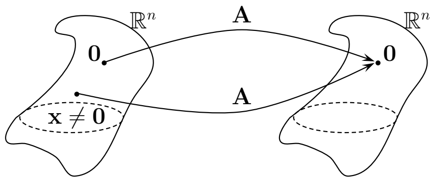

Figure 10.2: The zero vector $\mathbf{x} = \mathbf{0}$ is called the *trivial solution* to the the homogeneous equation $\mathbf{Ax} = \mathbf{0}$. If there exists $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{Ax} = \mathbf{0}$, we call $\mathbf{x} \neq \mathbf{0}$ for a *non-trivial solution* to the homogeneous equation. Non-trivial solutions to the homogeneous equation $\mathbf{Ax} = \mathbf{0}$ exist if and only if $\det\mathbf{A} = 0$, see Theorem 10.1.

Theorem 10.1. The homogeneous equation $\mathbf{Ax} = \mathbf{0}$ has non-trivial solutions if and only if $\det\mathbf{A} = 0$.

Proof: ($\Rightarrow$) Assume that the homogeneous equation $\mathbf{Ax} = \mathbf{0}$ has non-trivial solutions, that is, there exists at least one $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{Ax} = \mathbf{0}$. Because $\mathbf{x} = \mathbf{0}$ is also a solution to the homogeneous equation, we have at least two solutions. Hence, the solution is not unique and we can conclude from Cramer's Rule that $\det\mathbf{A} = 0$.
($\Leftarrow$) Assume that $\det\mathbf{A} = 0$. Cramer's Rule implies that no solution or infinitely many solutions exists. Because $\mathbf{x} = \mathbf{0}$ is a solution to the homogeneous equation, it implies that infinitely many solutions exist. Hence, the homogeneous equation $\mathbf{Ax} = \mathbf{0}$ must have (infinitely many) non-trivial solutions.

Q.E.D.

Example 10.4. Consider the matrix of coefficients from Ex. 9.3 and the associated homogeneous equation:

$$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{10.7}$$

The determinant of the matrix of coefficients

$$\begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} = 2 \cdot 2 - 1 \cdot 4 = 0, \tag{10.8}$$

so the homogeneous equation has non-trivial solutions according to Theorem 10.1. Let us apply Gaussian elimination to find the solutions. We form the augmented matrix and perform elementary row operations to reduce it to a simple form:

$$-2\text{R2} + \text{R1} \left(\begin{array}{cc|c} 2 & 4 & 0 \\ 1 & 2 & 0 \end{array}\right) \Leftrightarrow \tag{10.9a}$$

$$\text{Exchange rows} \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 2 & 0 \end{array}\right) \Leftrightarrow \tag{10.9b}$$

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \tag{10.9c}$$

that is,

$$x + 2y = 0 \Leftrightarrow x = -2y \Leftrightarrow \begin{cases} x = -2\lambda, \\ y = \lambda, \end{cases} \quad \text{where } \lambda \in \mathbb{R}, \tag{10.10}$$

where we have assigned an arbitrary value $\lambda \in \mathbb{R}$ to y. The latter form for the solution, we can write on a parametric vector form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} -2 \\ 1 \end{pmatrix} \Leftrightarrow \mathbf{x} = \lambda\mathbf{x}_1, \quad \text{where } \lambda \in \mathbb{R}. \tag{10.11}$$

that is, the solution to the homogeneous equation is a line in $\mathbb{R}^2$ with direction vector $\mathbf{x}_1$ passing through the point $\mathbf{0} \in \mathbb{R}^2$.

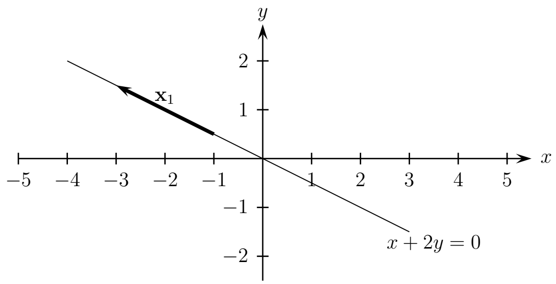

Figure 10.3: The line solution to the homogeneous Eq. (10.7). When the parameter λ runs through the real numbers, the points $\mathbf{x} = \lambda\mathbf{x}_1$ outline the line with direction vector $\mathbf{x}_1$ passing through the origin of $\mathbb{R}^2$. Compare this solution to the homogeneous equation with the solution to the inhomogeneous equation displayed in Fig. 9.4 on page 76

Example 10.5. Consider the matrix of coefficients from Ex. 9.5 and the associated homogeneous equation:

$$\begin{pmatrix} 1 & -2 & -3 \\ 1 & -4 & -13 \\ -3 & 5 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{10.12}$$

The determinant of the matrix of coefficients

$$\begin{vmatrix} 1 & -2 & -3 \\ 1 & -4 & -13 \\ -3 & 5 & 4 \end{vmatrix} = \begin{vmatrix} 1 & -2 & -3 \\ 0 & -2 & -10 \\ 0 & -1 & -5 \end{vmatrix} = \begin{vmatrix} 1 & -2 & -3 \\ 0 & 0 & 0 \\ 0 & -1 & -5 \end{vmatrix} = 0, \tag{10.13}$$

so the homogeneous equation has non-trivial solutions according to Theorem 10.1. Let us apply Gaussian elimination to find the solutions. We form the augmented matrix and perform elementary row operations to reduce it to a simple form:

$$-1\text{R1} + \text{R2 and } 3\text{R1} + \text{R3} \left(\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 1 & -4 & -13 & 0 \\ -3 & 5 & 4 & 0 \end{array}\right) \Leftrightarrow \tag{10.14a}$$

$$-2\text{R3} + \text{R2 and } -2\text{R3} + \text{R1} \left(\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & -2 & -10 & 0 \\ 0 & -1 & -5 & 0 \end{array}\right) \Leftrightarrow \tag{10.14b}$$

$$-\text{R3 and exchange R2 and R3} \left(\begin{array}{ccc|c} 1 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -5 & 0 \end{array}\right) \Leftrightarrow \tag{10.14c}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 7 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \tag{10.14d}$$

that is,

$$\begin{cases} x + 7z &= 0, \\ y + 5z &= 0. \end{cases} \tag{10.15a}$$

We assign an arbitrary value $\lambda \in \mathbb{R}$ to z and we find

$$\begin{cases} x &= -7\lambda, \\ y &= -5\lambda, \\ z &= \lambda, \end{cases} \Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda \begin{pmatrix} -7 \\ -5 \\ 1 \end{pmatrix} \Leftrightarrow \mathbf{x} = \lambda \mathbf{x}_1, \quad \text{where } \lambda \in \mathbb{R}, \tag{10.16a}$$

that is, the solution to the homogeneous equation is a line in $\mathbb{R}^3$ with direction vector $\mathbf{x}_1$ passing through the point $\mathbf{0} \in \mathbb{R}^3$ (namely when $\lambda = 0$).

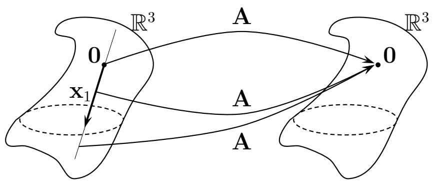

Figure 10.4: Schematic representation of the line solution to the homogeneous Eq. (10.12). When the parameter λ runs through the real numbers, the points $\mathbf{x} = \lambda \mathbf{x}_1$ outline the line with direction vector $\mathbf{x}_1$ passing through the origin of $\mathbb{R}^3$. For all $\mathbf{x}$ on the line, $\mathbf{Ax} = \mathbf{0}$. Shown explicitly are three such points, including the trivial solution $\mathbf{x} = \mathbf{0}$.

Example 10.6. Consider the matrix of coefficients from Ex. 9.6 and the associated homogeneous equation:

$$\begin{pmatrix} 2 & 4 & -6 \\ 3 & 6 & -9 \\ -1 & -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{10.17}$$

The determinant of the matrix of coefficients

$$\begin{vmatrix} 2 & 4 & -6 \\ 3 & 6 & -9 \\ -1 & -2 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 3 & 6 & -9 \\ -1 & -2 & 3 \end{vmatrix} = 0, \tag{10.18}$$

so the homogeneous equation has non-trivial solutions according to Theorem 10.1. Let us apply Gaussian elimination to find the solutions. We form the augmented matrix and perform elementary row operations to reduce it to a simple form:

$$2\text{R3} + \text{R1 and } 3\text{R3} + \text{R2} \left(\begin{array}{ccc|c} 2 & 4 & -6 & 0 \\ 3 & 6 & -9 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right) \Leftrightarrow \tag{10.19a}$$

$$-\text{R3, exchange R1 and R3} \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right) \Leftrightarrow \tag{10.19b}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Leftrightarrow \tag{10.19c}$$

$$x + 2y - 3z = 0, \tag{10.20}$$

We assign arbitrary values $\lambda_1 \in \mathbb{R}$ to y and $\lambda_2 \in \mathbb{R}$ to z and we find

$$\begin{cases} x &= -2\lambda_1 + 3\lambda_2, \\ y &= \lambda_1, \\ z &= \lambda_2, \end{cases} \Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda_1 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \Leftrightarrow \mathbf{x} = \lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2, \text{where } \lambda_1, \lambda_2 \in \mathbb{R}, \tag{10.21}$$

that is, the solution to the homogeneous equation is a plane in $\mathbb{R}^3$ passing through the point $\mathbf{0}$ and spanned by the two vectors $\mathbf{x}_1$ and $\mathbf{x}_2$, see Fig. 10.5

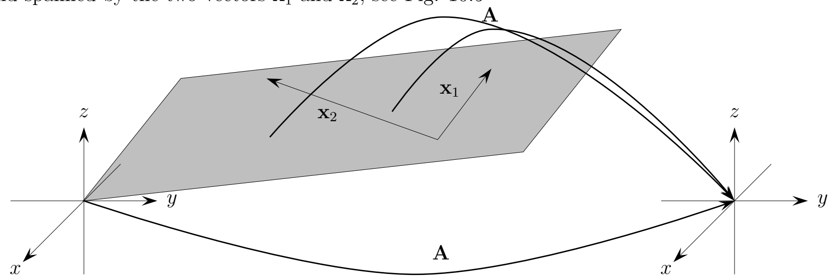

Figure 10.5: Schematic representation of the plane solution to the homogeneous Eq. (10.17). When the parameters λ_1, λ_2 runs through the real numbers, the points $\mathbf{x} = \lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2$ outline a plane that passes through the origin of $\mathbb{R}^3$. For all $\mathbf{x}$ on the plane, $\mathbf{Ax} = \mathbf{0}$. Shown explicitly are three such points, including the trivial solution $\mathbf{x} = \mathbf{0}$.

Theorem 10.2. If a system of n linear equations in n unknowns $\mathbf{Ax} = \mathbf{b}$ has infinitely many solutions

$$\mathbf{x} = \mathbf{x}_0 + \sum_{i=1}^{k} \lambda_i \mathbf{x}_i, \quad \text{where } \lambda_i \in \mathbb{R} \tag{10.22}$$

then the vector (point) $\mathbf{x}_0$ on the space of solution (line, plane, hyperplane) is a particular solution to the inhomogeneous equation and the vectors $\mathbf{x}_i, i = 1, 2, \ldots, k$ are non-trivial solutions to the associated homogeneous equation, that is

$$\mathbf{Ax}_0 = \mathbf{b}, \tag{10.23a}$$
$$\mathbf{Ax}_i = \mathbf{0} \quad \text{for all } i = 1, 2, \ldots, k. \tag{10.23b}$$

The dimensionality of the space of solutions to the homogeneous equations is given by k. This is also known as the dimensionality of the *null-space* $\mathcal{N}$ for the matrix $\mathbf{A}$, where the null-space is defined as the subspace $\mathcal{N} \subseteq \mathbb{R}^n$ for which $\mathbf{Ax} = \mathbf{0}$.

We notice that if we apply $\mathbf{A}$ on a solution $\mathbf{x}$, that consists of a particular solution $\mathbf{x}_0$ and a sum over solutions $\mathbf{x}_i$ to the associated homogeneous equation, we find

$$\begin{aligned}
\mathbf{Ax} &= \mathbf{A}\left(\mathbf{x}_0 + \sum_{i=1}^{k} \lambda_i \mathbf{x}_i\right) \\
&= \mathbf{Ax}_0 + \mathbf{A}\left(\sum_{i=1}^{k} \lambda_i \mathbf{x}_i\right) \quad \text{because } \mathbf{A}(\mathbf{x}+\mathbf{y}) = \mathbf{Ax} + \mathbf{Ay} \\
&= \mathbf{Ax}_0 + \sum_{i=1}^{k} \lambda_i \mathbf{A}(\mathbf{x}_i) \quad \text{because } \mathbf{A}(r\mathbf{x}) = r(\mathbf{Ax}) \\
&= \mathbf{b} + \sum_{i=1}^{k} \lambda_i \mathbf{0} \\
&= \mathbf{b}.
\end{aligned} \tag{10.24}$$

10.3 Linearly dependent and linearly independent vectors

Consider an $n \times n$ matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{i\,1} & a_{i\,2} & \cdots & a_{i\,j} & \cdots & a_{i\,n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix}. \tag{10.25}$$

Normally, we view the entries a_{ij} in the $n \times n$ matrix $\mathbf{A}$ as coefficients of the system of n linear equations in n unknowns. In the present Section, that is still the case but we will view the situation from a slightly different perspective.

Define $\mathbf{a}_j$ as the jth column vector in the matrix $\mathbf{A}$:

$$\mathbf{a}_j \stackrel{\text{def}}{=} \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{nj} \end{pmatrix} \in \mathbb{R}^n, \quad \text{for } j = 1, 2, \ldots, n. \tag{10.26}$$

Then the matrix $\mathbf{A}$ itself can be expressed as a collection of n column vectors:

$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_j & \cdots \mathbf{a}_n \end{pmatrix}. \tag{10.27}$$

Now, consider the homogeneous equation

$$\mathbf{Ax} = \mathbf{0}, \tag{10.28a}$$

which is equivalent to n linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1j}x_j + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2j}x_j + \cdots + a_{2n}x_n = 0, \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{i\,1}x_1 + a_{i\,2}x_2 + \cdots + a_{i\,j}x_j + \cdots + a_{i\,n}x_n = 0, \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nj}x_j + \cdots + a_{nn}x_n = 0. \end{cases} \tag{10.28b}$$

Using the column vectors for the $\mathbf{A}$ this system of linear equations is equivalent to

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_j\mathbf{a}_j + \cdots + x_n\mathbf{a}_n = \mathbf{0}. \tag{10.28c}$$

If $\det\mathbf{A} \neq 0$, the homogeneous equation (10.28) has a unique solution, namely the trivial solution $\mathbf{x} = \mathbf{0}$. This implies that the only solution to Eq. (10.28c) is $x_1 = x_2 = \ldots = x_n = 0$. When the only solution to Eq. (10.28c) is the trivial solution, we say that the set of column vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ are *linearly independent*. It implies that none of the vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ can be written as a linear combination of the others.

If $\det\mathbf{A} = 0$, the homogeneous equation (10.28) has non-trivial solutions $\mathbf{x} \neq \mathbf{0}$, see Theorem 10.1. This implies that there exists numbers $x_1, x_2, \ldots, x_n$ not all equal to zero such $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_j\mathbf{a}_j + \cdots + x_n\mathbf{a}_n = \mathbf{0}$. When Eq. (10.28c) has non-trivial solutions, we say that the set of column vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ are *linearly dependent*. It implies that at least one of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ can be written as a linear combination of the others. For example, assume that $x_1 \neq 0$. Then

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_j\mathbf{a}_j + \cdots + x_n\mathbf{a}_n = \mathbf{0} \Leftrightarrow \mathbf{a}_1 = \frac{-1}{x_1}\left(x_2\mathbf{a}_2 + \cdots + x_j\mathbf{a}_j + \cdots + x_n\mathbf{a}_n\right) \tag{10.29}$$

that is, we can write the vector $\mathbf{a}_1$ as a linear combination of the remaining vectors $\{\mathbf{a}_2, \ldots, \mathbf{a}_n\}$. The implies that the vector $\mathbf{a}_1$ lies in the hyperplane spanned by the other vectors $\{\mathbf{a}_2, \ldots, \mathbf{a}_n\}$.

The general definition of linearly dependent and linearly independent vectors is as follows:

Definition 10.3. A set of vectors $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m\}$ with $\mathbf{b}_i \in \mathbb{R}^n$ is *linearly dependent* if there exists numbers $c_1, c_2, \ldots, c_m$ not all equal to zero such that

$$c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_m\mathbf{b}_m = \mathbf{0}. \tag{10.30}$$

Definition 10.4. The set of vectors $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m\}$ is *linearly independent* if the only solution to the Eq. (10.30) is the trivial solution, that is, if

$$c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_m\mathbf{b}_m = \mathbf{0} \Rightarrow c_1 = c_2 = \cdots = c_m = 0. \tag{10.31}$$

Example 10.7. If two vectors $\mathbf{b}_1, \mathbf{b}_2 \in \mathbb{R}^n$ are parallel we have that one is a numerical multiple of the other. Hence, there exist a non-zero number $r \in \mathbb{R}$ such that

$$\mathbf{b}_1 = r\mathbf{b}_2 \Leftrightarrow \mathbf{b}_1 - r\mathbf{b}_2 = \mathbf{0}, \tag{10.32}$$

so the homogeneous equation $c_1\mathbf{b}_1 + c_2\mathbf{b}_2 = \mathbf{0}$ has a non-trivial solution, namely $c_1 = 1, c_2 = -r$. Hence two parallel vectors are linearly dependent.

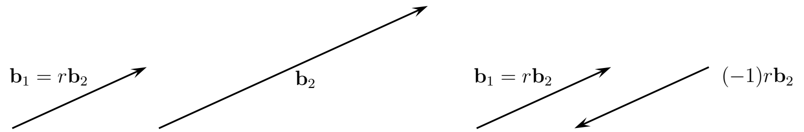

Figure 10.6: Two parallel vectors in $\mathbb{R}^n$ are linearly dependent. If $\mathbf{b}_1 = r\mathbf{b}_2$ then we have $\mathbf{b}_1 + (-1)r\mathbf{b}_2 = \mathbf{0}$.

Example 10.8. In $\mathbb{R}^2$, at most two vectors can be linearly independent. Say you have a set of three vectors in $\mathbb{R}^2$: $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. If two of those are parallel, the set is linearly dependent. So we assume that none of these vectors are parallel. One of the vectors can then be written as a sum of the two others because they are all in the same plane. Hence, we can find numbers $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 = \mathbf{0}, \tag{10.33}$$

and hence they are linearly dependent.

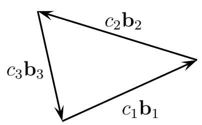

Figure 10.7: Geometrical representation of three vectors in $\mathbb{R}^2$, the plane of this page. If none of the vectors are parallel, then there exists non-trivial solutions to the equation $c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 = \mathbf{0}$ and hence there are at most two linearly independent vectors in $\mathbb{R}^2$.

The simplest set of n linearly independent vectors in $\mathbb{R}^n$ consists of the natural basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$. In $\mathbb{R}^n$, at most n vectors can be linearly independent. Indeed, any other vector $\mathbf{x} \in \mathbb{R}^n$ can be written as a linear combination of this set, $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$.

If you have a set of m vectors in $\mathbb{R}^n$ with $m > n$ you know the set is linearly dependent. If $m < n$, you will have to consider the solutions to Eq. (10.30) using Gaussian elimination to determine whether the set is linearly dependent or not. If $m = n$ there is an elegant method to reveal whether a set of n vectors in $\mathbb{R}^n$ is a linearly dependent or not.

Example 10.9. The fast-track to determine whether a set of n vectors in $\mathbb{R}^n$ is linearly dependent or linearly independent is to form a matrix of these vectors, using them as the columns or rows (one or the other) of the matrix. This matrix will be a square matrix and hence, the determinant of the matrix is well-defined. If the determinant of this matrix is non-zero, the vectors are linearly independent. If the determinant of this matrix is zero, the vectors are linearly dependent.

For example, consider the vectors

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} 7 \\ 3 \\ 0 \end{pmatrix}, \mathbf{b}_3 = \begin{pmatrix} -5 \\ -7 \\ 10 \end{pmatrix}. \tag{10.34}$$

Are these vectors linearly dependent? We are asking, whether there exists a non-trivial solution to the equation

$$c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 = \mathbf{0}. \tag{10.35}$$

Because this is 3 equations in 3 unknowns (c_1, c_2, c_3) we know that non-trivial solutions exists if and only if $\det\begin{pmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3\end{pmatrix} = 0$:

$$\det\begin{pmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3\end{pmatrix} = \begin{vmatrix} 1 & 7 & -5 \\ -2 & 3 & -7 \\ 5 & 0 & 10 \end{vmatrix} = \begin{vmatrix} 1 & 7 & -7 \\ -2 & 3 & -3 \\ 5 & 0 & 0 \end{vmatrix} = 5\begin{vmatrix} 7 & -7 \\ 3 & -3 \end{vmatrix} = 0 \tag{10.36}$$

so the vectors are linearly dependent. Indeed, inspection shows that

$$\mathbf{b}_3 = 2\mathbf{b}_1 - \mathbf{b}_2. \tag{10.37}$$

10.4 Scalar triple product as a determinant

Let us restrict ourselves to 3-dimensional space, that is, $\mathbb{R}^3$. Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ and consider the so-called scalar triple product (see p. 48 and p. 54) $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$. Three vectors are involved and the answer is a scalar quantity, hence the name. The scalar triple product is related to the

3×3 determinant with column vectors $\mathbf{c}, \mathbf{a}$ and $\mathbf{b}$, respectively:

$$\begin{aligned}
\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) &= \mathbf{c} \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \\
&= (c_x \mathbf{i} + c_y \mathbf{j} + c_z \mathbf{k}) \cdot \left(\begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \mathbf{k} \right) \\
&= c_x \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - c_y \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + c_z \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \\
&= \begin{vmatrix} c_x & c_y & c_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \\
&= \begin{vmatrix} c_x & a_x & b_x \\ c_y & a_y & b_y \\ c_z & a_z & b_z \end{vmatrix} \quad \text{see Property 6 of determinants, p. 65} \\
&= \det \begin{pmatrix} \mathbf{c} & \mathbf{a} & \mathbf{b} \end{pmatrix}. \qquad (10.38)
\end{aligned}$$

Note that this relationship is consistent with our previous observation that if $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = 0$ then the three vectors are co-planar. Similarly, $\det \begin{pmatrix} \mathbf{c} & \mathbf{a} & \mathbf{b} \end{pmatrix} = 0$ implies that the three vectors are linearly dependent which in turn implies that they are co-planar.

Note that using property 1 of determinants, that is, the sign of a determinant is reversed if two columns are interchanged we find

$$\det \begin{pmatrix} \mathbf{c} & \mathbf{a} & \mathbf{b} \end{pmatrix} = \begin{vmatrix} c_x & a_x & b_x \\ c_y & a_y & b_y \\ c_z & a_z & b_z \end{vmatrix} = \begin{vmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{vmatrix} = \begin{vmatrix} b_x & c_x & a_x \\ b_y & c_y & a_y \\ b_z & c_z & a_z \end{vmatrix} \qquad (10.39)$$

because each of these determinants results from two column exchanges from the original determinant. Consequently, we have that

$$\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) \qquad (10.40)$$

This shows that all scalar triple products where the same vectors are rotated in cyclic order are equal. However, the sign is reversed when the vectors are not in cyclic order although the absolute value is unchanged, that is,

$$\begin{aligned}
\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) &= -\mathbf{c} \cdot (\mathbf{b} \times \mathbf{a}) \\
|\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})| &= |\mathbf{c} \cdot (\mathbf{b} \times \mathbf{a})|. \qquad (10.41)
\end{aligned}$$

The absolute value of a scalar triple product has a geometrical interpretation and therefore, the absolute value of the determinant also has a geometrical interpretation. We find

$$\begin{aligned}
\left|\det \begin{pmatrix} \mathbf{c} & \mathbf{a} & \mathbf{b} \end{pmatrix}\right| &= |\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})| \\
&= |\mathbf{c}| \, |\cos\theta| \, |\mathbf{a} \times \mathbf{b}| \\
&= |\mathbf{c}| \, |\cos\theta| \, |\mathbf{a}| \, |\mathbf{b}| \, |\sin\phi| \\
&= (\text{Height in parallelepiped}) \times (\text{Area of base}) \qquad (10.42)
\end{aligned}$$

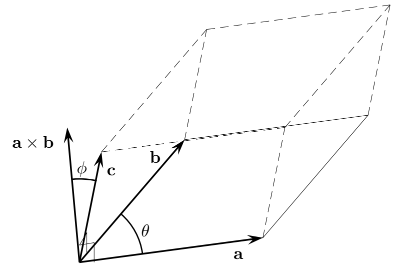

Figure 10.8: The absolute value of the triple scalar product $|\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})|$ yields the volume of a parallelepiped with edges $\mathbf{a}, \mathbf{b}, \mathbf{c}$. This result follows readily from $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = |c| \cos\phi \, |\mathbf{a} \times \mathbf{b}| =$ height $\times$ area of base.

is the volume of the parallelepiped spanned by the vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$
We will state the generalisation of this result without a proof:

Theorem 10.3. Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ be n vectors in $\mathbb{R}^n$. Then the volume of the parallelepiped with edges $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ is $\left|\det \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix}\right|$. If $\det \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix} > 0$, the ordered set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ form a right-handed system.

10.5 Summary

After studying Sec. 10, you should know that

- the equation $\mathbf{Ax} = \mathbf{0}$ is called the homogeneous equation
  - that $\mathbf{x} = \mathbf{0}$ is a <u>trivial solution</u> to the homogeneous equation
  - the the homogeneous equation $\mathbf{Ax} = \mathbf{0}$ has non-trivial solutions $\mathbf{x} \neq \mathbf{0}$ if and only if $\det \mathbf{A} = 0$
- the inhomogeneous equation $\mathbf{Ax} = \mathbf{b}$ has infinitely many solutions $\Leftrightarrow \mathbf{x} = \mathbf{x}_0 + \sum_{i=1}^{k} \lambda_i \mathbf{x}_i$ where $\mathbf{Ax}_0 = \mathbf{b}$ and $\mathbf{Ax}_i = \mathbf{0}, i = 1, \ldots, k$.
- a set of vectors $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m\}$ with $\mathbf{b}_i \in \mathbb{R}^n$ is *linearly dependent* if there exists numbers $c_1, c_2, \ldots, c_m$ <u>not all</u> equal to zero such that
$$c_1 \mathbf{b}_1 + c_2 \mathbf{b}_2 + \cdots + c_m \mathbf{b}_m = \mathbf{0}.$$
- a set of vectors $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m\}$ is *linearly independent* if
$$c_1 \mathbf{b}_1 + c_2 \mathbf{b}_2 + \cdots + c_m \mathbf{b}_m = \mathbf{0} \Rightarrow c_1 = c_2 = \cdots = c_m = 0.$$
- the scale triple product $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \det \begin{pmatrix} \mathbf{c} & \mathbf{a} & \mathbf{b} \end{pmatrix}$
- that $|\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})| =$ Volume of parallelepiped with edges $\mathbf{a}, \mathbf{b}, \mathbf{c}$

11 Linear functions and matrices: I

The aim is to discuss functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ and how they are represented graphically. We then turn our attention to linear functions from $\mathbb{R}^n$ to $\mathbb{R}^m$. Defining a function via a matrix naturally leads us to define matrix addition $\mathbf{A} + \mathbf{B}$, numerical multiplication $r\mathbf{A}$ and matrix multiplication $\mathbf{AB}$. We will discover that matrices form a vector space.

11.1 Functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ and their graphical representation

Definition 11.1. The *domain* of a function is the set on which it is defined. The *range* of a function is the set of values assumed by the function. We speak of functions "from $\mathbb{R}^n$ to $\mathbb{R}^m$" and write $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ to indicate a function whose domain is a subset of $\mathbb{R}^n$ and whose range is a subset of $\mathbb{R}^m$, see Fig. 11.1.

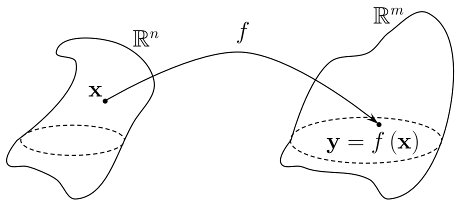

Figure 11.1: A function $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ has a domain in $\mathbb{R}^n$, that is, it is a function of n variables $x_1, x_2, \ldots, x_n$. The values of the function $\mathbf{y} = f(\mathbf{x}) \in \mathbb{R}^m$ so the range of the function is a subset of $\mathbb{R}^m$.

Note that because f has domain in $\mathbb{R}^n$, it is a function of n variables, $x_1, x_2, \ldots, x_n$. That the range of the function f is in $\mathbb{R}^m$ implies that the function value has m components.

Example 11.1. Consider a real-value function of <u>one</u> variable $f : \mathbb{R} \mapsto \mathbb{R}$, see Fig. 11.2. This is the most simple situation as both $n = 1$ and $m = 1$.

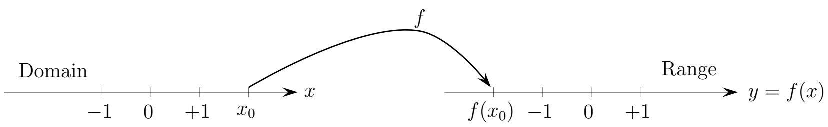

Figure 11.2: A real-valued function $f : \mathbb{R} \mapsto \mathbb{R}$. Both the domain and the range are subsets of the real numbers $\mathbb{R}$ Shown explicitly is the mapping of x_0 onto $f(x_0)$.

Such a function can be visualised graphically in the two-dimensional plane with the x-axis as the domain and the y-axis as the range, see Fig. 11.3

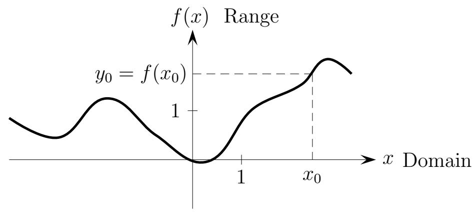

Figure 11.3: A real-value function $f : \mathbb{R} \mapsto \mathbb{R}$ can be represented graphically in the plane with the x-axis representing the domain of the function and the y-axis the range of the function. The graph of the function: $\{(x, y) \in \mathbb{R}^2 : y = f(x)\}$. Shown explicitly is the point $(x_0, f(x_0))$ on the graph.

Example 11.2. Consider a real-value function of <u>two</u> variables $f : \mathbb{R}^2 \mapsto \mathbb{R}$ and let $f(x, y) = z$. Such a function can be visualised in $\mathbb{R}^3$ with the domain of the function as the $x - y$ plane and the range $z = f(x, y)$ along the z-axis, see Fig. 11.4

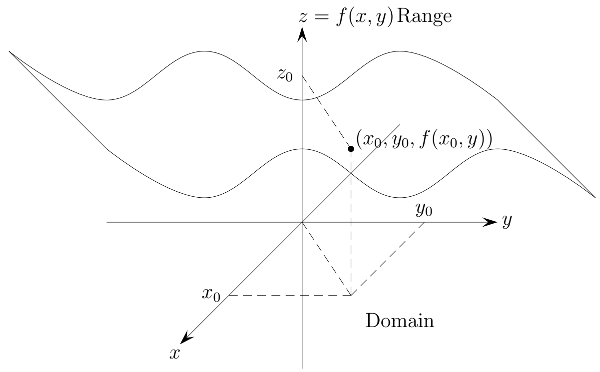

Figure 11.4: A real-value function $f : \mathbb{R}^2 \mapsto \mathbb{R}$ can be represented graphically in $\mathbb{R}^3$ with the $x - y$-plane representing the domain of the function and the z-axis the range of the function. The graph of the function: $\{(x, y, z) \in \mathbb{R}^3 : z = f(x, y)\}$. Shown explicitly is the point $(x_0, y_0, f(x_0, y_0))$ on the graph.

Example 11.3. Consider a function whose domain is $\mathbb{R}^3$ and whose values are vectors in $\mathbb{R}^2$, that is, $f : \mathbb{R}^3 \mapsto \mathbb{R}^2$, for example

$$y_1 = 2x_1 + 3x_2 + 4x_3 \tag{11.1a}$$
$$y_2 = x_1 - x_2 + 2x_3, \tag{11.1b}$$

which to every $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$ assigns a vector $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$. Indeed, we may write Eqs.

(11.1) on the matrix form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 4 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Leftrightarrow \mathbf{y} = \mathbf{B}\mathbf{x} \tag{11.2}$$

where $\mathbf{B}$ is a 2×3 matrix. Figure 11.5 is a schematic representation of such a function.

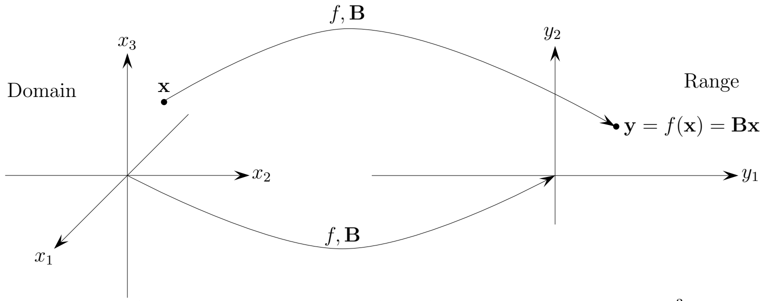

Figure 11.5: Schematic representation of the function whose domain is $\mathbb{R}^3$ and whose values is a vector in $\mathbb{R}^2$, that is, $f : \mathbb{R}^3 \mapsto \mathbb{R}^2$. For every $\mathbf{x} \in \mathbb{R}^3$ there exists a point $\mathbf{y} \in \mathbb{R}^2$ with $\mathbf{y} = \mathbf{B}\mathbf{x}$. For example, $x_1 = x_2 = x_3 = 1$ yields $(9, 2)$ while $x_1 = x_2 = x_3 = 0$ yields $(0, 0)$.

Clearly, we cannot represent graphically in a true way a function $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ when $n > 3$ or $m > 3$. In such cases, we rely on a schematic representation as displayed in Fig. 11.1.

11.2 Linear function $f : \mathbb{R}^n \mapsto \mathbb{R}^m$

Definition 11.2. A function f with domain in $\mathbb{R}^n$ and range in $\mathbb{R}^m$, that is, $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a linear function if

$$f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y}) \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, \tag{11.3a}$$

$$f(r\mathbf{x}) = r f(\mathbf{x}) \quad \forall r \in \mathbb{R}, \forall \mathbf{x} \in \mathbb{R}^n. \tag{11.3b}$$

Observation: A linear function $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ satisfies that

$$\begin{aligned} f(\mathbf{0}) &= f(\mathbf{x} - \mathbf{x}) \\ &= f(\mathbf{x} + (-1)\mathbf{x}) \\ &= f(\mathbf{x}) + f((-1)\mathbf{x}) && \text{using property (11.3a)} \\ &= f(\mathbf{x}) - f(\mathbf{x}) && \text{using property (11.3b)} \\ &= \mathbf{0}. \end{aligned} \tag{11.4}$$

Hence, for a linear function $f(\mathbf{0}) = \mathbf{0}$, that is, it must map $\mathbf{0}$ onto $\mathbf{0}$. Therefore, the real-valued function $f : \mathbb{R} \mapsto \mathbb{R}$ defined by $f(x) = ax + b$ is not, strictly speaking, a linear function (unless $b = 0$) because $f(0) = b$. When $b \neq 0$, it is an *affine* function which is a linear function followed by a translation:

Definition 11.3.* A function $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ is an *affine function* if it is the composition of a linear function $g : \mathbb{R}^n \mapsto \mathbb{R}^m$ followed by a translation $t_{\mathbf{b}} : \mathbb{R}^m \mapsto \mathbb{R}^m$ which simply add the vector $\mathbf{b}$, that is,

$$f = t_{\mathbf{b}} \circ g : \mathbb{R}^n \xrightarrow{g} \mathbb{R}^m \xrightarrow{t_{\mathbf{b}}} \mathbb{R}^m$$

such that for $\mathbf{x} \in \mathbb{R}^n$ we have

$$f(\mathbf{x}) = t_{\mathbf{b}}(g(\mathbf{x})) = g(\mathbf{x}) + \mathbf{b}.$$

11.3 Functions and matrices

Let us consider two functions $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ and $g : \mathbb{R}^n \mapsto \mathbb{R}^n$ defined by the two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ such that

$$f(\mathbf{x}) = \mathbf{A}\mathbf{x} \quad \forall \mathbf{x} \in \mathbb{R}^n \tag{11.5a}$$

$$g(\mathbf{x}) = \mathbf{B}\mathbf{x} \quad \forall \mathbf{x} \in \mathbb{R}^n. \tag{11.5b}$$

Addition: The sum of two functions is well defined and it would be convenient if:

$$\begin{aligned} f(\mathbf{x}) + g(\mathbf{x}) &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{x} \\ &\stackrel{?}{=} (\mathbf{A} + \mathbf{B})\mathbf{x}. \end{aligned} \tag{11.6}$$

Therefore, we need to define the operation of adding two matrices, that is, we need to define the sum of two matrices $\mathbf{A} + \mathbf{B}$.

Numerical multiple: Similarly, the multiplication of a function by a real number $r \in \mathbb{R}$ is well-defined and it would be convenient if:

$$\begin{aligned} r f(\mathbf{x}) &= r(\mathbf{A}\mathbf{x}) \\ &\stackrel{?}{=} (r\mathbf{A})\mathbf{x}. \end{aligned} \tag{11.7}$$

Therefore, we need to define the operation of numerical multiple of a matrix, that is, we need to define what we mean by $r\mathbf{A}$.

Composition: Finally, the composition of two functions f and g is well-defined and it would be convenient if:

$$\begin{aligned} f \circ g(\mathbf{x}) &= f(g(\mathbf{x})) \\ &= \mathbf{A}(\mathbf{B}\mathbf{x}) \\ &\stackrel{?}{=} (\mathbf{A}\mathbf{B})\mathbf{x}. \end{aligned} \tag{11.8}$$

Therefore, we need to define the operation of matrix multiplication, that is, $\mathbf{A}\mathbf{B}$ such that Eq. (11.8) is fulfilled.

11.4 Operations with matrices

Sum of two matrices $\mathbf{A}+\mathbf{B}$

Two matrices can be added if and only if they have the same shape.

Definition 11.4. If $\mathbf{A}$ is an $m\times n$ matrix with elements a_{ij} and $\mathbf{B}$ is an $m\times n$ matrix with elements b_{ij}, then their *sum* $\mathbf{A}+\mathbf{B}$ is defined as the $m\times n$ matrix with the ijth element $a_{ij}+b_{ij}$, that is, the sum of the corresponding elements in the two matrices:

$$\mathbf{A}+\mathbf{B}=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \cdots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{pmatrix}+\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n}\\ b_{21} & b_{22} & \cdots & b_{2n}\\ \vdots & \vdots & \cdots & \vdots\\ b_{m1} & b_{m2} & \cdots & b_{mn}\end{pmatrix}$$
$$=\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n}\\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n}\\ \vdots & \vdots & \cdots & \vdots\\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn}\end{pmatrix}. \tag{11.9}$$

Example 11.4. If $\mathbf{A}=\begin{pmatrix}2 & 5\\ -1 & 6\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}1 & -1\\ 3 & 9\end{pmatrix}$, the matrices can be added as they have the same shape:

$$\mathbf{A}+\mathbf{B}=\begin{pmatrix}2 & 5\\ -1 & 6\end{pmatrix}+\begin{pmatrix}1 & -1\\ 3 & 9\end{pmatrix}=\begin{pmatrix}2+1 & 5+(-1)\\ -1+3 & 6+9\end{pmatrix}=\begin{pmatrix}3 & 4\\ 2 & 15\end{pmatrix}. \tag{11.10}$$

Numerical multiple $r\mathbf{A}$

Definition 11.5. If $\mathbf{A}$ is an $m\times n$ matrix with elements a_{ij} and $r\in\mathbb{R}$ is a real number, then the *numerical multiple* $r\mathbf{A}$ is defined as the $m\times n$ matrix with the ijth elements ra_{ij}, that is, simply multiply all elements in the matrix with r:

$$r\mathbf{A}=r\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \cdots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{pmatrix}=\begin{pmatrix} ra_{11} & ra_{12} & \cdots & ra_{1n}\\ ra_{21} & ra_{22} & \cdots & ra_{2n}\\ \vdots & \vdots & \cdots & \vdots\\ ra_{m1} & ra_{m2} & \cdots & ra_{mn}\end{pmatrix}. \tag{11.11}$$

Convention: Note that we write $-\mathbf{A}$ for the multiple $(-1)\mathbf{A}$ and we write $\mathbf{A}-\mathbf{B}$ as an abbreviation for $\mathbf{A}+(-1)\mathbf{B}$.

Example 11.5. If $\mathbf{A}=\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ -1 & -2 & -3\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}2 & 4 & 6\\ -1 & 3 & 5\\ 4 & -2 & -3\end{pmatrix}$ we find

$$3\mathbf{A}=3\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ -1 & -2 & -3\end{pmatrix}=\begin{pmatrix}3\cdot 1 & 3\cdot 2 & 3\cdot 3\\ 3\cdot 4 & 3\cdot 5 & 3\cdot 6\\ 3\cdot(-1) & 3\cdot(-2) & 3\cdot(-3)\end{pmatrix}=\begin{pmatrix}3 & 6 & 9\\ 12 & 15 & 18\\ -3 & -6 & -9\end{pmatrix} \tag{11.12}$$

and

$$\mathbf{A}-\mathbf{B}=\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ -1 & -2 & -3\end{pmatrix}-\begin{pmatrix}2 & 4 & 6\\ -1 & 3 & 5\\ 4 & -2 & -3\end{pmatrix}=\begin{pmatrix}-1 & -2 & -3\\ 5 & 2 & 1\\ -5 & 0 & 0\end{pmatrix} \tag{11.13a}$$

$$3\mathbf{A}+\mathbf{A}=\begin{pmatrix}3 & 6 & 9\\ 12 & 15 & 18\\ -3 & -6 & -9\end{pmatrix}+\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ -1 & -2 & -3\end{pmatrix}=\begin{pmatrix}4 & 8 & 12\\ 16 & 20 & 24\\ -4 & -8 & -12\end{pmatrix}=4\mathbf{A} \tag{11.13b}$$

Properties of matrix addition and numerical multiple

Let $\mathbf{0}$ denote the $m\times n$ null-matrix, that is, the matrix where all the entries are zero:

$$\mathbf{0}=\begin{pmatrix}0 & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \cdots & \\ 0 & 0 & 0 & \cdots & 0\end{pmatrix}. \tag{11.14}$$

The following formulas hold for any $m\times n$ matrices $\mathbf{A},\mathbf{B},\mathbf{C}$ and arbitrary real numbers $r,s\in\mathbb{R}$:

1. $r\mathbf{A}+s\mathbf{A}=(r+s)\mathbf{A}$ — Distributive law for numerical multiple
2. $r\mathbf{A}+r\mathbf{B}=r\,(\mathbf{A}+\mathbf{B})$ — Distributive law for numerical multiple
3. $r\,(s\mathbf{A})=(rs)\,\mathbf{A}$ — Associative law for numerical multiple
4. $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$ — Commutative law for addition.
5. $(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$ — Associative law for addition.
6. $\mathbf{A}+\mathbf{0}=\mathbf{A}$ — Neutral element for addition.
7. $\mathbf{A}+(-\mathbf{A})=\mathbf{0}$ — Inverse element for addition.

These laws for the operations of addition and numerical multiple with matrices are analogous to the familiar laws for handling real numbers and can easily be proven. We also note that these laws are equivalent to the laws for handling vectors and we see that the $m\times n$ matrices forms a vector space, see definition 2.10 on page 11. Indeed, you may think of matrices as a generalisation of vectors. Vectors are special matrices, either column vectors $n\times 1$ matrices or row vectors $1\times n$ matrices.

Matrix multiplication AB

Definition 11.6. Two matrices **A** and **B** can be multiplied if and only if the number of columns in **A** equals the number of rows in **B**. If **A** is an $m \times p$ matrix with elements a_{ij} and **B** is an $p \times n$ matrix with elements b_{ij}, then the *matrix product* **AB** is defined as the $m \times n$ matrix with the ijth element $\sum_{k=1}^{p} a_{ik} b_{kj}$, that is, the sum of the product of the elements from the ith row in **A** with the elements from the jth column in **B**:

$$\begin{aligned}\mathbf{AB} &= \begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2p} \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mp} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pn} \end{pmatrix} \\ &= \begin{pmatrix} \sum_{k=1}^{p} a_{1k} b_{k1} & \sum_{k=1}^{p} a_{1k} b_{k2} & \cdots & \sum_{k=1}^{p} a_{1k} b_{kn} \\ \sum_{k=1}^{p} a_{2k} b_{k1} & \sum_{k=1}^{p} a_{2k} b_{k2} & \cdots & \sum_{k=1}^{p} a_{2k} b_{kn} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=1}^{p} a_{mk} b_{k1} & \sum_{k=1}^{p} a_{mk} b_{k2} & \cdots & \sum_{k=1}^{p} a_{mk} b_{kn} \end{pmatrix}. \end{aligned} \tag{11.15}$$

Schematically, the entries in a matrix product are found by the mechanism illustrated below. The process of matrix multiplication is sometimes called row-by-column multiplication for matrices.

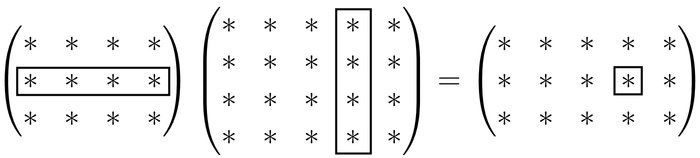

Figure 11.6: Schematic representation of matrix multiplication. The ijth entry in the matrix product **AB** is the sum of the product of the elements from the ith row of **A** and the jth column of **B**. Show in the diagram is the second row of **A** and the 4th column of **B** that enters into the 24th element of the matrix product shown on the right-hand side.

Matrix multiplication is, in general, non-commutative, that is

$$\mathbf{AB} \neq \mathbf{BA}. \tag{11.16}$$

Indeed, it is possible that two matrices can be multiplied in one order but not the other. For example, if **A** is an $m \times p$ matrix and **B** is an $p \times n$ matrix, the product **AB** is well defined and it is an $m \times n$ matrix. The matrix product **BA** of the $p \times n$ and $m \times p$ matrix would only be well-defined if $n = m$. In this case, **BA** would be a $p \times p$ matrix which is of another shape than **AB** that is an $m \times n$, unless $m = n = p$. However, even for square matrices where the shape of **AB** is the same as for **BA**, in general $\mathbf{AB} \neq \mathbf{BA}$.

Example 11.6. Consider the 2×2 matrices $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 5 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix}$ and the 2×3 matrix $\mathbf{C} = \begin{pmatrix} 2 & 3 & 4 \\ 1 & -1 & 2 \end{pmatrix}$. Then the matrix products **AB** and **BA** are well-defined and they are 2×2 matrix. Likewise, the matrix product **AC** is well-defined and it is an 2×3 matrix. We find

$$\mathbf{AB} = \begin{pmatrix} 3 & -1 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 2 & 16 \end{pmatrix}, \tag{11.17a}$$

$$\mathbf{BA} = \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 10 & 4 \\ 18 & 5 \end{pmatrix}, \tag{11.17b}$$

$$\mathbf{AC} = \begin{pmatrix} 3 & -1 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 10 & 10 \\ 12 & 13 & 24 \end{pmatrix}. \tag{11.17c}$$

Properties of matrix multiplication

Let **I** denote the $n \times n$ identity-matrix, that is, the matrix where all the entries are zero except at the main diagonal where the entries are 1:

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} = \operatorname{diag}(1, 1, 1, \ldots, 1). \tag{11.18}$$

The following properties hold for any **A**, **B**, **C** matrices for which the indicated matrix operations are defined and for arbitrary $r \in \mathbb{R}$:

1. $(\mathbf{A} + \mathbf{B})\,\mathbf{C} = \mathbf{AC} + \mathbf{BC}$ — Matrix multiplication is distributive over addition
2. $\mathbf{A}\,(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ — Matrix multiplication is distributive over addition
3. $(r\mathbf{A})\,\mathbf{B} = r\,(\mathbf{AB}) = \mathbf{A}\,(r\mathbf{B})$
4. $\mathbf{A}\,(\mathbf{BC}) = (\mathbf{AB})\,\mathbf{C}$ — Matrix multiplication is associative
5. $\mathbf{0A} = \mathbf{A0} = \mathbf{0}$ — Zero's element for matrix multiplication
6. $\mathbf{IA} = \mathbf{AI} = \mathbf{A}$ — Neutral element for matrix multiplication

Proof of property 2*: Let **A** be an $p \times q$ matrix, **B** be an $q \times r$ matrix, and **C** be an $q \times r$ matrix. Then **AB** is an $p \times r$ matrix and **AC** an $p \times r$ We calculate the ijth entry on the left-hand side:

$$\begin{aligned}
(\mathbf{A}(\mathbf{B}+\mathbf{C}))_{ij} &= \sum_{k=1}^{q} a_{ik}(\mathbf{B}+\mathbf{C})_{kj} \\
&= \sum_{k=1}^{q} a_{ik}(b_{kj}+c_{kj}) \\
&= \sum_{k=1}^{q} (a_{ik}b_{kj}+a_{ik}c_{kj}) \\
&= \sum_{k=1}^{q} a_{ik}b_{kj} + \sum_{k=1}^{q} a_{ik}c_{kj} \\
&= (\mathbf{AB})_{ij} + (\mathbf{AC})_{ij} \qquad (11.19)
\end{aligned}$$

Q.E.D.

Proof of property 4*: Let $\mathbf{A}$ be an $p \times q$ matrix, $\mathbf{B}$ be an $q \times r$ matrix, and $\mathbf{C}$ be an $r \times s$ matrix. Then $\mathbf{AB}$ is an $p \times r$ matrix and $\mathbf{BC}$ an $q \times s$ matrix. We calculate the ijth entry on both side of the equation.

$$\begin{aligned}
(\mathbf{A}(\mathbf{BC}))_{ij} &= \sum_{k=1}^{q} a_{ik}(\mathbf{BC})_{kj} \\
&= \sum_{k=1}^{q} a_{ik} \sum_{l=1}^{r} b_{kl}c_{lj} \qquad (11.20)
\end{aligned}$$

$$\begin{aligned}
((\mathbf{AB})\mathbf{C})_{ij} &= \sum_{l=1}^{r} (\mathbf{AB})_{il}c_{lj} \\
&= \sum_{l=1}^{r}\sum_{k=1}^{q} a_{ik}b_{kl}c_{lj}. \qquad (11.21)
\end{aligned}$$

By comparison, these two term are identical.

Q.E.D.

11.5 Summary

After studying Sec. 11, you should

- be able to explain what is meant by a function $f : \mathbb{R}^n \to \mathbb{R}^m$
- be able to explain how they are represented graphically in a coordinate system for $n = 1, 2$ and $m = 1$ and schematically for any other n and m.
- know what is meant by a linear function $f : \mathbb{R}^n \to \mathbb{R}^m$.
- know that for a linear function $f(\mathbf{0}) = \mathbf{0}$
- be able to explain when addition and matrix multiplication is defined between two matrices
- be able to add two matrices and multiply a matrix by a real number
- know the properties of matrix addition and numerical multiplication
- be able to multiply an $m \times p$ matrix $\mathbf{A}$ with an $p \times n$ matrix $\mathbf{B}$ to obtain the $m \times n$ matrix $\mathbf{AB}$
- $\mathbf{A} : \mathbb{R}^p \to \mathbb{R}^m$; $\mathbf{B} : \mathbb{R}^n \to \mathbb{R}^p$; $\mathbb{R}^n \xrightarrow{\mathbf{B}} \mathbb{R}^p \xrightarrow{\mathbf{A}} \mathbb{R}^m$, that is $\mathbf{AB} : \mathbb{R}^n \to \mathbb{R}^m$.
- know the properties of matrix multiplication

12 Linear functions and matrices: II

The aim is to discuss linear functions from $\mathbb{R}^n$ to $\mathbb{R}^m$. First, we will show that a linear function maps a straight line onto a straight line. Then we will show that for every linear function $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ there exists an associated $m \times n$ matrix $\mathbf{A} = (f(\mathbf{e}_1), f(\mathbf{e}_2), \ldots, f(\mathbf{e}_n))$ where $\mathbf{e}_j$ is the jth natural basis vector of $\mathbb{R}^n$ such that $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$. We will the apply this theorem to find the associated matrices for various linear functions from $\mathbb{R}^n$ to $\mathbb{R}^n$:

- identity function in $\mathbb{R}^n$
- simple scaling (expansion and contraction) in $\mathbb{R}^n$
- rotation about the origin in $\mathbb{R}^2$
- rotation about a positive axis in $\mathbb{R}^3$.

12.1 Introduction

We recall that a linear function f with domain in $\mathbb{R}^n$ and range in $\mathbb{R}^m$, that is, $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ satisfies

$$f(\mathbf{x}+\mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y}) \quad \forall\, \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, \tag{12.1a}$$
$$f(r\mathbf{x}) = r f(\mathbf{x}) \quad \forall\, r \in \mathbb{R}, \forall\, \mathbf{x} \in \mathbb{R}^n, \tag{12.1b}$$

see Definition 11.2 on page 97.

Before we state the central Theorem of Sec. 12, we make the following observation, that will be useful later on.

Theorem 12.1. A linear function $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ maps a straight line in $\mathbb{R}^n$ onto a straight line in $\mathbb{R}^m$ if the direction vector for the line in $\mathbb{R}^n$ is not in the null-space for f.

Proof: The parametric vector equation for a straight line in $\mathbb{R}^n$ through the point $\mathbf{x}_0 \in \mathbb{R}^n$ along the direction vector $\mathbf{x}_1 \in \mathbb{R}^n$ is given by

$$\mathbf{x} = \mathbf{x}_0 + \lambda \mathbf{x}_1, \quad \text{where } \lambda \in \mathbb{R}. \tag{12.2}$$

Hence, the mapping of this straight line under the linear function f:

$$\begin{aligned} f(\mathbf{x}) &= f(\mathbf{x}_0 + \lambda\mathbf{x}_1) \\ &= f(\mathbf{x}_0) + f(\lambda\mathbf{x}_1) && \text{using property (12.1a) of a linear function} \\ &= f(\mathbf{x}_0) + \lambda f(\mathbf{x}_1) && \text{using property (12.1b) of a linear function.} \end{aligned} \tag{12.3}$$

Assuming that $f(\mathbf{x}_1) \neq \mathbf{0}$, this is a parametric vector equation for a straight line in $\mathbb{R}^m$ passing through the point $f(\mathbf{x}_0) \in \mathbb{R}^m$ with direction vector $f(\mathbf{x}_1) \in \mathbb{R}^m$.

Q.E.D.

12.2 Linear function $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ and its associated $m \times n$ matrix A

Theorem 12.2. A function f from $\mathbb{R}^n$ to $\mathbb{R}^m$ is linear if and only if there exists an $m \times n$ matrix $\mathbf{A} : f(\mathbf{x}) = \mathbf{A}\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

Proof: ($\Leftarrow$) We first show that if $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ is defined by $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$, where $\mathbf{A}$ is an $m \times n$ matrix, then f is a linear function. Therefore, let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $r \in \mathbb{R}$. We find

$$\begin{aligned} f(\mathbf{x}+\mathbf{y}) &= \mathbf{A}(\mathbf{x}+\mathbf{y}) && \text{definition of } f \\ &= \mathbf{A}\mathbf{x} + \mathbf{A}\mathbf{y} && \text{property 2 (page 102) for matrix multiplication} \\ &= f(\mathbf{x}) + f(\mathbf{y}) && \text{definition of } f \end{aligned} \tag{12.4a}$$

and

$$\begin{aligned} f(r\mathbf{x}) &= \mathbf{A}(r\mathbf{x}) && \text{definition of } f \\ &= r(\mathbf{A}\mathbf{x}) && \text{property 3 (page 102) for matrix multiplication} \\ &= r f(\mathbf{x}) && \text{definition of } f. \end{aligned} \tag{12.4b}$$

($\Rightarrow$) We now assume that f is a linear function from $\mathbb{R}^n$ to $\mathbb{R}^m$. Let $\mathbf{e}_j, j = 1, 2, \ldots, n$ denote the jth natural basis vector of $\mathbb{R}^n$, that is, the column vector that has 1 at entry j and zero otherwise. Let $\mathbf{x} \in \mathbb{R}^n$ denote any vector. Then we can express the vector $\mathbf{x}$ in terms of the natural basis vectors:

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n, \quad \forall\, \mathbf{x} \in \mathbb{R}^n. \tag{12.5}$$

Because f is a linear function, we find

$$\begin{aligned} f(\mathbf{x}) &= f(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n) \\ &= f(x_1\mathbf{e}_1) + f(x_2\mathbf{e}_2) + \cdots + f(x_n\mathbf{e}_n) && \text{using property (12.1a) of a linear fct.} \\ &= x_1 f(\mathbf{e}_1) + x_2 f(\mathbf{e}_2) + \cdots + x_n f(\mathbf{e}_n) && \text{using property (12.1b) of a linear fct.} \end{aligned} \tag{12.6}$$

We note that $f(\mathbf{e}_j) \in \mathbb{R}^m$ is an m-dimensional column vector. If we now define

$$\mathbf{a}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} \stackrel{def}{=} f(\mathbf{e}_j) \in \mathbb{R}^m \quad \text{for } j = 1, 2, \ldots, n, \tag{12.7}$$

we find

$$\begin{aligned} f(\mathbf{x}) &= x_1 f(\mathbf{e}_1) + x_2 f(\mathbf{e}_2) + \cdots + x_n f(\mathbf{e}_n) \\ &= x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n \\ &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= \mathbf{A}\mathbf{x} \end{aligned} \tag{12.8}$$

Hence, we have shown that if $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a linear function, then the associated $m \times n$ matrix whose jth column is $f(\mathbf{e}_j)$ yields $f(\mathbf{x}) = \mathbf{A}\mathbf{x} \quad \forall \mathbf{x} \in \mathbb{R}^n$.

Q.E.D.

This is a remarkable theorem! It states that the vector space of all linear functions $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ is one-to-one with the vector space of $m \times n$ matrices: Each linear function has an associated matrix and vice versa. For a given linear function f, Theorem 12.2 even explains how to construct the matrix $\mathbf{A}$. Find $f(\mathbf{e}_j)$ for $j = 1, 2, \ldots, n$ and use these as the column vectors to build the associated $m \times n$ matrix $\mathbf{A}$.

12.3 Application of Theorem 12.2 for linear functions from $\mathbb{R}^n$ to $\mathbb{R}^n$

In Sec. 12.3, we are going to apply Theorem 12.2 to find the associated matrices for a selection of linear functions from $\mathbb{R}^n$ to $\mathbb{R}^n$. We will start with the simple linear functions of the identity and simple scaling in $\mathbb{R}^n$ and then look at rotations in $\mathbb{R}^2$ and $\mathbb{R}^3$.

The identity function

Definition 12.1. The function $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ for which

$$f(\mathbf{x}) = \mathbf{x} \quad \forall \mathbf{x} \in \mathbb{R}^n \tag{12.9}$$

is called the *identity function*.

Clearly, the identity function is a linear function.

$$f(\mathbf{x}+\mathbf{y}) = \mathbf{x}+\mathbf{y} = f(\mathbf{x}) + f(\mathbf{y}) \quad \forall \mathbf{x},\mathbf{y} \in \mathbb{R}^n, \tag{12.10a}$$

$$f(r\mathbf{x}) = r\mathbf{x} = rf(\mathbf{x}) \quad \forall r \in \mathbb{R}, \forall \mathbf{x} \in \mathbb{R}^n. \tag{12.10b}$$

To construct the matrix associated with the identity function, we need to consider how f is mapping the natural basis vectors $\mathbf{e}_j \in \mathbb{R}^n, j = 1, 2, \ldots, n$. That is easy because $f(\mathbf{e}_j) = \mathbf{e}_j$, see Fig. 12.1.

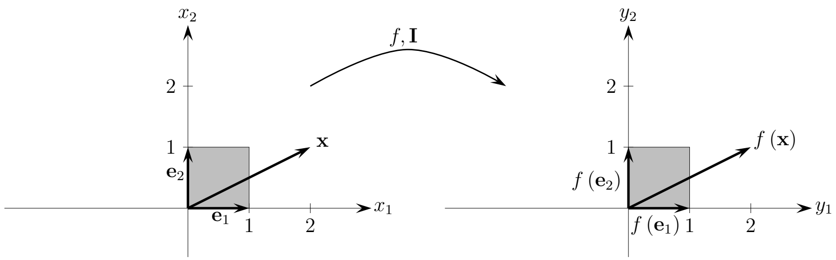

Figure 12.1: The identity function $f(\mathbf{x}) = \mathbf{x}$ from $\mathbb{R}^2$ to $\mathbb{R}^2$. We notice that $f(\mathbf{e}_1) = \mathbf{e}_1$ and $f(\mathbf{e}_2) = \mathbf{e}_2$ so the 2×2 matrix associated with the identity function has columns $\mathbf{e}_1$ and $\mathbf{e}_2$. Notice that this implies $f(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) = x_1 f(\mathbf{e}_1) + x_2 f(\mathbf{e}_2) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$.

Hence, the associated $n \times n$ matrix for the identity function:

$$\mathbf{I} = \begin{pmatrix} f(\mathbf{e}_1) & f(\mathbf{e}_2) & \ldots & f(\mathbf{e}_n) \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \ldots & \mathbf{e}_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{12.11}$$

Simple scaling – expansion and contraction – in $\mathbb{R}^n$

Say that we want to design a function $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ such that it contracts the scale along the x-axis by a factor of 2 and expands the scale along the y-axis by a factor of 4, see Fig. 12.2

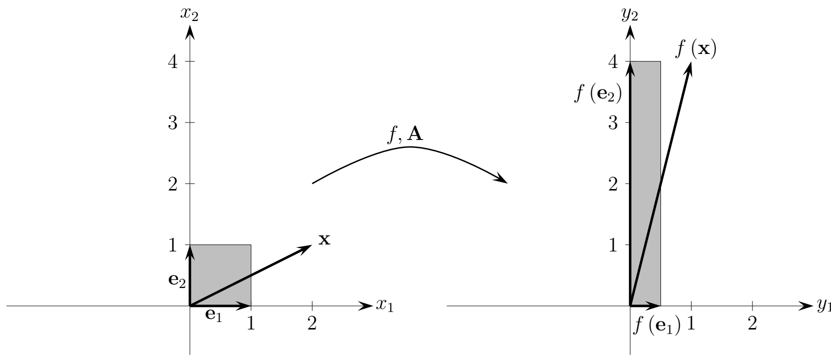

Figure 12.2: The function $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ that contracts the scale along the x-axis by a factor 2 and expand the scale along the y-axis by a factor 4. We notice that $f(\mathbf{e}_1) = \frac{1}{2}\mathbf{e}_1$ and $f(\mathbf{e}_2) = 4\mathbf{e}_2$ so the 2×2 matrix associated with simple scaling has columns $\frac{1}{2}\mathbf{e}_1$ and $4\mathbf{e}_2$. Notice that this implies $f(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) = x_1 f(\mathbf{e}_1) + x_2 f(\mathbf{e}_2) = \frac{1}{2}x_1\mathbf{e}_1 + 4x_2\mathbf{e}_2$. Shown explicitly is that $(2, 1)$ maps onto $(1, 4)$.

We find the mapping of the two natural basis vectors in $\mathbb{R}^2$ (see Fig. 12.2):

$$f(\mathbf{e}_1) = \frac{1}{2}\mathbf{e}_1 = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \tag{12.12a}$$

$$f(\mathbf{e}_2) = 4\mathbf{e}_2 = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \tag{12.12b}$$

such that the associated matrix is

$$\mathbf{A} = \begin{pmatrix} f(\mathbf{e}_1) & f(\mathbf{e}_2) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 4 \end{pmatrix}. \tag{12.13}$$

Notice that the unit square in the domain is mapped onto a rectangle with area 2, so when applying the linear function whose associated matrix is given by Eq. (12.13), the area is multiplied by $\det\mathbf{A} = \frac{1}{2} \cdot 4 - 0 = 2$.

In general, the $n \times n$ diagonal matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} = \operatorname{diag}(a_{11}, a_{22}, a_{33}, \ldots, a_{nn}) \tag{12.14}$$

describes the function which scales the natural basis vector $\mathbf{e}_j$ with the factor a_{jj}. If $a_{jj} > 1$, the scale is expanded. If $a_{jj} < 1$, the scale is contracted. If a_{jj} the scale is left unchanged. Notice that the determinant

$$\det\mathbf{A} = a_{11}a_{22}a_{33} \cdots a_{nn} = \Pi_{j=1}^{n} a_{jj}. \tag{12.15}$$

Rotation about the origin $\mathcal{O}$ in $\mathbb{R}^2$

Definition 12.2. We define the *positive direction* of rotation about the origin in $\mathbb{R}^2$ as anti-clockwise (counter-clockwise) and hence the *negative direction* of rotation about the origin in $\mathbb{R}^2$ as clockwise.

Consider a vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$. What are the coordinates of this point after a rotation about the origin by some angle θ?

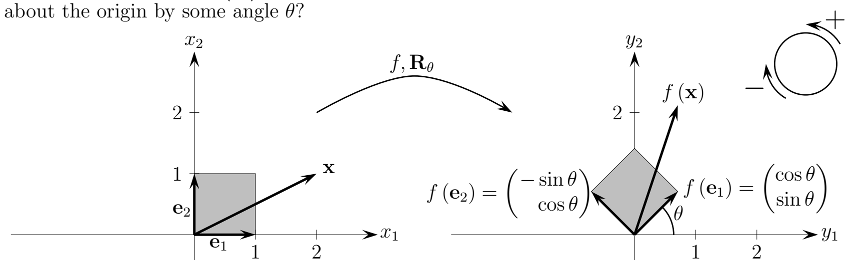

Figure 12.3: Rotation about the origin $\mathcal{O}$ in $\mathbb{R}^2$ by an angle θ. We notice that $f(\mathbf{e}_1) = (\cos\theta, \sin\theta)$ and $f(\mathbf{e}_2) = \left(\cos(\frac{\pi}{2} + \theta), \sin(\frac{\pi}{2} + \theta)\right) = (-\sin\theta, \cos\theta)$. The diagram in the upper right corner is designed to show that the positive direction of rotation about the origin is anti-clock-wise while the negative direction is clockwise. Shown explicitly is that $(2, 1)$ maps onto $\left(\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$.

Clearly, this is also a linear function. Rotating the resultant sum of two vectors yield the same results as rotating the the vectors first and then adding them. Also, rotating the vector $r\mathbf{x}$ yields the same result as rotating $\mathbf{x}$ and then multiplying with the factor r. Therefore, we may find the 2×2 matrix associated with this linear function and then apply that matrix on $\mathbf{x}$.

We find the mapping of the two natural basis vectors in $\mathbb{R}^2$. From Fig. 12.3, we identify

$$f(\mathbf{e}_1) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{12.16a}$$

$$f(\mathbf{e}_2) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \tag{12.16b}$$

such that the associated matrix is

$$\mathbf{R}_\theta = \begin{pmatrix} f(\mathbf{e}_1) & f(\mathbf{e}_2) \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{12.17}$$

Hence, the matrix $\mathbf{R}_\theta$ given by Eq. (12.17) represents the rotation of a vector about the origin by an angle θ where $\theta > 0$ is an anti-clockwise rotation. For example, say you want to rotate the vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ by an angle θ into the vector $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$, the coordinates of this vector can be found by applying the matrix $\mathbf{R}_\theta$:

$$\mathbf{y} = \mathbf{R}_\theta\mathbf{x} \Leftrightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta \end{pmatrix}. \tag{12.18}$$

An example is shown in Fig. 12.3 with $\theta = \frac{\pi}{4}$ and $\mathbf{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ yielding

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} \end{pmatrix} \approx \begin{pmatrix} 0.707 \\ 2.121 \end{pmatrix}. \tag{12.19}$$

Notice that the unit square in the domain is mapped onto a square with area 1, so when applying the linear function whose associated matrix is given by Eq. (12.17), the area is multiplied by $\det\mathbf{R}_\theta = \cos^2\theta + \sin^2\theta = 1$.

Rotation about an axis in $\mathbb{R}^3$

In three dimensions, we may consider a rotation about the positive x_3-axis by an angle θ, see Fig. 12.4. We find the mapping of the three natural basis vectors in $\mathbb{R}^3$. From Fig. 12.4, we identify

$$f(\mathbf{e}_1) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \tag{12.20a}$$

$$f(\mathbf{e}_2) = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} \tag{12.20b}$$

$$f(\mathbf{e}_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{12.20c}$$

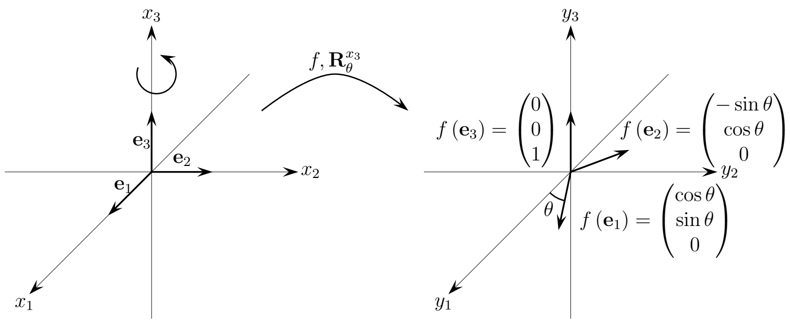

Figure 12.4: An anti-clockwise rotation by an angle θ about the positive x_3-axis in a right-handed coordinate system.

such that the associated matrix is

$$\mathbf{R}_\theta^{x_3} = \begin{pmatrix} f(\mathbf{e}_1) & f(\mathbf{e}_2) & f(\mathbf{e}_3) \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{12.21}$$

Hence, the matrix $\mathbf{R}_\theta^{x_3}$ given by Eq. (12.21) represents the rotation of a vector about the positive x_3-axis by an angle θ where $\theta > 0$ is an anti-clockwise rotation. For example, say you want to rotate the vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$ by an angle θ around the positive x_3-axis into the vector $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{R}^3$, the coordinates of this vector can be found by applying the matrix $\mathbf{R}_\theta^{x_3}$:

$$\mathbf{y} = \mathbf{R}_\theta^{x_3}\mathbf{x} \Leftrightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta \\ x_3 \end{pmatrix}. \tag{12.22}$$

Notice that also this mapping conserves the volume because $\det\mathbf{R}_\theta^{x_3} = \cos^2\theta + \sin^2\theta = 1$.

Example 12.1. One might also consider anti-clockwise rotations by an angle θ about the positive x_1-axis in $\mathbb{R}^3$ and about the positive x_2-axis in $\mathbb{R}^3$.

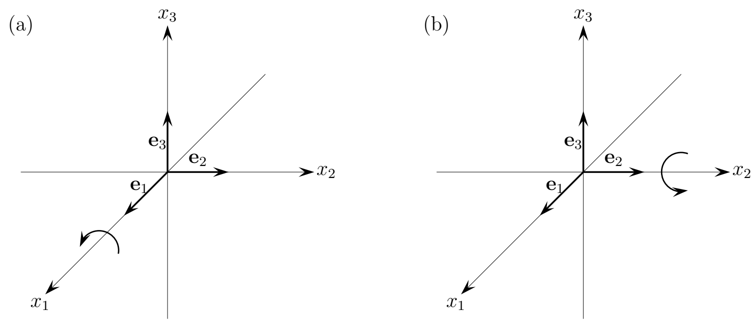

Figure 12.5: (a) An anti-clockwise rotation about the positive x_1-axis in a right-handed coordinate system. (b) An anti-clockwise rotation about the positive x_2-axis in a right-handed coordinate system.

It is left as an exercise to the reader to show

$$\mathbf{R}_\theta^{x_1} = \begin{pmatrix} f(\mathbf{e}_1) & f(\mathbf{e}_2) & f(\mathbf{e}_3) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \tag{12.23a}$$

$$\mathbf{R}_\theta^{x_2} = \begin{pmatrix} f(\mathbf{e}_1) & f(\mathbf{e}_2) & f(\mathbf{e}_3) \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}. \tag{12.23b}$$

Theorem 12.3. A linear function f from $\mathbb{R}^n$ to $\mathbb{R}^n$ with matrix $\mathbf{A}$ multiplies the volumes by the factor $|\det\mathbf{A}|$.

Proof: See classwork 6 ◡̈.

12.4 Summary

After studying Sec. 12, you should

- know that a linear function maps a straight line onto a straight line
- know the theorem that a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear $\Leftrightarrow$ The exists an $m \times n$ matrix $\mathbf{A}$ such that $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = (f(\mathbf{e}_1), \dots, f(\mathbf{e}_n))$
- be able to apply the theorem to find the associated matrices for a various linear functions from $\mathbb{R}^n$ to $\mathbb{R}^n$:
  - simple scaling (expansion and contraction), yielding the diagonal matrix $\text{diag}(a_1, \dots, a_n)$
  - rotation about origin in two dimensions, yielding the 2×2 rotation matrix $\mathbf{R}_\theta$.
  - rotation about axis in three dimensions, yielding the 3×3 rotation matrix $\mathbf{R}_\theta^{x_i}$.

13 Matrices: Compositions, Inverse and Transpose

The aim is to

- discuss the composition of linear functions $g \circ f$ and the associated matrix
- discuss the notion of invertible matrices
- introduce the transpose $\mathbf{A}^t$ of a matrix $\mathbf{A}$
- introduce the matrix of co-factors $\mathbf{C}$ of the matrix $\mathbf{A}$
- give the condition for a matrix $\mathbf{A}$ to be invertible
- introduce and apply a formula to determine the inverse $\mathbf{A}^{-1}$

13.1 Composition of linear functions

Theorem 13.1. Let $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear function with the associated $m \times n$ matrix $\mathbf{A}$ and let $g : \mathbb{R}^m \mapsto \mathbb{R}^p$ be a linear function with the associated $p \times m$ matrix $\mathbf{B}$, that is,

$$f(\mathbf{x}) = \mathbf{A}\mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n \tag{13.1a}$$
$$g(\mathbf{y}) = \mathbf{B}\mathbf{y} \quad \text{for all } \mathbf{y} \in \mathbb{R}^m. \tag{13.1b}$$

Then the composition $g \circ f : \mathbb{R}^n \xrightarrow[\mathbf{A}]{f} \mathbb{R}^m \xrightarrow[\mathbf{B}]{g} \mathbb{R}^p$ is given by the $p \times n$ matrix $\mathbf{BA}$, that is,

$$g \circ f(\mathbf{x}) = (\mathbf{BA})\,\mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n. \tag{13.2}$$

Proof: Because $\mathbf{B}$ is an $p \times m$ matrix and $\mathbf{A}$ is an $m \times n$ matrix, the matrix product $\mathbf{BA}$ is an $(p \times m)(m \times n) = p \times n$ matrix, so it is associated with a linear mapping from $\mathbb{R}^n \mapsto \mathbb{R}^p$. We find

$$g \circ f(\mathbf{x}) = g(f(\mathbf{x})) = g(\mathbf{A}\mathbf{x}) = \mathbf{B}(\mathbf{A}\mathbf{x}) = (\mathbf{BA})\,\mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n. \tag{13.3}$$

Q.E.D.

Example 13.1. Consider a rotation f in $\mathbb{R}^2$ by an angle θ about the origin followed by a scaling g where $\mathbf{e}_1$ is scale by the factor a_{11} and $\mathbf{e}_2$ by the factor a_{22}. The 2×2 matrices associated with these two linear functions are (see Sec. 12.3)

$$\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{13.4a}$$
$$\mathbf{A} = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}, \tag{13.4b}$$

see Fig. 13.1 for an illustration with $\theta = \frac{\pi}{4}$, $a_{11} = 3$ and $a_{22} = \frac{1}{2}$.

The 2×2 matrix for the composition $g \circ f$ of the rotation f followed by the scaling g is, according to Theorem 13.1, given by the matrix product

$$\mathbf{AR}_\theta = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} a_{11}\cos\theta & -a_{11}\sin\theta \\ a_{22}\sin\theta & a_{22}\cos\theta \end{pmatrix} \tag{13.5}$$

with determinant

$$\det(\mathbf{AR}_\theta) = \begin{vmatrix} a_{11}\cos\theta & -a_{11}\sin\theta \\ a_{22}\sin\theta & a_{22}\cos\theta \end{vmatrix} = a_{11}a_{22}\left(\cos^2\theta + \sin^2\theta\right) = \det\mathbf{A}\,\det\mathbf{R}_\theta = a_{11}a_{22}. \tag{13.6}$$

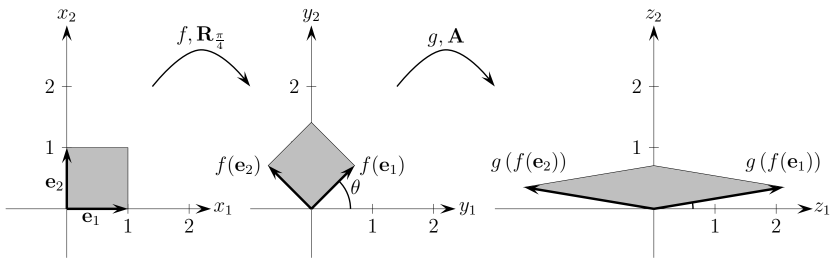

Figure 13.1: Rotation by an angle $\theta = \frac{\pi}{4}$ about the origin $\mathcal{O}$ in $\mathbb{R}^2$, $\mathbf{R}_{\frac{\pi}{4}}$, followed by a scaling, $\mathbf{A}$ expanding by $a_{11} = 3$ along the y_1-axis and contracting by $a_{22} = \frac{1}{2}$ along the y_2-axis. The rotation leaves the area invariant. The scaling multiplies the area by the factor $|a_{11}a_{22}|$. Hence, the unit square in the $x_1 - x_2$ plane transform into a rhombus in the $z_1 - z_2$ plane with area $\frac{3}{2}$.

Inserting $\theta = \frac{\pi}{4}$, $a_{11} = 3$ and $a_{22} = \frac{1}{2}$ into the 2×2 matrix for the composition $g \circ f$ of the rotation f followed by the scaling g we find

$$\mathbf{AR}_{\frac{\pi}{4}} = \begin{pmatrix} \frac{3}{\sqrt{2}} & -\frac{3}{\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{pmatrix} \tag{13.7}$$

so the basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ in the $x_1 - x_2$ plane maps into

$$\begin{pmatrix} \frac{3}{\sqrt{2}} & -\frac{3}{\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \end{pmatrix} \approx \begin{pmatrix} 2.12 \\ 0.35 \end{pmatrix}, \tag{13.8a}$$
$$\begin{pmatrix} \frac{3}{\sqrt{2}} & -\frac{3}{\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{3}{\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \end{pmatrix} \approx \begin{pmatrix} -2.12 \\ 0.35 \end{pmatrix}. \tag{13.8b}$$

Notice that the determinant

$$\det\left(\mathbf{AR}_{\frac{\pi}{4}}\right) = \begin{vmatrix} \frac{3}{\sqrt{2}} & -\frac{3}{\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{vmatrix} = \frac{3}{4} + \frac{3}{4} = \frac{3}{2}. \tag{13.9}$$

13.2 Inverse of a matrix I

As we have seen, the 2×2 matrix

$$\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{13.10}$$

describes a rotation by an angle θ about the origin in $\mathbb{R}^2$.

Let us consider the composition of a rotation by an angle θ about the origin in $\mathbb{R}^2$ followed by another rotation of an angle $-\theta$ about the origin in $\mathbb{R}^2$. Using that the cos is an even function, that is, $\cos(-\theta) = \cos\theta$ and that the sin is an odd function, that is, $\sin(-\theta) = -\sin\theta$, and that $\cos^2\theta + \sin^2\theta = 1$, we find

$$\begin{aligned} \mathbf{R}_{-\theta}\mathbf{R}_\theta &= \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \mathbf{I}. \end{aligned} \tag{13.11}$$

Clearly, this composition is equivalent to the identity function, see Fig. 13.2.

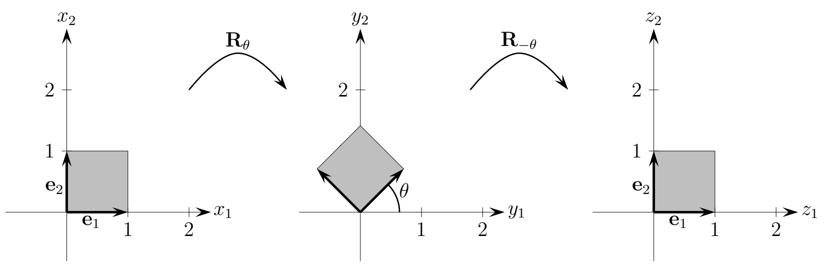

Figure 13.2: Rotation by an angle θ about the origin $\mathcal{O}$ in $\mathbb{R}^2$, $\mathbf{R}_\theta$, followed by another rotation by an angle $-\theta$ about the origin $\mathcal{O}$ in $\mathbb{R}^2$, $\mathbf{R}_{-\theta}$. The composition $\mathbf{R}_{-\theta}\mathbf{R}_\theta : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is clearly the identity function. The diagram uses $\theta = \frac{\pi}{4}$.

A slightly different view is to consider the mapping $\mathbf{R}_{-\theta}$ as a mapping from the $y_1 - y_2$ plane back to the $x_1 - x_2$ plane where the mapping $\mathbf{R}_\theta$ originated, see Fig. 13.3. In this way, we would naturally view the mapping $\mathbf{R}_{-\theta}$ as the inverse mapping of $\mathbf{R}_\theta$.

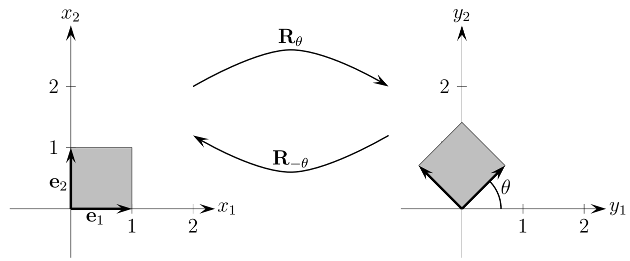

Figure 13.3: Rotation by an angle θ about the origin $\mathcal{O}$ in $\mathbb{R}^2$, $\mathbf{R}_\theta$, followed by another rotation by an angle $-\theta$ about the origin $\mathcal{O}$ in $\mathbb{R}^2$, $\mathbf{R}_{-\theta}$. The composition $\mathbf{R}_{-\theta}\mathbf{R}_\theta : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is clearly the identity function. We may view $\mathbf{R}_{-\theta}$ as the inverse function of $\mathbf{R}_\theta$. The diagram uses $\theta = \frac{\pi}{4}$.

Definition 13.1. Let $\mathbf{A}$ be a square $n \times n$ matrix. If there exists a square $n \times n$ matrix $\mathbf{B}$ such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}, \tag{13.12}$$

where $\mathbf{I}$ denotes the $n \times n$ identity matrix we say that the matrix $\mathbf{A}$ is *invertible* and that $\mathbf{B}$ is the *inverse matrix* of $\mathbf{A}$ and we write

$$\mathbf{B} = \mathbf{A}^{-1}. \tag{13.13}$$

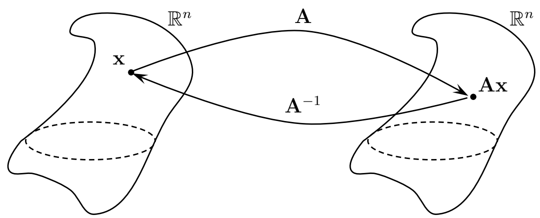

Figure 13.4: Schematic representation of the $n \times n$ matrix $\mathbf{A}$ and the $n \times n$ matrix $\mathbf{A}^{-1}$, the inverse matrix of $\mathbf{A}$: $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

Example 13.2. We identify that

$$\mathbf{R}_\theta^{-1} = \mathbf{R}_{-\theta} \tag{13.14}$$

because we have

$$\mathbf{R}_\theta\mathbf{R}_{-\theta} = \mathbf{R}_{-\theta}\mathbf{R}_\theta = \mathbf{I}. \tag{13.15}$$

Example 13.3. Consider the two 2×2 matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix}. \tag{13.16}$$

We find

$$\mathbf{AB} = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}\begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{13.17a}$$

$$\mathbf{BA} = \begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{13.17b}$$

so we have shown that

$$\mathbf{B} = \mathbf{A}^{-1} = \begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix}. \tag{13.18}$$

Uniqueness of the inverse

For real non-zero numbers, the inverse exists and it is obvious that the inverse is unique. For example, the inverse to 7 is $\frac{1}{7}$. No other real number satisfy that $7 \cdot \frac{1}{7} = 1$. However, it is not obvious that the inverse of a matrix is unique. Therefore, given that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I} \tag{13.19}$$

one may rightfully ask whether $\mathbf{B}$ is the only possible inverse matrix of $\mathbf{A}$? Luckily, the answer is a “Yes”.

Theorem 13.2. A matrix $\mathbf{A}$ has at most one inverse.

Proof: Suppose there are two matrices $\mathbf{B}$ and $\mathbf{C}$ such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}, \tag{13.20a}$$

$$\mathbf{AC} = \mathbf{CA} = \mathbf{I}. \tag{13.20b}$$

Our strategy is to show that these two matrices $\mathbf{B}$ and $\mathbf{C}$ are identical. Because both $\mathbf{AB} = \mathbf{I}$ and $\mathbf{AC} = \mathbf{I}$ we must have that

$$\begin{aligned}
\mathbf{AB} = \mathbf{AC} &\Rightarrow \mathbf{B}(\mathbf{AB}) = \mathbf{B}(\mathbf{AC}) \\
&\Rightarrow (\mathbf{BA})\,\mathbf{B} = (\mathbf{BA})\,\mathbf{C} \quad \text{property 4 (page 102) for matrix multiplication} \\
&\Rightarrow (\mathbf{I})\,\mathbf{B} = (\mathbf{I})\,\mathbf{C} \quad \text{using Eq. (13.20a)} \\
&\Rightarrow \mathbf{IB} = \mathbf{IC} \\
&\Rightarrow \mathbf{B} = \mathbf{C} \quad \text{property 6 (page 102) for matrix multiplication.}
\end{aligned} \tag{13.21}$$

Q.E.D.

13.3 Transpose of a matrix

Definition 13.2. Let $\mathbf{A}$ denote an $m \times n$ matrix with elements a_{ij}. The *transpose* $\mathbf{A}^t$ of the matrix $\mathbf{A}$ is the $n \times m$ matrix obtained by exchanging rows and columns, that is, the ijth entry in the transpose $a^t_{ij} = a_{ji}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \Rightarrow \mathbf{A}^t = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ a_{13} & a_{23} & \cdots & a_{m3} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}. \tag{13.22}$$

Notice that $(\mathbf{A}^t)^t = \mathbf{A}$.

Example 13.4. If $\mathbf{A}$ is a 2×2 matrix, its transpose $\mathbf{A}^t$ is a 2×2 matrix. If $\mathbf{B}$ is a 2×3 matrix, its transpose $\mathbf{B}^t$ is a 3×2 matrix. If $\mathbf{C}$ is a 1×3 matrix, its transpose $\mathbf{C}^t$ is a 3×1 matrix. For example,

$$\mathbf{A} = \begin{pmatrix} 2 & -3 \\ 5 & 7 \end{pmatrix} \Leftrightarrow \mathbf{A}^t = \begin{pmatrix} 2 & 5 \\ -3 & 7 \end{pmatrix}. \tag{13.23a}$$

$$\mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \Leftrightarrow \mathbf{B}^t = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}. \tag{13.23b}$$

$$\mathbf{C} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \Leftrightarrow \mathbf{C}^t = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}. \tag{13.23c}$$

Observation: We can write the scalar product between two vectors in $\mathbb{R}^n$ as a matrix product between two matrices using the concept of a transpose. If

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \quad \text{and} \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n \tag{13.24}$$

are two n-dimensional column vectors, then their scalar product can be expressed as a matrix product, using the notion of the transpose:

$$\begin{aligned}
\mathbf{x} \cdot \mathbf{y} &= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \\
&= \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \qquad (1 \times n)(n \times 1) = 1 \times 1 \\
&= \mathbf{x}^t \mathbf{y}.
\end{aligned} \tag{13.25}$$

13.4 Inverse of a matrix II

We now return to our discussion on square matrices and the concept of an inverse matrix. First, we recall that the determinant of an $n \times n$ matrix $\mathbf{A}$ is defined by (expanded by the ith row)

$$\det \mathbf{A} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det \mathbf{A}_{ij} = \sum_{j=1}^{n} a_{ij} \underbrace{\left((-1)^{i+j} \det \mathbf{A}_{ij}\right)}_{C_{ij}}, \tag{13.26}$$

where $\mathbf{A}_{ij}$ is the ijth minor of the matrix $\mathbf{A}$.

Definition 13.3. Let $\mathbf{A}$ be an $n \times n$ matrix. Then the $n \times n$ matrix $\mathbf{C}$ with elements

$$C_{ij} = (-1)^{i+j} \det \mathbf{A}_{ij} \quad \text{for } i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, n, \tag{13.27}$$

is call the *matrix of co-factors* of the matrix $\mathbf{A}$.

Theorem 13.3. Let $\mathbf{A}$ be an $n \times n$ matrix. If $\det \mathbf{A} \neq 0$ then $\mathbf{A}$ is invertible and

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \mathbf{C}^t \tag{13.28}$$

where $\mathbf{C}$ is the matrix of co-factors of $\mathbf{A}$ and $\mathbf{C}^t$ its transpose.

Theorem 13.3 tell us that an $n \times n$ matrix $\mathbf{A}$ has an inverse when its determinant is non-zero, that is $\det \mathbf{A} \neq 0$. In addition, Theorem 13.3 yields a procedure to find the inverse if it exists. Let us apply Theorem 13.3 on 2×2 matrices and a 3×3 matrix.

Inverse of 2×2 matrices

Let $\mathbf{A}$ be a 2×2

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{13.29}$$

The determinant

$$\det \mathbf{A} = a_{11}a_{22} - a_{21}a_{12}. \tag{13.30}$$

If $\det \mathbf{A} = a_{11}a_{22} - a_{21}a_{12} \neq 0$ then $\mathbf{A}$ is invertible and

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}^t = \frac{1}{\det \mathbf{A}} \begin{pmatrix} +a_{22} & -a_{21} \\ -a_{12} & +a_{11} \end{pmatrix}^t = \frac{1}{\det \mathbf{A}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}. \tag{13.31}$$

Hence, the procedure to find the inverse matrix for a 2×2 matrix $\mathbf{A}$ is as follows: If the determinant is non-zero $\det \mathbf{A} \neq 0$, the inverse exists and to find $\mathbf{A}^{-1}$:

1. Exchange the elements in the leading diagonal.
2. Reverse the sign of the off-diagonal elements.
3. Divide by $\det \mathbf{A}$.

Example 13.5. Let us apply this procedure on the 2×2 rotation matrix

$$\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{13.32}$$

We note that $\det \mathbf{R}_\theta = 1 \neq 0$, so the matrix has an inverse and

$$\mathbf{R}_\theta^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \mathbf{R}_{-\theta}. \tag{13.33}$$

Example 13.6. Let us apply this procedure on the 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}. \tag{13.34}$$

We note that $\det \mathbf{A} = 1 \neq 0$, so the matrix has an inverse and

$$\mathbf{A}^{-1} = \begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix}. \tag{13.35}$$

Inverse of 3×3 matrix

Example 13.7. Consider a 3×3 matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 4 & 3 \\ 1 & -2 & -2 \\ -3 & 3 & 2 \end{pmatrix}. \tag{13.36}$$

First, we determine whether $\mathbf{A}$ is invertible. To do so, we calculate the determinant

$$\det \mathbf{A} = \begin{vmatrix} 2 & 4 & 3 \\ 1 & -2 & -2 \\ -3 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 8 & 7 \\ 1 & -2 & -2 \\ 0 & -3 & -4 \end{vmatrix} = -1 \cdot \begin{vmatrix} 8 & 7 \\ -3 & -4 \end{vmatrix} = 11. \tag{13.37}$$

Because $\det \mathbf{A} = 11 \neq 0$, the matrix is invertible and we find

$$\begin{aligned} \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \mathbf{C}^t &= \frac{1}{11} \begin{pmatrix} \begin{vmatrix} -2 & -2 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ -3 & 2 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ -3 & 3 \end{vmatrix} \\ -\begin{vmatrix} 4 & 3 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ -3 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 4 \\ -3 & 3 \end{vmatrix} \\ \begin{vmatrix} 4 & 3 \\ -2 & -2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 1 & -2 \end{vmatrix} \end{pmatrix}^t \\ &= \frac{1}{11} \begin{pmatrix} 2 & 4 & -3 \\ 1 & 13 & -18 \\ -2 & 7 & -8 \end{pmatrix}^t \\ &= \frac{1}{11} \begin{pmatrix} 2 & 1 & -2 \\ 4 & 13 & 7 \\ -3 & -18 & -8 \end{pmatrix}. \end{aligned} \tag{13.38}$$

Let us check:

$$\frac{1}{11} \begin{pmatrix} 2 & 4 & 3 \\ 1 & -2 & -2 \\ -3 & 3 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 & -2 \\ 4 & 13 & 7 \\ -3 & -18 & -8 \end{pmatrix} = \frac{1}{11} \begin{pmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{13.39}$$

13.5 Summary

After studying Sec. 13, you should be able to

- qualify that if $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$ and $g(\mathbf{y}) = \mathbf{B}\mathbf{y}$ then $(g \circ f)\mathbf{x} = (\mathbf{BA})\mathbf{x}$.
- define an *invertible matrix*: $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$ then $\mathbf{B} = \mathbf{A}^{-1}$
- show that a square matrix has at most one inverse
- find the transpose $\mathbf{A}^t$ of a matrix $\mathbf{A}$
- write a dot-product of vectors as matrix multiplication using the transpose: $\mathbf{x}\cdot\mathbf{y} = \mathbf{x}^t\mathbf{y}$
- determiner whether $\mathbf{A}$ is invertible: yes if $\det\mathbf{A} \neq 0$
- find the *matrix of co-factors* $\mathbf{C}$ of a $n \times n$ matrix $\mathbf{A}$ with $C_{ij} = (-1)^{i+j}\det\mathbf{A}_{ij}$
- find the inverse using $\mathbf{A}^{-1} = \frac{1}{\det\mathbf{A}}\mathbf{C}^t$

14 Matrices: Inverse, transpose, Cramer's Rule and rank

The aim is to

- introduce the notions of the *adjoint matrix* adj $\mathbf{A}$ and a *singular matrix*
- *prove the theorem for the inverse matrix: If $\det\mathbf{A} \neq 0 \Rightarrow \mathbf{A}^{-1} = \frac{1}{\det\mathbf{A}}\mathbf{C}^t$.
- finding the solution to a system of n equations in n unknowns using the notion of an invertible matrix: $\mathbf{Ax} = \mathbf{b} \Leftrightarrow \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$
- discuss the properties of the inverse
- discuss the properties of the transpose
- prove Cramer's rule
- *discuss rank of a linear function f and the identity: $\dim(\text{Null-space}) + \dim(\text{Range}) = \dim(\text{Domain})$.

14.1 Inverse matrices

Recall that the determinant of an $n \times n$ matrix $\mathbf{A}$ is defined by (expanded by the ith row)

$$\det\mathbf{A} = \sum_{j=1}^{n}(-1)^{i+j}a_{ij}\det\mathbf{A}_{ij} = \sum_{j=1}^{n}a_{ij}\underbrace{\left((-1)^{i+j}\det\mathbf{A}_{ij}\right)}_{C_{ij}} = \sum_{j=1}^{n}a_{ij}C_{ij}, \tag{14.1}$$

where $\mathbf{A}_{ij}$ is the ijth minor of the matrix $\mathbf{A}$ and C_{ij} is the ijth co-factor of the matrix $\mathbf{A}$.

Definition 14.1. The *adjoint matrix* adj$\mathbf{A}$ of a square $n \times n$ matrix $\mathbf{A}$ is the transpose of $\mathbf{C}$, the matrix of co-factors of $\mathbf{A}$:

$$\text{adj}\mathbf{A} = \mathbf{C}^t. \tag{14.2}$$

We can now re-state Theorem 13.3 as follows

Theorem 14.1. Let $\mathbf{A}$ be an $n \times n$ matrix. If $\det\mathbf{A} \neq 0$ then $\mathbf{A}$ is invertible and

$$\mathbf{A}^{-1} = \frac{1}{\det\mathbf{A}}\mathbf{C}^t = \frac{\text{adj}\mathbf{A}}{\det\mathbf{A}}. \tag{14.3}$$

Proof *: We need to show that $\mathbf{A}\frac{1}{\det\mathbf{A}}\mathbf{C}^t = \mathbf{I}$, that is, $\mathbf{AC}^t = \text{diag}(\det\mathbf{A}, \det\mathbf{A}, \ldots, \det\mathbf{A})$. Let the $n \times n$ matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}. \tag{14.4}$$

We now consider the transpose of the matrix of co-factors:

$$\mathbf{C}^t = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}^t = \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix}. \tag{14.5}$$

We find

$$\mathbf{AC}^t = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix} = \begin{pmatrix} \det \mathbf{A} & 0 & \cdots & 0 \\ 0 & \det \mathbf{A} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \det \mathbf{A} \end{pmatrix}. \tag{14.6}$$

To prove the last equal sign in Eq. (14.6), we have to show that the diagonal elements equal the determinant of $\mathbf{A}$, $\det \mathbf{A}$, and that the off-diagonal elements equal zero.
Diagonal elements ii: We notice that the iith, entry is given by

$$\left(\mathbf{AC}^t\right)_{ii} = \sum_{j=1}^{n} a_{ij} C^t_{ji} = \sum_{j=1}^{n} a_{ij} C_{ij} = \det \mathbf{A} \quad \text{for } i = 1, 2, \ldots, n \tag{14.7}$$

because the last sum in Eq. (14.7) is, by definition, the determinant of $\mathbf{A}$ expanded by row i. Therefore, all the diagonal entries equal $\det \mathbf{A}$.
Off-diagonal elements ik, $i \neq k$: Consider the off-diagonal element ik in the matrix product:

$$\left(\mathbf{AC}^t\right)_{ik} = \sum_{j=1}^{n} a_{ij} C^t_{jk} = \sum_{j=1}^{n} a_{ij} C_{kj}. \tag{14.8}$$

If the coefficients in front of C_{kj} had been a_{kj}, this would have been the determinant of $\mathbf{A}$ (expanded by row k). However, the coefficients in front of C_{kj} are the elements from row i in $\mathbf{A}$. Therefore, Eq. (14.8) actually represents the expansion of a determinant whose kth and ith row are identical. Such a determinant is zero and we can conclude that

$$\sum_{j=1}^{n} a_{ij} C_{kj} = 0 \quad \text{for all } i \neq k. \tag{14.9}$$

Hence, assuming $\det \mathbf{A} \neq 0$ we find

$$\mathbf{A} \frac{1}{\det \mathbf{A}} \mathbf{C}^t = \mathbf{I}. \tag{14.10}$$

Because the inverse is unique, we can conclude that it exists if $\det \mathbf{A} \neq 0$ and that

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \mathbf{C}^t. \tag{14.11}$$

Q.E.D.

Definition 14.2. We call the matrix $\mathbf{A}$ for *singular* if $\det \mathbf{A} = 0$ or if $\mathbf{A}$ is not a square matrix.

Hence, a matrix is singular if and only if it has no inverse.

14.2 Inverse matrices and system of linear equations

Consider a system of n linear equations in n unknowns

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1j}x_j + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2j}x_j + \cdots + a_{2n}x_n = b_2, \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n = b_i, \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nj}x_j + \cdots + a_{nn}x_n = b_n, \end{cases}$$
$$\Updownarrow$$
$$\mathbf{Ax} = \mathbf{b}. \tag{14.12}$$

We know from Theorem 14.1 that when $\det \mathbf{A} \neq 0$, $\mathbf{A}$ is invertible so $\mathbf{A}^{-1}$ exists. Therefore, applying $\mathbf{A}^{-1}$ on both sides of Eq. (14.12) from the left, we find

$$\mathbf{Ax} = \mathbf{b} \Leftrightarrow \mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{b} \Leftrightarrow \mathbf{Ix} = \mathbf{A}^{-1}\mathbf{b} \Leftrightarrow \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}. \tag{14.13}$$

Hence, we find a unique solution to the system of n linear equations in n unknowns. This is consistent with Cramer's Rule (see Theorem 7.4 page 58) that such a system has a unique solution if and only if $\det \mathbf{A} \neq 0$.
We can interpret Eq. (14.13) as follows: When the determinant of the matrix of coefficients $\mathbf{A}$ to a system of n equations in n unknowns is non-zero, $\det \mathbf{A} \neq 0$, then for every $\mathbf{b} \in \mathbb{R}^n$ there exists a unique $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{Ax} = \mathbf{b}$. Hence, the associated linear function is one-to-one because every $\mathbf{x}$ in the domain $\mathbb{R}^n$ is associated with one and only one $\mathbf{b}$ in the range $\mathbb{R}^n$. In particular, of course, the homogeneous equation $\mathbf{Ax} = \mathbf{0}$ has a unique solution, namely the trivial solution $\mathbf{x} = \mathbf{0}$. The dimension of the domain for the function is n and the dimension for the range of the function is also n. The dimension of the null-space is zero.

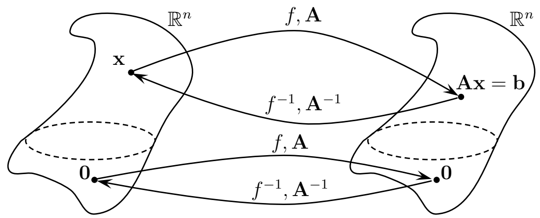

Figure 14.1: The function $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a linear function with an associated $n \times n$ matrix $\mathbf{A}$. If $\det \mathbf{A} \neq 0$, the function is one-to-one and an inverse function $f^{-1} : \mathbb{R}^n \mapsto \mathbb{R}^n$ exists with an associated $n \times n$ matrix $\mathbf{A}^{-1}$. The system of n linear equations in n unknowns $\mathbf{Ax} = \mathbf{b}$ has a unique solution. In particular, the homogeneous equation $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.

14.3 Properties of the inverse

In the following, all matrices that appear are square $n \times n$ matrices. The inverse of a matrix satisfies the following properties:

1. $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$
2. $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
   - $(\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_k)^{-1} = \mathbf{A}_k^{-1}\cdots\mathbf{A}_2^{-1}\mathbf{A}_1^{-1}$ for $k \in \mathbb{Z}^+$
3. $(\mathbf{A}^t)^{-1} = (\mathbf{A}^{-1})^t$
4. $\det \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}$.

Proof of property 2: According to the definition, we have to prove that $(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{I}$ and $(\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) = \mathbf{I}$. That is easily done using that matrix multiplication is associative:

$$(\mathbf{AB})\left(\mathbf{B}^{-1}\mathbf{A}^{-1}\right) = \mathbf{A}\left(\mathbf{BB}^{-1}\right)\mathbf{A}^{-1} = \mathbf{AIA}^{-1} = \mathbf{AA}^{-1} = \mathbf{I}, \tag{14.14a}$$

$$\left(\mathbf{B}^{-1}\mathbf{A}^{-1}\right)(\mathbf{AB}) = \mathbf{B}^{-1}\left(\mathbf{A}^{-1}\mathbf{A}\right)\mathbf{B} = \mathbf{B}^{-1}\mathbf{IB} = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I} \tag{14.14b}$$

so because the inverse is unique, we conclude that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

Q.E.D.

Proof of property 4: This proof makes use of the property that the determinant of a product of matrices equals the product of the determinants of the matrices, that is,

$$\det(\mathbf{AB}) = \det \mathbf{A} \cdot \det \mathbf{B}. \tag{14.15}$$

and that the determinant of the unit matrix is unity, $\det \mathbf{I} = 1$. We find

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \Rightarrow \det\left(\mathbf{A}^{-1}\mathbf{A}\right) = \det \mathbf{I} \Rightarrow \det \mathbf{A}^{-1}\cdot \det\mathbf{A} = \det \mathbf{I} \Rightarrow \det \mathbf{A}^{-1} = \frac{1}{\det\mathbf{A}}. \tag{14.16}$$

Q.E.D.

14.4 Properties of the transpose

The transpose of a matrix satisfies the following properties:

1. $(\mathbf{A}^t)^t = \mathbf{A}$
2. $(\mathbf{AB})^t = \mathbf{B}^t\mathbf{A}^t$
   - $(\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_k)^t = \mathbf{A}_k^t\cdots\mathbf{A}_2^t\mathbf{A}_1^t$ for $k \in \mathbb{Z}^+$
3. $(\mathbf{A}^t)^{-1} = (\mathbf{A}^{-1})^t$
4. $\det \mathbf{A}^t = \det \mathbf{A}$.

Proof of property 2*: We first check the shapes of the matrices. If $\mathbf{A}$ is an $m \times p$ matrix, $\mathbf{A}^t$ is an $p \times m$ matrix. If $\mathbf{B}$ is an $p \times n$ matrix, $\mathbf{B}^t$ is an $n \times p$ matrix. Then $\mathbf{AB}$ is an $m \times n$ matrix. Therefore, $(\mathbf{AB})^t$ is an $n \times m$ matrix and so is $\mathbf{B}^t\mathbf{A}^t$. We now evaluate the ijth term of $(\mathbf{AB})^t$, making use of the fact that for the transpose, $a^t_{ij} = a_{ji}$:

$$(\mathbf{AB})^t_{ij} = (\mathbf{AB})_{ji} = \sum_{k=1}^{p} a_{jk}b_{ki} = \sum_{k=1}^{p} a^t_{kj}b^t_{ik} = \sum_{k=1}^{p} b^t_{ik}a^t_{kj} = \left(\mathbf{B}^t\mathbf{A}^t\right)_{ij}. \tag{14.17}$$

Q.E.D.

14.5 Proof of Cramer's Rule

Theorem 14.2. Cramer's rule: A system of n linear equations in n unknowns $x_1, x_2, \ldots, x_n$ written on the matrix form

$$\mathbf{Ax} = \mathbf{b} \tag{14.18}$$

where $\mathbf{A}$ is the $n \times n$ matrix of coefficients of the system has a unique solution if and only if $\det\mathbf{A} \neq 0$. If a unique solution $(x_1, x_2, \ldots, x_n)$ exists, it is given by

$$x_j = \frac{\det\mathbf{B}^j}{\det\mathbf{A}} \quad \text{for } j = 1, 2, \ldots, n \tag{14.19}$$

where the matrix $\mathbf{B}^j$ is obtained from the matrix $\mathbf{A}$ by replacing its jth column with column vector $\mathbf{b}$ making up the right-hand side of the system of equations.

Proof: Let $\mathbf{A}$ denote an $n \times n$ matrix and consider the system of n linear equations in n unknowns: $\mathbf{Ax} = \mathbf{b}$. Let $\mathbf{I}^j(\mathbf{x})$ denote the matrix obtained from the identity matrix $\mathbf{I}$ by replacing its jth column with column vector $\mathbf{x}$. Showing explicitly row j and column j, we find

$$\begin{aligned}\mathbf{AI}^j(\mathbf{x}) &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jj} & \cdots & a_{jn} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & x_1 & \cdots & 0 \\ 0 & 1 & \cdots & x_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & x_j & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & x_n & \cdots & 1 \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{j1} & a_{j2} & \cdots & b_j & \cdots & a_{jn} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n & \cdots & a_{nn} \end{pmatrix} \\ &= \mathbf{B}^j, \end{aligned} \tag{14.20}$$

because the product of row i and column j equals b_i: $\sum_{j=1}^{n} a_{ij}x_j = b_i$ for $i = 1, 2, \ldots, n$.

Now we take the determinant on both sides of the equation. Using $\det \mathbf{I}^j(\mathbf{x}) = x_j$, we have

$$\det\left(\mathbf{A}\mathbf{I}^j(\mathbf{x})\right) = \det\mathbf{B}^j \Leftrightarrow \det\mathbf{A} \cdot \det\mathbf{I}^j(\mathbf{x}) = \det\mathbf{B}^j \Leftrightarrow \det\mathbf{A} \cdot x_j = \det\mathbf{B}^j \tag{14.21}$$

from which it follows that if $\det\mathbf{A} \neq 0$ then

$$x_j = \frac{\det\mathbf{B}^j}{\det\mathbf{A}} \quad \text{for } j = 1, 2, \ldots, n. \tag{14.22}$$

Q.E.D.

14.6 Rank of a linear function*

Consider the set of vectors $\mathcal{S} = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ where $\mathbf{a}_j \in \mathbb{R}^m$.

Definition 14.3. A vector $\mathbf{b} \in \mathbb{R}^m$ is a *linear combination* of the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ if there exist numbers $x_1, x_2, \ldots, x_n$ such that

$$\mathbf{b} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n. \tag{14.23}$$

Example 14.1. Let

$$\mathbf{a}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \mathbf{a}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}, \mathbf{a}_4 = \begin{pmatrix}1\\1\\1\end{pmatrix}. \tag{14.24}$$

Then the vector $\mathbf{b} = \begin{pmatrix}2\\2\\0\end{pmatrix}$ is a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$ because

$$\mathbf{b} = 2\mathbf{a}_1 + 2\mathbf{a}_2. \tag{14.25}$$

It is also a linear combination of $\mathbf{a}_3$ and $\mathbf{a}_4$ because

$$\mathbf{b} = 2\mathbf{a}_4 - 2\mathbf{a}_3. \tag{14.26}$$

However, $\mathbf{b}$ is not a linear combination of $\mathbf{a}_2$ and $\mathbf{a}_3$ because any such linear combination $x_2\mathbf{a}_2 + x_3\mathbf{a}_3$ will have zero on the first entry while the first entry in $\mathbf{b}$ is $2 \neq 0$.

Definition 14.4. We define the *span of a set $\mathcal{S}$ of vectors* to the be set consisting of all linear combinations of elements from $\mathcal{S}$.

Example 14.2. The set of vectors $\mathcal{S} = \left\{\begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}0\\3\end{pmatrix}\right\}$ spans $\mathbb{R}^2$ because all $\mathbf{x} \in \mathbb{R}^2$ can be written as a linear combination of the two vectors.

Example 14.3. The set of vectors $\mathcal{S} = \left\{\begin{pmatrix}1\\3\end{pmatrix}, \begin{pmatrix}2\\6\end{pmatrix}\right\}$ spans a one-dimensional line through the origin and with direction vector (proportional to) $\begin{pmatrix}1\\3\end{pmatrix}$. The two vectors do not span $\mathbb{R}^2$ because they are linearly dependent.

Definition 14.5. A set of vectors which is linearly independent and spans a space $\mathscr{V}$ is call a *basis* for $\mathscr{V}$.

Definition 14.6. The *dimension* $\dim\mathscr{V}$ of a vector space $\mathscr{V}$ that has a finite spanning set is the number of elements in any basis for that space.

Example 14.4. The set of vectors $\mathcal{S} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ where $\mathbf{e}_j \in \mathbb{R}^n$ is the jth natural basis vector for $\mathbb{R}^n$ span the whole of $\mathbb{R}^n$. Because they form a linearly independent set of vectors that span $\mathbb{R}^n$, there are a basis for $\mathbb{R}^n$ and hence $\dim\mathbb{R}^n = n$.

Example 14.5. Let us re-visit Ex. 10.5. The null-space for the matrix is spanned by one vector $\mathbf{x}_1 = \begin{pmatrix}-7\\-5\\1\end{pmatrix}$. It forms a basis for the null-space. Hence, the dimension of the null-space in Ex. 10.5 is 1, that is, $\dim\mathcal{N} = 1$.

Example 14.6. Let us re-visit Ex. 10.6. The null-space for the matrix is spanned by the two vectors $\mathbf{x}_1 = \begin{pmatrix}-2\\1\\0\end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix}3\\0\\1\end{pmatrix}$. These two vectors are linearly independent and they form a basis for the null-space. Hence, the dimension of the null-space in Ex. 10.6 is 2, that is, $\dim\mathcal{N} = 2$.

Definition 14.7. The *rank* of a linear function $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ is the dimension of the range of f, $\dim(\text{Range of } f)$.

Theorem 14.3. Let $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a linear function with an associated $n \times n$ matrix $\mathbf{A}$. Then the rank of f is equal to the number of linearly independent columns in $\mathbf{A}$.

Proof: Let $\mathbf{x}$ be any vector in $\mathbb{R}^n$. Then we can write $\mathbf{x}$ as a linear combination of the natural basis vectors $\mathbf{e}_j \in \mathbb{R}^n$, $j = 1, 2, \ldots, n$:

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n. \tag{14.27}$$

Because f is linear, we find

$$\begin{aligned} f(\mathbf{x}) &= f(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n) \\ &= f(x_1\mathbf{e}_1) + f(x_2\mathbf{e}_2) + \cdots + f(x_n\mathbf{e}_n) \\ &= x_1 f(\mathbf{e}_1) + x_2 f(\mathbf{e}_2) + \cdots + x_n f(\mathbf{e}_n) \\ &= x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n \end{aligned} \tag{14.28}$$

where $\mathbf{a}_j \in \mathbb{R}^n$, is the jth column vector in the matrix $\mathbf{A}$. When $x_1, x_2, \ldots, x_n$ runs through all point in $\mathbb{R}^n$, we see that the range of f is the span of the column vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$. Hence, the dimension of the range of f is the dimension of the span of the vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$. The dimension of the span is simply the number of independent vectors in the set $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$.

Theorem 14.4. Let $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a linear function from $\mathbb{R}^n$ into a subspace $\mathscr{V} \in \mathbb{R}^n$. Then

$$\dim(\text{Null-space}) + \dim(\text{Range}) = \dim(\text{Domain}). \tag{14.29}$$

Example 14.7. In Ex. 10.5, the dimension of the domain for the linear function is 3, the dimension of the null-space is 1. Hence, we can conclude that the dimension for the range of the mapping must be 2. Indeed, inspecting the three column vectors in the matrix

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} -2 \\ -4 \\ 5 \end{pmatrix}, \mathbf{a}_3 = \begin{pmatrix} -3 \\ -13 \\ 4 \end{pmatrix} \tag{14.30}$$

we find that

$$\mathbf{a}_3 = 7\mathbf{a}_1 + 5\mathbf{a}_2 \tag{14.31}$$

and hence the number if linearly independent columns in $\mathbf{A}$ is two.

Example 14.8. In Ex. 10.6, the dimension of the domain for the linear function is 3, the dimension of the null-space is 2. Hence, we can conclude that the dimension for the range of the mapping must be 1. Indeed, inspecting the three column vectors in the matrix

$$\mathbf{a}_1 = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 4 \\ 6 \\ -2 \end{pmatrix}, \mathbf{a}_3 = \begin{pmatrix} -6 \\ -9 \\ 3 \end{pmatrix} \tag{14.32}$$

we find that

$$\mathbf{a}_2 = 2\mathbf{a}_1 \text{ and } \mathbf{a}_3 = -3\mathbf{a}_1 \tag{14.33}$$

and hence the number if linearly independent columns in $\mathbf{A}$ is one.

Example 14.9. In Ex. 9.4, the dimension of the domain for the linear function is 3, the dimension of the null-space is 0. Hence, we can conclude that the dimension for the range of the mapping must be 3. Indeed, because the determinant of the associated matrix is non-zero, the column vectors are linearly independent and we conclude that the range has dimension 3.

14.7 Summary

After studying Sec. 14, you should be able to

- find the *adjoint matrix* adj $\mathbf{A} = \mathbf{C}^t$
- determine whether a matrix is singular: $\det\mathbf{A} = 0$ or non-square
- find the solution to a system of n equations in n unknowns using the notion of an invertible matrix, $\mathbf{Ax} = \mathbf{b} \Leftrightarrow \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$
- use the properties of the inverse and the transpose:

$$\begin{aligned}
&\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A} && \left(\mathbf{A}^{t}\right)^{t} = \mathbf{A} \\
&\left(\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_k\right)^{-1} = \mathbf{A}_k^{-1}\cdots\mathbf{A}_2^{-1}\mathbf{A}_1^{-1} && \left(\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_k\right)^{t} = \mathbf{A}_k^{t}\cdots\mathbf{A}_2^{t}\mathbf{A}_1^{t} \\
&\left(\mathbf{A}^{t}\right)^{-1} = \left(\mathbf{A}^{-1}\right)^{t} && \left(\mathbf{A}^{t}\right)^{-1} = \left(\mathbf{A}^{-1}\right)^{t} \\
&\det\mathbf{A}^{-1} = \frac{1}{\det\mathbf{A}} && \det\mathbf{A}^{t} = \det\mathbf{A}
\end{aligned}$$

15 The eigenvalue problem: I

The aim is to consider a linear function $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ with its associated $n \times n$ matrix $\mathbf{A}$ and discuss the eigenvalue problem $f(\mathbf{x}) = \lambda\mathbf{x} \Leftrightarrow \mathbf{Ax} = \lambda\mathbf{x}$. We will show how to

- determine the eigenvalues $\lambda_i, i = 1, 2, \ldots, n$.
- determine the associated eigenvectors $\mathbf{x}_i, i = 1, 2, \ldots, n$.
- construct the matrix of eigenvectors $\mathbf{S} = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \ldots & \mathbf{x}_n \end{pmatrix}$
- diagonalise the matrix $\mathbf{A}$ by the construction: $\mathbf{S}^{-1}\mathbf{AS} = \mathbf{\Lambda} = \text{diag}\left(\lambda_1, \lambda_2, \ldots, \lambda_n\right)$.

15.1 Eigenvalues and eigenvectors

Consider a linear function $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ with its associated $n \times n$ matrix $\mathbf{A}$ such that $f(\mathbf{x}) = \mathbf{Ax}$ for all $\mathbf{x} \in \mathbb{R}^n$.

Definition 15.1. If there exists $\mathbf{x} \neq \mathbf{0}$ such that

$$f(\mathbf{x}) = \lambda\mathbf{x} \tag{15.1a}$$

or, equivalently,

$$\mathbf{Ax} = \lambda\mathbf{x}, \tag{15.1b}$$

then $\mathbf{x}$ is called an *eigenvector* for the linear function f (matrix $\mathbf{A}$) and λ is called the associated *eigenvalue*.

Eigenvectors $\mathbf{x}$ for a transformation are very special in that the function f (matrix $\mathbf{A}$) maps them into a multiple of themselves, namely $\lambda\mathbf{x}$. Hence, the mapping of $\mathbf{x}$ is in the same direction if $\lambda > 0$ or in the opposite direction if $\lambda < 0$ or mapped into the zero-vector if $\lambda = 0$, see Fig. 15.1. Therefore, clearly, the domain and the range of the linear transformation f (matrix $\mathbf{A}$) have to be the same space (dimension) for the notion of eigenvectors and eigenvalues to be well-defined.

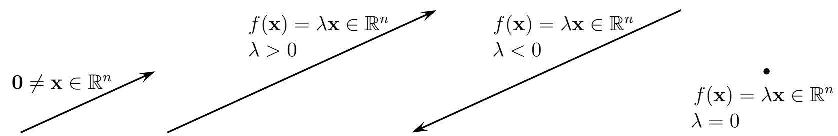

Figure 15.1: Schematic representation of a linear function $f : \mathbb{R}^n \mapsto \mathbb{R}^n$. A vector $\mathbf{x} \neq \mathbf{0}$ is an eigenvector for f with eigenvalue λ if $f(\mathbf{x}) = \lambda\mathbf{x}$. If $\lambda > 0$, the mapping of $\mathbf{x}$ is pointing in the same direction as $\mathbf{x}$ and its magnitude is multiplied by λ. If $\lambda < 0$, the mapping of $\mathbf{x}$ is pointing in the opposite direction of $\mathbf{x}$ and its magnitude is multiplied by $|\lambda|$. If $\lambda = 0$, the mapping of $\mathbf{x}$ is the zero-vector, that is, $\mathbf{x}$ belongs to the null-space of the function f.

Example 15.1. Consider $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ given by the 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}. \tag{15.2}$$

Then we observe that

$$\underbrace{\begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\mathbf{x}_1} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = \underbrace{4}_{\lambda_1} \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\mathbf{x}_1} \Leftrightarrow \mathbf{A}\mathbf{x}_1 = \lambda_1 \mathbf{x}_1, \tag{15.3}$$

so $\mathbf{x}_1$ is an eigenvector for f (matrix $\mathbf{A}$) with the eigenvalue $\lambda_1 = 4$.

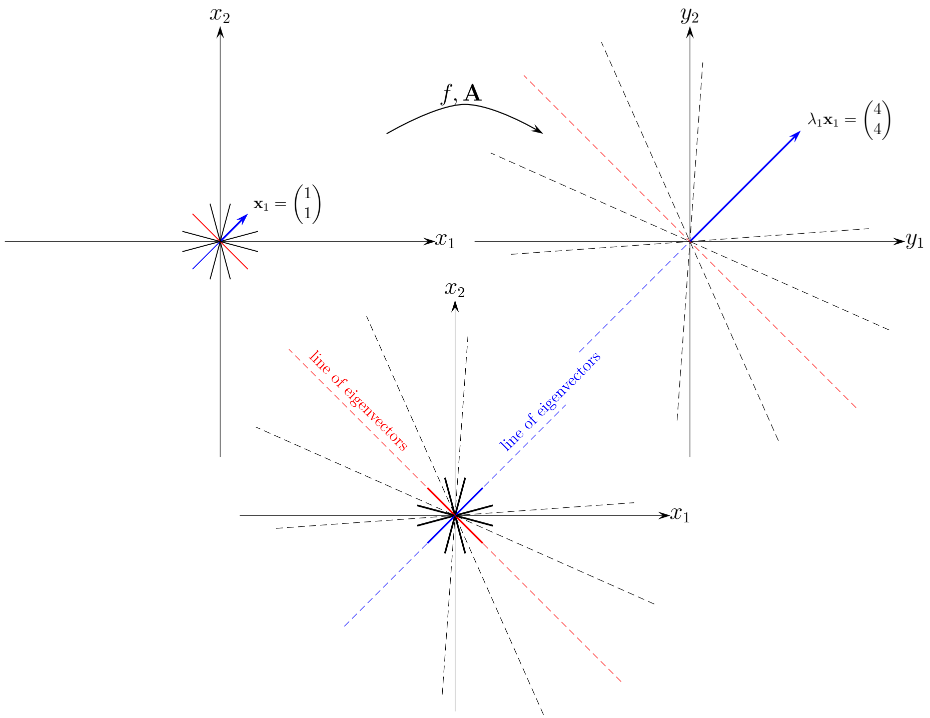

Figure 15.2: The linear function $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ defined by the matrix in Eq. (15.2). A linear function maps lines onto lines. Shown are the lines with angles $\theta = 15°, 45°, 75°, 105°, 135°, 165°$ measured from the positive x-axis. There are two special directions for f given by $\mathbf{x}_1$ along $\theta = 45°$ (blue) and $\mathbf{x}_2$ along $\theta = 135°$ (red) where $f(\mathbf{x}_1) = \lambda_1 \mathbf{x}_1$ and $f(\mathbf{x}_2) = \lambda_2 \mathbf{x}_2$, respectively.

All the vectors along the special direction of $\mathbf{x}_1$ are mapped onto vectors parallel with that direction. Checking that, say $3\mathbf{x}_1 = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ is also an eigenvector for f with eigenvalue $\lambda_1 = 4$ we find:

$$\underbrace{\begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} 3 \\ 3 \end{pmatrix}}_{3\mathbf{x}_1} = \begin{pmatrix} 12 \\ 12 \end{pmatrix} = \underbrace{4}_{\lambda_1} \underbrace{\begin{pmatrix} 3 \\ 3 \end{pmatrix}}_{3\mathbf{x}_1} \Leftrightarrow \mathbf{A}\,(3\mathbf{x}_1) = \lambda_1\,(3\mathbf{x}_1)\,, \tag{15.4}$$

so $3\mathbf{x}_1$ is an eigenvector for f (matrix $\mathbf{A}$) with the same eigenvalue $\lambda_1 = 4$ as $\mathbf{x}_1$.

Theorem 15.1. If $\mathbf{x}$ is an eigenvector for f (matrix $\mathbf{A}$) with eigenvalue λ then $c\mathbf{x}, c \neq 0$ is also an eigenvector for f (matrix $\mathbf{A}$) with the same eigenvalue λ.

Proof: Let $\mathbf{x} \neq \mathbf{0}$ denote an eigenvector for f with eigenvalue λ, that is, $f(\mathbf{x}) = \lambda\mathbf{x}$. When $c \neq 0$, we have $c\mathbf{x} \neq \mathbf{0}$ and we find that

$$\begin{aligned} f(c\mathbf{x}) &= cf(\mathbf{x}) && \text{using that } f \text{ is linear} \\ &= c\lambda\mathbf{x} && \text{because } \mathbf{x} \text{ is a eigenvector for } f \text{ with eigenvalue } \lambda \\ &= \lambda\,(c\mathbf{x}) && \text{associative law for scalar multiplication.} \end{aligned} \tag{15.5}$$

Hence, $c\mathbf{x} \neq \mathbf{0}$ is an eigenvector for f (matrix $\mathbf{A}$) with eigenvalue λ.

Q.E.D.

15.2 Finding the eigenvalues – the characteristic equation

How can we determine the eigenvalues λ in a systematic way? The eigenvalue problem is defined by the equation $f(\mathbf{x}) = \lambda\mathbf{x}$ or, equivalently, $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$. Note that both the eigenvalue λ *and* the eigenvector $\mathbf{x} \neq \mathbf{0}$ are unknowns! Let us re-arrange the equation:

$$\begin{aligned} \mathbf{A}\mathbf{x} = \lambda\mathbf{x} &\Leftrightarrow \mathbf{A}\mathbf{x} = \lambda\mathbf{I}\mathbf{x} && \text{where } \mathbf{I} \text{ is the } n \times n \text{ identity matrix} \\ &\Leftrightarrow \mathbf{A}\mathbf{x} - \lambda\mathbf{I}\mathbf{x} = \mathbf{0} \\ &\Leftrightarrow (\mathbf{A} - \lambda\mathbf{I})\,\mathbf{x} = \mathbf{0} && \text{using linearity.} \end{aligned} \tag{15.6}$$

Equation (15.6) is a *homogeneous equation* where the matrix of coefficients is $(\mathbf{A} - \lambda\mathbf{I})$. We are seeking *non-trivial* solutions $\mathbf{x} \neq \mathbf{0}$ to Eq. (15.6) and they exists if and only if

$$\boxed{\det(\mathbf{A} - \lambda\mathbf{I}) = 0.} \tag{15.7}$$

Equation (15.7) is known as the *characteristic equation* for the matrix $\mathbf{A}$ and the left-hand-side as the *characteristic* or *secular determinant* of $\mathbf{A}$.

Theorem 15.2. The eigenvalues λ of a linear function $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ are the solutions to the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ for the $n \times n$ matrix $\mathbf{A}$ associated with f.

Example 15.2. Consider the 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}. \tag{15.8}$$

In order to determine the eigenvalues for $\mathbf{A}$ we must solve the characteristic equation

$$\begin{aligned}
\det(\mathbf{A}-\lambda\mathbf{I}) = 0 &\Leftrightarrow \begin{vmatrix} 5-\lambda & -1 \\ -1 & 5-\lambda \end{vmatrix} = 0 \\
&\Leftrightarrow (5-\lambda)^2 - 1 = 0 \\
&\Leftrightarrow \lambda^2 - \underbrace{10}_{a_{11}+a_{22}}\lambda + \underbrace{24}_{\det\mathbf{A}} = 0 \\
&\Leftrightarrow \lambda = \frac{10 \pm \sqrt{10^2 - 4\cdot 1\cdot 24}}{2} \\
&\Leftrightarrow \lambda = \begin{cases} 4, \\ 6. \end{cases} \qquad (15.9)
\end{aligned}$$

Hence, the 2×2 matrix $\mathbf{A}$ has 2 eigenvalues, namely

$$\lambda_1 = 4, \qquad (15.10a)$$
$$\lambda_2 = 6. \qquad (15.10b)$$

Introducing the trace of a matrix $\mathbf{A}$ as the sum of the entries in the main-diagonal, that is, $\operatorname{Trace}\mathbf{A} = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii}$, we notice that

$$\lambda_1 \cdot \lambda_2 = \det\mathbf{A}, \qquad (15.11a)$$
$$\lambda_1 + \lambda_2 = \operatorname{Trace}\mathbf{A}. \qquad (15.11b)$$

The rules in Equation (15.11) generalise for $n \times n$ matrices to the following:

$$\lambda_1 \cdot \lambda_2 \cdot \cdots \cdot \lambda_n = \det\mathbf{A}, \qquad (15.12a)$$
$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \operatorname{Trace}\mathbf{A}. \qquad (15.12b)$$

15.3 Finding the eigenvectors – the homogeneous equation

How can we determine the eigenvectors associated with the eigenvalues in a systematic way? We simply have to solve the homogeneous Eq. (15.6) with respect to the unknown vectors $\mathbf{x}$.

Example 15.3. We re-write Eq. (15.6) on component form

$$\begin{aligned}
(\mathbf{A}-\lambda\mathbf{I})\,\mathbf{x} = \mathbf{0} &\Leftrightarrow \begin{pmatrix} 5-\lambda & -1 \\ -1 & 5-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
&\Leftrightarrow \begin{cases} (5-\lambda)x - y = 0 \\ -x + (5-\lambda)y = 0. \end{cases} \qquad (15.13)
\end{aligned}$$

Now we have to consider each of the two eigenvalues separately. For clarity, we will use the notation x_1, y_1 for the eigenvector(s) belonging to the eigenvalue λ_1 and x_2, y_2 for the eigenvector(s) belonging to the eigenvalue λ_2.

$\underline{\lambda_1 = 4}$:
Inserting the value $\lambda_1 = 4$ into Eq. (15.13), we find

$$\begin{cases} x_1 - y_1 = 0 \\ -x_1 + y_1 = 0 \end{cases} \Leftrightarrow y_1 = x_1. \qquad (15.14)$$

Letting $x_1 = c_1, c_1 \neq 0$, we find that

$$\mathbf{x}_1 = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{for } c_1 \neq 0 \qquad (15.15)$$

are the eigenvectors associated with the eigenvalue $\lambda_1 = 4$. Because Theorem 15.1 inform us that if $\mathbf{x}_1$ is an eigenvector for $\mathbf{A}$ with eigenvalue λ_1 then so is $c_1\mathbf{x}_1$, $c_1 \neq 0$, we often omit the constant c_1 in Eq. (15.15) and choose the simplest vector as a representative for the eigenvector associated with the eigenvalue λ_1, in this case

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \qquad (15.16)$$

$\underline{\lambda_2 = 6}$:
Inserting the value $\lambda_2 = 6$ into Eq. (15.13), we find

$$\begin{cases} -x_2 - y_2 = 0 \\ -x_2 - y_2 = 0 \end{cases} \Leftrightarrow y_2 = -x_2. \qquad (15.17)$$

Letting $x_2 = c_2, c_2 \neq 0$, we find that

$$\mathbf{x}_2 = c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{for } c_2 \neq 0 \qquad (15.18)$$

are the eigenvectors associated with the eigenvalue $\lambda_2 = 6$. Because Theorem 15.1 inform us that if $\mathbf{x}_2$ is an eigenvector for $\mathbf{A}$ with eigenvalue λ_2 then so is $c_2\mathbf{x}_2$, $c_2 \neq 0$, we often omit the constant c_2 in Eq. (15.18) and choose the simplest vector as a representative for the eigenvector associated with the eigenvalue λ_2, in this case

$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \qquad (15.19)$$

Let us check that we didn't make a mistake in the calculation of $\mathbf{x}_2$:

$$\begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix} = 6\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad (15.20)$$

confirming that indeed $\mathbf{x}_2$ is the eigenvector for $\mathbf{A}$ associated with the eigenvalue $\lambda_2 = 6$.

Procedure to find the eigenvalues and the associated eigenvectors

To solve the eigenvalue problem $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ where $\mathbf{x} \neq \mathbf{0}$:

1. Find the eigenvalues λ by solving the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$.
2. For each eigenvalue λ_i, find the associated eigenvector $\mathbf{x}_i$ by solving the homogeneous equation $(\mathbf{A} - \lambda_i\mathbf{I})\,\mathbf{x}_i = \mathbf{0}$.

15.4 Matrix of eigenvectors and diagonalisation of the matrix A

Let $\mathbf{A}$ denote a general 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{15.21}$$

for a linear transformation $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ and assume that this transformation has two eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ associated with the eigenvalues λ_1 and λ_2, respectively, that is,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \lambda_1 x_1 \\ \lambda_1 y_1 \end{pmatrix} \tag{15.22a}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \lambda_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \lambda_2 x_2 \\ \lambda_2 y_2 \end{pmatrix}. \tag{15.22b}$$

Equations (15.22) can conveniently be written as

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}}_{\mathbf{S}} = \begin{pmatrix} \lambda_1 x_1 & \lambda_2 x_2 \\ \lambda_1 y_1 & \lambda_2 y_2 \end{pmatrix} = \underbrace{\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}}_{\mathbf{S}} \underbrace{\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}}_{\mathbf{\Lambda}} \tag{15.23}$$

The matrix $\mathbf{S}$ has the eigenvector $\mathbf{x}_1$ as column 1 and the eigenvector $\mathbf{x}_2$ as column 2. The matrix $\mathbf{\Lambda}$ is a diagonal matrix with the eigenvalues in the main diagonal. Hence, introducing the *matrix of eigenvectors*

$$\mathbf{S} = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \tag{15.24}$$

and the *diagonal matrix of eigenvalues*

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{15.25}$$

we have shown that

$$\mathbf{AS} = \mathbf{S\Lambda}. \tag{15.26}$$

Therefore, if we assume that the matrix of eigenvectors $\mathbf{S}$ is non-singular, that is, $\det \mathbf{S} \neq 0$, such that its inverse exists, then by multiplying with $\mathbf{S}^{-1}$ from the left, we find

$$\boxed{\mathbf{S}^{-1}\mathbf{AS} = \mathbf{\Lambda}.} \tag{15.27}$$

We say that the matrix of eigenvectors $\mathbf{S}$ diagonalise the matrix $\mathbf{A}$ in a *similarity transformation*.

Example 15.4. The 2×2 matrix

$$\begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix} \tag{15.28}$$

has eigenvalues $\lambda_1 = 4$ and $\lambda = 6$ with associated eigenvectors

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{associated with the eigenvalue } \lambda_1 = 4 \tag{15.29a}$$

$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{associated with the eigenvalue } \lambda_2 = 6. \tag{15.29b}$$

Therefore, the matrix of eigenvectors $\mathbf{S}$ is given by

$$\mathbf{S} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{15.30}$$

This matrix is invertible because $\det \mathbf{S} = -1 - 1 = -2 \neq 0$ and we find

$$\mathbf{S}^{-1} = \frac{1}{-2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{15.31}$$

We find

$$\mathbf{AS} = \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 4 & -6 \end{pmatrix} \tag{15.32}$$

and by applying $\mathbf{S}^{-1}$ from the left, we find

$$\mathbf{S}^{-1}\mathbf{AS} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 4 & 6 \\ 4 & -6 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 8 & 0 \\ 0 & 12 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \mathbf{\Lambda}. \tag{15.33}$$

Procedure to diagonalise a matrix A

1. Find the eigenvalues λ by solving the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$.
2. For each eigenvalue λ_i, find the associated eigenvector $\mathbf{x}_i$ by solving the homogeneous equation $(\mathbf{A} - \lambda_i\mathbf{I})\,\mathbf{x}_i = \mathbf{0}$.
3. Construct the matrix of eigenvectors $\mathbf{S} = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{pmatrix}$ whose jth column is the jth eigenvector $\mathbf{x}_j$.
4. Then $\mathbf{S}^{-1}\mathbf{AS} = \mathbf{\Lambda}$ where $\mathbf{\Lambda} = \mathrm{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is the diagonal matrix with the eigenvalues along the main diagonal.

15.5 Summary

We considered a linear function $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ with its associated matrix $\mathbf{A}$ and discussed the eigenvalue problem $f(\mathbf{x}) = \lambda\mathbf{x} \Leftrightarrow \mathbf{Ax} = \lambda\mathbf{x}$. After studying Sec. 15, you should be able to

- define the *eigenvalue problem* $f(\mathbf{x}) = \lambda\mathbf{x} \Leftrightarrow \mathbf{Ax} = \lambda\mathbf{x}, \mathbf{x} \neq \mathbf{0}$
- show that the *eigenvalues* λ are solutions to the *characteristic eq.* $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

For linear functions $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $f : \mathbb{R}^3 \to \mathbb{R}^3$, you should be able to

- apply the *characteristic equation* $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ to find the eigenvalues λ_i
- find the associated eigenvectors by solving the *homogeneous equation* $(\mathbf{A} - \lambda_i\mathbf{I})\,\mathbf{x}_i = \mathbf{0}$
- construct the *matrix of eigenvectors* $\mathbf{S} = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{pmatrix}$
- diagonalise the matrix $\mathbf{A}$ by $\mathbf{S}^{-1}\mathbf{AS} = \mathbf{\Lambda} = \mathrm{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

16 The eigenvalue problem: II

The aim is to continue discussing the eigenvalue problem $f(\mathbf{x}) = \lambda\mathbf{x}$ for a linear function $f : \mathbb{R}^n \to \mathbb{R}^n$ with its associated $n \times n$ matrix $\mathbf{A}$. In particular, we will

- show that for a symmetric matrix, that is, $\mathbf{A} = \mathbf{A}^t$, the eigenvectors corresponding to distinct eigenvalues are orthogonal (perpendicular)
- how that relates to the construction of an orthogonal matrix of eigenvectors, $\mathbf{S}^{-1} = \mathbf{S}^t$
- dissect the diagonalisation $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{\Lambda}$ of $\mathbf{A}$ mathematically and geometrically
- briefly discuss matrices with degenerated eigenvalues and imaginary eigenvalues.

Summary of Example 15.1

First, let us summarise Ex. 15.1 and then continue investigating this example in further detail. Given the linear function $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ with its associated 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}, \tag{16.1}$$

we considered the eigenvalue problem (see Fig. 16.1)

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \quad \text{for } \mathbf{x} \neq \mathbf{0}. \tag{16.2}$$

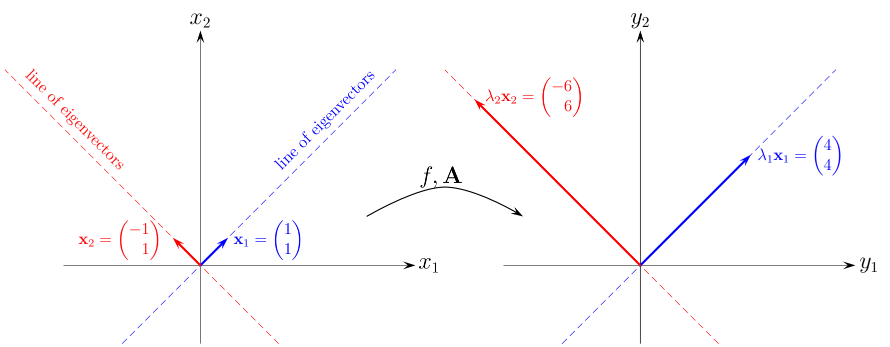

Figure 16.1: The linear function $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ defined by the matrix in Eq. (15.2). There are two special directions for f given by $\mathbf{x}_1$ (blue) and $\mathbf{x}_2$ (red) where $f(\mathbf{x}_1) = \lambda_1\mathbf{x}_1$ and $f(\mathbf{x}_2) = \lambda_2\mathbf{x}_2$, respectively with $\lambda_1 = 4$ and $\lambda_2 = 6$. Notice that we have chosen the eigenvector obtained with $c_2 = -1$ in Eq. (15.18) as a representative of a direction vector for the line of eigenvectors. This is done to simplify what follows in Sec. 16.

Solving the characteristic equation yielded the two eigenvalues

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \Leftrightarrow \begin{cases} \lambda_1 = 4, \\ \lambda_2 = 6, \end{cases} \tag{16.3a}$$

and then solving the homogeneous equation yielded the two associated (lines of) eigenvectors

$$(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{x} = 0 \Leftrightarrow \begin{cases} \mathbf{x}_1 = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & c_1 \neq 0, \text{ for } \lambda_1 \\ \mathbf{x}_2 = c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}, & c_2 \neq 0, \text{ for } \lambda_2. \end{cases} \tag{16.3b}$$

Choosing for simplicity $c_1 = c_2 = 1$, we can construct the matrix of eigenvectors

$$\mathbf{S} = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \Leftrightarrow \mathbf{S}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \tag{16.4a}$$

Finally, we showed that the matrix $\mathbf{A}$ could be diagonalise by the procedure

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \mathbf{\Lambda}. \tag{16.4b}$$

It is worth noticing that if we construct $\tilde{\mathbf{S}} = \begin{pmatrix} \mathbf{x}_2 & \mathbf{x}_1 \end{pmatrix}$, i.e., exchange the columns, we will find

$$\tilde{\mathbf{S}}^{-1}\mathbf{A}\tilde{\mathbf{S}} = \begin{pmatrix} 6 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix} = \tilde{\mathbf{\Lambda}}. \tag{16.5}$$

16.1 Symmetric matrices and orthogonal eigenvectors

We make the observation that the eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal (perpendicular) because

$$\mathbf{x}_1 \cdot \mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -1 + 1 = 0. \tag{16.6}$$

This is not a simple coincidence. This is an implication of the matrix $\mathbf{A}$ being symmetric:

Theorem 16.1. Let $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a linear function with a (real) symmetric matrix $\mathbf{A}$, that is, $\mathbf{A} = \mathbf{A}^t$. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are eigenvectors for f (matrix $\mathbf{A}$) corresponding to distinct eigenvalues $\lambda_1 \neq \lambda_2$ then $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal (perpendicular).

Proof: Let $\mathbf{A}\mathbf{x}_1 = \lambda_1\mathbf{x}_1$ and $\mathbf{A}\mathbf{x}_2 = \lambda_2\mathbf{x}_2$ with $\lambda_1 \neq \lambda_2$. We want to show that $\mathbf{x}_1 \cdot \mathbf{x}_2 = 0$.

$$\begin{aligned}
\lambda_1(\mathbf{x}_1 \cdot \mathbf{x}_2) &= (\lambda_1\mathbf{x}_1) \cdot \mathbf{x}_2 \\
&= (\mathbf{A}\mathbf{x}_1) \cdot \mathbf{x}_2 && \text{because } \mathbf{A}\mathbf{x}_1 = \lambda_1\mathbf{x}_1 \\
&= (\mathbf{A}\mathbf{x}_1)^t\,\mathbf{x}_2 && \text{because } \mathbf{x} \cdot \mathbf{y} = \mathbf{x}^t\mathbf{y}\text{, see page 118} \\
&= \left(\mathbf{x}_1^t\mathbf{A}^t\right)\mathbf{x}_2 && \text{because } (\mathbf{A}\mathbf{B})^t = \mathbf{B}^t\mathbf{A}^t\text{, see page 125} \\
&= \mathbf{x}_1^t\,(\mathbf{A}\mathbf{x}_2) && \text{because } \mathbf{A} = \mathbf{A}^t \\
&= \mathbf{x}_1^t\,(\lambda_2\mathbf{x}_2) && \text{because } \mathbf{A}\mathbf{x}_2 = \lambda_2\mathbf{x}_2 \\
&= \lambda_2(\mathbf{x}_1 \cdot \mathbf{x}_2) && \text{because } \mathbf{x} \cdot \mathbf{y} = \mathbf{x}^t\mathbf{y} \text{ and } \lambda_2 \text{ just a number.}
\end{aligned} \tag{16.7}$$

Therefore, we have $(\lambda_1 - \lambda_2)(\mathbf{x}_1 \cdot \mathbf{x}_2) = 0$. Because $\lambda_1 \neq \lambda_2$, we can conclude that $\mathbf{x}_1 \cdot \mathbf{x}_2 = 0$. Q.E.D.

Let us remind ourselves, that the Kronecker delta

$$\delta_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases} \tag{16.8}$$

Hence, if we have a set of normalised (i.e., unit) vectors $\{\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \dots, \hat{\mathbf{x}}_n\}$ that are pairwise orthogonal (perpendicular), that is,

$$\hat{\mathbf{x}}_i \cdot \hat{\mathbf{x}}_i = 1 \quad \text{for } i = 1, 2, \dots, n \text{ – the vectors are unit vectors,} \tag{16.9a}$$
$$\hat{\mathbf{x}}_i \cdot \hat{\mathbf{x}}_j = 0 \quad \text{for } i \neq j \text{ – the vectors are pairwise orthogonal,} \tag{16.9b}$$

we may simply write $\hat{\mathbf{x}}_i \cdot \hat{\mathbf{x}}_j = \delta_{ij}$. Therefore, as a direct consequence of Theorem 16.1 we have

Theorem 16.2. If $\mathbf{A}$ is a (real) symmetric matrix, that is, $\mathbf{A} = \mathbf{A}^t$, with *distinct* eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ then the set of associated normalised eigenvectors $\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \dots, \hat{\mathbf{x}}_n$ is *orthonormal*, that is,

$$\hat{\mathbf{x}}_i \cdot \hat{\mathbf{x}}_j = \delta_{ij} \tag{16.10}$$

and the matrix of normalised and orthogonal eigenvectors

$$\mathbf{S} = \begin{pmatrix} \hat{\mathbf{x}}_1 & \hat{\mathbf{x}}_2 & \cdots & \hat{\mathbf{x}}_n \end{pmatrix} \tag{16.11}$$

is an *orthogonal* matrix, that is,

$$\mathbf{S}^{-1} = \mathbf{S}^t. \tag{16.12}$$

Example 16.1. The eigenvectors in Eq. (16.3b) with $c_1 = c_2 = 1$ have magnitudes $|\mathbf{x}_1| = |\mathbf{x}_2| = \sqrt{2}$. Therefore, normalising the eigenvectors from Eq. (16.3b) we have

$$\hat{\mathbf{x}}_1 = \frac{\mathbf{x}_1}{|\mathbf{x}_1|} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad \text{associated with } \lambda_1 \tag{16.13a}$$

$$\hat{\mathbf{x}}_2 = \frac{\mathbf{x}_2}{|\mathbf{x}_2|} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad \text{associated with } \lambda_2 \tag{16.13b}$$

such that the matrix of normalised and orthogonal eigenvectors is an orthogonal matrix (see Classwork 7)

$$\mathbf{S} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{16.14}$$

Because $\det \mathbf{S} = 1$, $\mathbf{S}^{-1}$ exists, and indeed we find that the inverse is the transpose:

$$\mathbf{S}^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \mathbf{S}^t. \tag{16.15}$$

Let us make a brute force check:

$$\mathbf{S}\mathbf{S}^t = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}. \tag{16.16}$$

16.2 Diagonalisation - a closer inspection

One of the key results for the eigenvalue problem is that if the matrix of eigenvectors $\mathbf{S} = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{pmatrix}$ is non-singular, that is, $\det \mathbf{S} \neq 0$ then we may diagonalise the matrix matrix $\mathbf{A}$ by

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{\Lambda} = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n). \tag{16.17}$$

Hence, the matrix product (composite function) $\mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ act as a simple scaling (expansion or contraction), see page 108. Why is that? How can we understand this important result in more detail? We will address this issue in $\mathbb{R}^2$ but what follows can easily be extended to $\mathbb{R}^n$.

First we notice that the matrix of eigenvectors $\mathbf{S}$ maps the natural basis vectors of $\mathbb{R}^2$ onto the eigenvectors of the matrix $\mathbf{A}$:

$$\underbrace{\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}}_{\mathbf{S}} \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\mathbf{e}_1} = \underbrace{\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}}_{\mathbf{x}_1} \Leftrightarrow \mathbf{S}\mathbf{e}_1 = \mathbf{x}_1, \tag{16.18a}$$

$$\underbrace{\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}}_{\mathbf{S}} \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\mathbf{e}_2} = \underbrace{\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}}_{\mathbf{x}_2} \Leftrightarrow \mathbf{S}\mathbf{e}_2 = \mathbf{x}_2, \tag{16.18b}$$

and applying $\mathbf{S}^{-1}$ from the left, we find that $\mathbf{S}^{-1}$ maps the eigenvectors of the matrix $\mathbf{A}$ onto the natural basis vectors of $\mathbb{R}^2$:

$$\mathbf{e}_1 = \mathbf{S}^{-1}\mathbf{x}_1 \quad \text{and} \quad \mathbf{e}_2 = \mathbf{S}^{-1}\mathbf{x}_2. \tag{16.19}$$

Mathematical interpretation of diagonalisation

Let us now consider an arbitrary vector $\mathbf{x} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in \mathbb{R}^2$ with coordinates a_1 and a_2, that is, $\mathbf{x} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$. Let us dissect how the matrix product $\mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ acts on $\mathbf{x}$:

$$\begin{aligned}
\left(\mathbf{S}^{-1}\mathbf{A}\mathbf{S}\right)\mathbf{x} &= \mathbf{S}^{-1}\mathbf{A}\,(\mathbf{S}\mathbf{x}) \\
&= \mathbf{S}^{-1}\mathbf{A}\,(\mathbf{S}\,(a_1\mathbf{e}_1 + a_2\mathbf{e}_2)) && \text{because } \mathbf{x} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 \\
&= \mathbf{S}^{-1}\mathbf{A}\,(a_1\mathbf{S}\mathbf{e}_1 + a_2\mathbf{S}\mathbf{e}_2) && \text{linearity of } \mathbf{S} \\
&= \mathbf{S}^{-1}\mathbf{A}\,(a_1\mathbf{x}_1 + a_2\mathbf{x}_2) && \text{because } \mathbf{S}\mathbf{e}_1 = \mathbf{x}_1 \text{ and } \mathbf{S}\mathbf{e}_2 = \mathbf{x}_2 \\
&= \mathbf{S}^{-1}\,(a_1\mathbf{A}\mathbf{x}_1 + a_2\mathbf{A}\mathbf{x}_2) && \text{linearity of } \mathbf{A} \\
&= \mathbf{S}^{-1}\,(a_1\lambda_1\mathbf{x}_1 + a_2\lambda_2\mathbf{x}_2) && \text{because } \mathbf{A}\mathbf{x}_1 = \lambda_1\mathbf{x}_1 \text{ and } \mathbf{A}\mathbf{x}_2 = \lambda_2\mathbf{x}_2 \\
&= a_1\lambda_1\mathbf{S}^{-1}\mathbf{x}_1 + a_2\lambda_2\mathbf{S}^{-1}\mathbf{x}_2 && \text{linearity of } \mathbf{S}^{-1} \\
&= \lambda_1 a_1\mathbf{e}_1 + \lambda_2 a_2\mathbf{e}_2 \\
&= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \\
&= \mathbf{\Lambda}\mathbf{x}.
\end{aligned} \tag{16.20}$$

The diagonal matrix Λ represents a simple scaling (extension or contraction) with the factors λ_1 and λ_2. One may think of the matrix $\mathbf{\Lambda}$ as representing the matrix $\mathbf{A}$ expressed in the basis of $\{\mathbf{x}_1, \mathbf{x}_2\}$ rather than the natural basis $\{\mathbf{e}_1, \mathbf{e}_2\}$.

Geometrical interpretation of diagonalisation

Example 16.2. We note that the symmetric matrix

$$\mathbf{A} = \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix} \tag{16.21}$$

has an matrix of eigenvectors that is orthogonal when the eigenvectors are normalised:

$$\mathbf{S} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\Bigg|_{\theta=+\frac{\pi}{4}} = \mathbf{R}_{+\frac{\pi}{4}}, \tag{16.22}$$

$$\mathbf{S}^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\Bigg|_{\theta=-\frac{\pi}{4}} = \mathbf{R}_{-\frac{\pi}{4}}. \tag{16.23}$$

Hence, we realise that applying the matrix $\mathbf{S}$ rotates the vectors about the origin by an angle $\theta = \frac{\pi}{4}$ (anti-clockwise rotation by angle $\frac{\pi}{4}$) while applying the matrix $\mathbf{S}^{-1}$ rotates the vectors about the origin by an angle $\theta = -\frac{\pi}{4}$ (clockwise rotation by angle $\frac{\pi}{4}$).

Therefore, the transformation $\mathbf{S}$ maps $\mathbf{e}_1$ onto the unit eigenvector $\hat{\mathbf{x}}_1$ for $\mathbf{A}$ and $\mathbf{e}_2$ onto the unit eigenvector $\hat{\mathbf{x}}_2$ for $\mathbf{A}$, that is, $\mathbf{x} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$ is mapped onto $\mathbf{S}\mathbf{x} = a_1\hat{\mathbf{x}}_1 + a_2\hat{\mathbf{x}}_2$, see Fig. 16.2(b).

Now, it is straightforward to apply $\mathbf{A}$ as we know that it acts on its eigenvectors by simple scaling. The transformation $\mathbf{S}$ makes sure that we are expressing $\mathbf{x}$ in the most natural basis for $\mathbf{A}$, namely the eigenvectors. Hence $\mathbf{ASx} = a_1\lambda_1\hat{\mathbf{x}}_1 + a_2\lambda_2\hat{\mathbf{x}}_2$, Fig. 16.2(c).

Finally, we act upon this vector with $\mathbf{S}^{-1}$ which simple rotates the vectors about the origin by an angle $\theta = \frac{\pi}{4}$ (clockwise) such that the unit eigenvectors $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ are mapped back onto the natural basis vectors $\mathbf{e}_1$ and $\mathbf{e}_1$, respectively: $\mathbf{S}^{-1}\mathbf{ASx} = a_1\lambda_1\mathbf{e}_1 + a_2\lambda_2\mathbf{e}_2 = \mathbf{\Lambda x}$, Fig. 16.2(d).

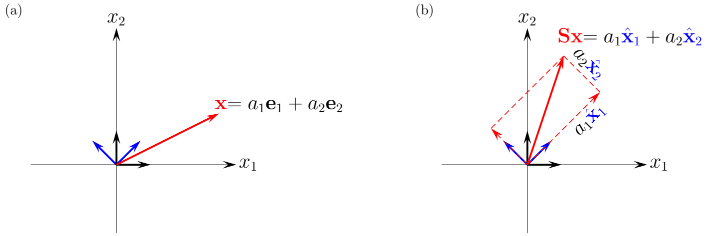

Figure 16.2: The linear function $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ defined by the matrix in Eq. (15.2). In all the graphs (a)-(d), the natural basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ for $\mathbb{R}^2$ and the unit eigenvectors $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ with associated eigenvalues λ_1 and λ_2 of the matrix $\mathbf{A}$ are displayed. (a) An arbitrary vector $\mathbf{x} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$ with coordinates a_1 and a_2 w.r.t. the natural basis vectors. (b) Applying the mapping $\mathbf{S}$, the vector $\mathbf{x}$ is rotated about the origin by an angle $\theta = \frac{\pi}{4}$. The vector $\mathbf{Sx} = a_1\hat{\mathbf{x}}_1 + a_2\hat{\mathbf{x}}_2$ has coordinates a_1 and a_2 w.r.t. the eigenvectors for the matrix $\mathbf{A}$.

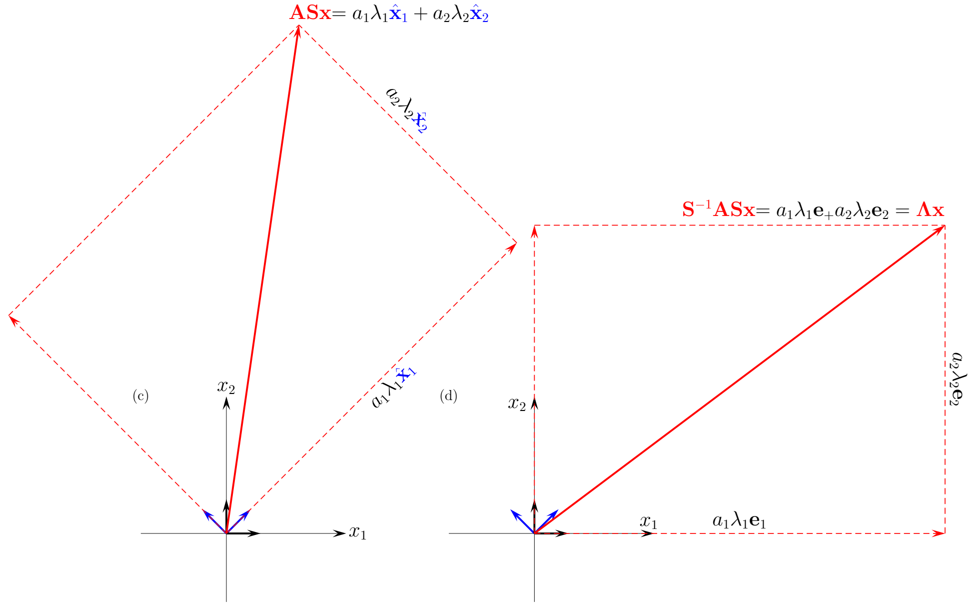

Figure 16.2: Continued. (c) Applying the mapping $\mathbf{A}$, the vector $\mathbf{Sx}$ is expanded by a factor λ_1 along the direction $\hat{\mathbf{x}}_1$ and by a factor λ_2 along the direction $\hat{\mathbf{x}}_2$. The vector $\mathbf{ASx} = a_1\lambda_1\hat{\mathbf{x}}_1 + a_2\lambda_2\hat{\mathbf{x}}_2$ has coordinates $a_1\lambda_1$ and $a_2\lambda_2$ w.r.t. the eigenvectors for the matrix $\mathbf{A}$. (d) Applying the mapping $\mathbf{S}^{-1}$, the vector $\mathbf{ASx}$ is rotated about the origin by an angle $\theta = -\frac{\pi}{4}$. The vector $\mathbf{S}^{-1}\mathbf{ASx} = a_1\lambda_1\mathbf{e}_1 + a_2\lambda_2\mathbf{x}_2$ has coordinates $a_1\lambda_1$ and $a_2\lambda_2$ w.r.t. the natural basis for $\mathbb{R}^2$. Hence the mapping $\mathbf{S}^{-1}\mathbf{AS} = \mathbf{\Lambda}$.

16.3 Eigenvalues and eigenvectors for special matrices

The identity matrix

Consider the 2×2 identity matrix

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{16.24}$$

Clearly,

$$\mathbf{Ix} = \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^2. \tag{16.25}$$

Hence, all vectors $\mathbf{x} \in \mathbb{R}^2$ except $\mathbf{x} = \mathbf{0}$ are eigenvectors of $\mathbf{I}$ with eigenvalue $\lambda = 1$.

Indeed, solving the characteristic equation we find that

$$\det(\mathbf{I} - \lambda\mathbf{I}) = 0 \Leftrightarrow \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} \Leftrightarrow (1-\lambda)^2 = 0 \Leftrightarrow \lambda = 1 \quad \text{double root}, \tag{16.26}$$

that is, *all* the eigenvalues equal 1. The eigenvalues are *degenerated* in this case two-fold.

As usual, note that

$$\lambda_1 \cdot \lambda_2 = 1 = \det \mathbf{I}, \tag{16.27a}$$
$$\lambda_1 + \lambda_2 = 2 = \text{Trace}\, \mathbf{I}. \tag{16.27b}$$

To determine the eigenvectors, we have to solve the associated homogeneous equation which we write on component form:

$$(\mathbf{I} - \lambda\mathbf{I})\,\mathbf{x} = \mathbf{0} \Leftrightarrow \begin{pmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\Leftrightarrow \begin{cases} (1-\lambda)x = 0, \\ (1-\lambda)y = 0. \end{cases} \tag{16.28}$$

Inserting the value $\lambda = 1$, we find

$$\begin{cases} 0 = 0, \\ 0 = 0. \end{cases} \tag{16.29}$$

There are no constraints at all. Therefore, we may choose the eigenvectors arbitrarily. However, the convention is to choose two linear independent and orthogonal unit vectors, namely $\hat{\mathbf{x}}_1 = \mathbf{e}_1$ and $\hat{\mathbf{x}}_2 = \mathbf{e}_2$ as representative eigenvectors for $\mathbf{I}$. Any other eigenvector can be written as a linear combination of those two so the eigenvectors $\{\mathbf{e}_1, \mathbf{e}_2\}$ will span the space of eigenvectors.

Also note the the matrix of eigenvectors $\mathbf{S} = \mathbf{I}$ and clearly $\mathbf{S}^{-1}\mathbf{I}\mathbf{S} = \mathbf{I} = \det(\lambda_1, \lambda_2)$ with $\lambda_1 = \lambda_2 = 1$.

The rotation matrix

Consider the 2×2 rotation matrix

$$\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{16.30}$$

When $\theta \neq 0$, clearly this matrix has *no* eigenvectors in $\mathbb{R}^2$ because it rotates the vectors about the origin by an angle θ and therefore can not map a vector onto a numerical multiple of itself. Well, let us apply the machinery anyway to see where that will take us. To make the situation as trivial as possible, let us consider $\theta = \frac{\pi}{2}$ where

$$\mathbf{R}_{\frac{\pi}{2}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{16.31}$$

Note that $\mathbf{R}^t_{\frac{\pi}{2}} = -\mathbf{R}_{\frac{\pi}{2}}$. We say that the matrix is *anti-symmetric*. Being anti-symmetric is, in some sense, the opposite of being symmetric.

Definition 16.1. A $n \times n$ matrix $\mathbf{A}$ is *symmetric* if $\mathbf{A} = \mathbf{A}^t$. The matrix is *anti-symmetric* if $\mathbf{A} = -\mathbf{A}^t$.

To determine the eigenvalues, we solve the characteristic equation

$$\det\left(\mathbf{R}_{\frac{\pi}{2}} - \lambda\mathbf{I}\right) = 0 \Leftrightarrow \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} \Leftrightarrow \lambda^2 + 1 = 0 \Leftrightarrow \lambda = \begin{cases} +i, \\ -i. \end{cases} \tag{16.32}$$

that is, the eigenvalues are imaginary numbers!

Note, however that, as usual

$$\lambda_1 \cdot \lambda_2 = +1 = \det \mathbf{R}_{\frac{\pi}{2}}, \tag{16.33a}$$
$$\lambda_1 + \lambda_2 = 0 = \text{Trace}\, \mathbf{R}_{\frac{\pi}{2}}. \tag{16.33b}$$

Therefore, clearly, $\mathbf{R}_{\frac{\pi}{2}}$ does not have any real eigenvectors in $\mathbb{R}^2$. However, if we defined the matrix in the complex space $\mathbb{C}^2$, we could continue. To determine the eigenvectors, we have to solve the associated homogeneous equation which we write on component form:

$$\left(\mathbf{R}_{\frac{\pi}{2}} - \lambda\mathbf{I}\right)\mathbf{x} = \mathbf{0} \Leftrightarrow \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\Leftrightarrow \begin{cases} -\lambda x - y = 0 \\ x - \lambda y = 0. \end{cases} \tag{16.34}$$

<u>$\lambda_1 = +i$:</u>
Inserting the value $\lambda_1 = +i$, we find

$$\begin{cases} -ix_1 - y_1 = 0, \\ x_1 - iy_1 = 0. \end{cases} \quad \Leftrightarrow y_1 = -ix_1. \tag{16.35}$$

Letting $x_1 = c_1, c_1 \neq 0$, we find that

$$\mathbf{x}_1 = c_1 \begin{pmatrix} 1 \\ -i \end{pmatrix} \quad \text{for } c_1 \neq 0. \tag{16.36}$$

<u>$\lambda_2 = -i$:</u>
Inserting the value $\lambda_2 = -i$, we find

$$\begin{cases} ix_2 - y_2 = 0, \\ x_2 + iy_2 = 0. \end{cases} \quad \Leftrightarrow y_2 = ix_2. \tag{16.37}$$

Letting $x_2 = c_2, c_2 \neq 0$, we find that

$$\mathbf{x}_2 = c_2 \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{for } c_2 \neq 0. \tag{16.38}$$

We will not dwell further on matrices with complex eigenvalues, as we are only interested in real spaces but, of course, a whole new wonderful world would reveal itself were we allowed to look into the space $\mathbb{C}^2$ or in general $\mathbb{C}^n$. ◡̈

16.4 Summary

We considered a linear function $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ with its associated matrix $\mathbf{A}$ and further discussed the eigenvalue problem $f(\mathbf{x}) = \lambda\mathbf{x} \Leftrightarrow \mathbf{A}\mathbf{x} = \lambda\mathbf{x}$. After studying Sec. 16, you should

- know that for a (real) symmetric matrix $\mathbf{A}$, that is, $\mathbf{A} = \mathbf{A}^t$, the eigenvectors $\mathbf{x}_1, \mathbf{x}_2$ corresponding to distinct eigenvalues $\lambda_1 \neq \lambda_2$ are perpendicular, $\mathbf{x}_1 \cdot \mathbf{x}_2 = 0$
- be able to construct an *orthonormal set of eigenvectors*, $\hat{\mathbf{x}}_i \cdot \hat{\mathbf{x}}_j = \delta_{ij}$
- be able to construct an *orthogonal* matrix of eigenvectors $\mathbf{S} = \begin{pmatrix} \hat{\mathbf{x}}_1 & \hat{\mathbf{x}}_2 & \cdots & \hat{\mathbf{x}}_n \end{pmatrix}$ where $\mathbf{S}^{-1} = \mathbf{S}^t$
- be able to interpret $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{\Lambda}$ mathematically and geometrically
- know that some matrices have degenerated eigenvalues or complex eigenvalues.

17 Applications of eigenvalues and eigenvectors

The aim is to discuss a few applications of the eigenvalues and the associated eigenvectors of a linear function $f : \mathbb{R}^n \to \mathbb{R}^n$ with its $n \times n$ matrix $\mathbf{A}$. In particular, we consider

- powers $\mathbf{A}^k$ of a diagonalisable matrix $\mathbf{A}$
- trace of $\mathbf{A}^k$ of a diagonalisable matrix $\mathbf{A}$
- difference equation and powers of a matrix, exemplified by the Fibonacci sequence
- system of n coupled linear differential equations in n unknowns

17.1 Powers and trace of powers of a diagonalisable matrix

Consider a (real) symmetric $n \times n$ matrix $\mathbf{A}$. For such a matrix we know that there exists a matrix of eigenvectors $\mathbf{S}$ such that

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{\Lambda} \tag{17.1}$$

where $\mathbf{\Lambda} = \text{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a diagonal matrix with the eigenvalues of the matrix $\mathbf{A}$ as its entries.

Say that we want to evaluate the kth power of the matrix $\mathbf{A}$. If k is small, we might do this by brute force but as soon as k becomes large, this is not a viable option anymore. Indeed, when solving problems in physics, in particular in statistical mechanics, one often has to calculate

$$\mathbf{A}^k = \underbrace{\mathbf{A}\mathbf{A}\cdots\mathbf{A}}_{k \text{ factors}}, \tag{17.2}$$

k being at the order of Avogadro's number $N_A = 6.022 \cdot 10^{23}$. What can we do?

First, we notice that from Eq. (17.1) we have

$$\mathbf{A} = \mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}. \tag{17.3}$$

Hence,

$$\begin{aligned} \mathbf{A}^2 &= \left(\mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}\right)\left(\mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}\right) \\ &= \mathbf{S}\mathbf{\Lambda}\left(\mathbf{S}^{-1}\mathbf{S}\right)\mathbf{\Lambda}\mathbf{S}^{-1} && \text{product is associative} \\ &= \mathbf{S}\mathbf{\Lambda}\mathbf{I}\mathbf{\Lambda}\mathbf{S}^{-1} && \text{because } \mathbf{S}^{-1}\mathbf{S} = \mathbf{I} \\ &= \mathbf{S}\mathbf{\Lambda}\mathbf{\Lambda}\mathbf{S}^{-1} \\ &= \mathbf{S}\mathbf{\Lambda}^2\mathbf{S}^{-1} \end{aligned} \tag{17.4}$$

and by simple generalisation

$$\begin{aligned} \mathbf{A}^k &= \underbrace{\left(\mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}\right)\left(\mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}\right)\cdots\left(\mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}\right)}_{k \text{ factors}} \\ &= \mathbf{S}\mathbf{\Lambda}^k\mathbf{S}^{-1}. \end{aligned} \tag{17.5}$$

A diagonal matrix is easy to raise to a power k. For example, for a 2×2 diagonal matrix

$$\mathbf{\Lambda}^2 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix} \tag{17.6a}$$

and similarly,

$$\mathbf{\Lambda}^k = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cdots \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix} \tag{17.6b}$$

or, for a general $n \times n$ diagonal matrix $\mathbf{\Lambda}$,

$$\mathbf{\Lambda}^k = \text{diag}\left(\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k\right). \tag{17.7}$$

Hence, according to the formula

$$\mathbf{A}^k = \mathbf{S}\mathbf{\Lambda}^k\mathbf{S}^{-1}, \tag{17.8}$$

the task has been reduced to determine the eigenvalues and eigenvectors for the matrix $\mathbf{A}$ such that you can construct $\mathbf{S}, \mathbf{S}^{-1}$ and $\mathbf{\Lambda}^k$ and evaluate the right-hand-side of Eq. (17.8).

Trace of the matrix $\mathbf{A}^k$

Often, it is not the power of a matrix $\mathbf{A}^k$ one is looking for but rather the trace of that matrix, that is, Trace $\mathbf{A}^k$. We might use Eq. (17.8), calculate the right-hand-side and then take the trace, that is, the sum over the diagonal elements. However, we can do better than that. Much better. In fact, we will shortly realise that if you want to evaluate Trace $\mathbf{A}^k$, we do *not* need to find the eigenvectors and evaluate $\mathbf{S}$ and $\mathbf{S}^{-1}$ but we just need the eigenvalues themselves.

First we show that the trace of the product of two matrices is independent of the order of their multiplication.

Theorem 17.1. Let $\mathbf{A}$ and $\mathbf{B}$ denote two $n \times n$ matrices. Then Trace $(\mathbf{AB})$ = Trace $(\mathbf{BA})$.

Proof: The trace is the sum over all diagonal elements. Therefore,

$$\text{Trace}\,(\mathbf{AB}) = \sum_{i=1}^{n} (\mathbf{AB})_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji} = \sum_{j=1}^{n}\sum_{i=1}^{n} b_{ji}a_{ij} = \sum_{j=1}^{n} (\mathbf{BA})_{jj} = \text{Trace}\,(\mathbf{BA}). \tag{17.9}$$

Q.E.D.

Theorem 17.2. If $\mathbf{A}$ is a (real) symmetric $n \times n$ matrix then the trace of the kth power of the matrix $\mathbf{A}$ equals the sum of the kth power of the n associated eigenvalues, that is,

$$\text{Trace}\,\mathbf{A}^k = \lambda_1^k + \lambda_2^k + \cdots + \lambda_n^k = \sum_{j=1}^{n} \lambda_j^k. \tag{17.10}$$

Proof: Because $\mathbf{A}$ is a real symmetric $n \times n$ matrix, there exist a matrix of eigenvectors such that $\mathbf{S}^{-1}\mathbf{AS} = \mathbf{\Lambda}$ where $\mathbf{\Lambda}$ is the diagonal matrix of eigenvalues for the matrix $\mathbf{A}$. Therefore,

we also have $\mathbf{A}^k = \mathbf{S}\mathbf{\Lambda}^k\mathbf{S}^{-1}$ so that

$$\begin{aligned}
\text{Trace}\,\mathbf{A}^k &= \text{Trace}\,\left(\mathbf{S}\mathbf{\Lambda}^k\mathbf{S}^{-1}\right) \\
&= \text{Trace}\,\left(\mathbf{S}\left(\mathbf{\Lambda}^k\mathbf{S}^{-1}\right)\right) \\
&= \text{Trace}\,\left(\left(\mathbf{\Lambda}^k\mathbf{S}^{-1}\right)\mathbf{S}\right) && \text{using Trace}\,(\mathbf{AB}) = \text{Trace}\,(\mathbf{BA}) \\
&= \text{Trace}\,\left(\mathbf{\Lambda}^k\left(\mathbf{S}^{-1}\mathbf{S}\right)\right) && \text{product is associative} \\
&= \text{Trace}\,\left(\mathbf{\Lambda}^k\mathbf{I}\right) && \text{because } \mathbf{S}^{-1}\mathbf{S} = \mathbf{I} \\
&= \text{Trace}\,\mathbf{\Lambda}^k \\
&= \sum_{j=1}^{n} \lambda_j^k.
\end{aligned} \tag{17.11}$$

Q.E.D.

Note that often one is considering $k \gg 1$. Therefore, the dominating term in the Trace $\mathbf{A}^k = \sum_{j=1}^{n} \lambda_j^k$ will be the term with the largest eigenvalue. It is the direction along the eigenvector associated with the largest eigenvalue that will determine the outcome or result when $k \gg 1$. Therefore, it is common to order the eigenvalues in monotonic order, say $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. If $\lambda_1 > \lambda_2$, we have that for $k \gg 1$, Trace $\mathbf{A}^k \approx \lambda_1^k$.

17.2 Difference equations and powers of a matrix

Fibonacci's numbers

The *Fibonacci numbers* are the numbers in the sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, \ldots$$

The first two numbers are 0 and 1. The next term in the sequence is the sum of the two previous terms. So entry 3 is $0 + 1 = 1$, entry 4 is $1 + 1 = 2$, entry 5 is $2 + 3 = 5$ and so on. Mathematising this, we have that the $k + 1$ entry, F_{k+1}, is the sum of entry k, F_k, and entry $k - 1$, F_{k-1}, that is,

$$F_{k+1} = F_k + F_{k-1}, \tag{17.12}$$

with the initial condition $F_0 = 0$ and $F_1 = 1$. It is known than the ratio for large n tends to the Golden ratio ϕ, that is

$$\lim_{k\to\infty} \frac{F_k}{F_{k-1}} = \frac{1 + \sqrt{5}}{2} = \phi \approx 1.618. \tag{17.13}$$

What is entry number 100 in the Fibonacci sequence? How fast do the entries in the Fibonacci sequence grow? Why does the ratio of two consecutive numbers in the sequence tend to the Golden ratio as k becomes large? The answers to all these questions are given by the eigenvalues and eigenvectors of a 2×2 matrix.

First, let of reformulate the definition of the Fibonacci sequence in terms of linear algebra. We might add a second trivial equation to the definition of the Fibonacci numbers that will facilitate the formulation of this problem in terms of linear algebra. Adding the trivial identity $F_k = F_k$ to Eq. (17.12) we find

$$F_{k+1} = F_k + F_{k-1}, \tag{17.14a}$$
$$F_k = F_k. \tag{17.14b}$$

Hence we have

$$\underbrace{\begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix}}_{\mathbf{u}_k} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} F_k \\ F_{k-1} \end{pmatrix}}_{\mathbf{u}_{k-1}} \quad \text{for } k \geq 1. \tag{17.15}$$

It is therefore convenient to introduce $\mathbf{u}_k$, a 2×1 vector in $\mathbb{R}^2$ with entries

$$\mathbf{u}_k = \begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix} \quad \text{with initial condition} \quad \mathbf{u}_0 = \begin{pmatrix} F_1 \\ F_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{17.16}$$

and the matrix 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \tag{17.17}$$

such that Eq. (17.15) reads

$$\mathbf{u}_k = \mathbf{A}\mathbf{u}_{k-1}. \tag{17.18}$$

Equation (17.18) is an example of a difference equation $\mathbf{u}_k = g(\mathbf{u}_{k-1}, \mathbf{u}_{k-2}, \ldots, \mathbf{u}_0)$ where the kth term is given in terms of the previous terms, here only the $(k-1)$th term. In principle, it is easily solved:

$$\mathbf{u}_k = \mathbf{A}\mathbf{u}_{k-1} = \mathbf{A}^2\mathbf{u}_{k-2} = \cdots = \mathbf{A}^k\mathbf{u}_0. \tag{17.19}$$

Hence, the task is reduced to calculate the kth power of the real and symmetric matrix $\mathbf{A}$.

Finding $\mathbf{A}^k$

In order to determine the eigenvalues for $\mathbf{A}$ we must solve the characteristic equation

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) = 0 &\Leftrightarrow \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \\ &\Leftrightarrow \lambda^2 - \lambda - 1 = 0 \\ &\Leftrightarrow \lambda = \frac{1 \pm \sqrt{1 - 4 \cdot 1 \cdot (-1)}}{2} \\ &\Leftrightarrow \lambda = \frac{1 \pm \sqrt{5}}{2}. \end{aligned} \tag{17.20}$$

Hence, the 2×2 matrix $\mathbf{A}$ has 2 eigenvalues, namely

$$\lambda_1 = \frac{1 + \sqrt{5}}{2} = \phi \approx 1.618, \tag{17.21a}$$
$$\lambda_2 = \frac{1 - \sqrt{5}}{2} = (1 - \phi) \approx -0.618. \tag{17.21b}$$

Note that, as usual

$$\lambda_1 \cdot \lambda_2 = \det \mathbf{A} = -1, \tag{17.22a}$$
$$\lambda_1 + \lambda_2 = \operatorname{Trace} \mathbf{A} = 1. \tag{17.22b}$$

Finding the eigenvectors

We have to solve the homogeneous equation $(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{x} = \mathbf{0}$ with respect to the unknown vectors $\mathbf{x}$.

Example 17.1. We rewrite the homogeneous equation on component form

$$\begin{aligned} (\mathbf{A} - \lambda\mathbf{I})\,\mathbf{x} = \mathbf{0} &\Leftrightarrow \begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &\Leftrightarrow \begin{cases} (1-\lambda)x + y = 0, \\ x - \lambda y = 0. \end{cases} \end{aligned} \tag{17.23}$$

Now we have to consider each of the two eigenvalues separately. For clarity, we will use the notation x_1, y_1 for the eigenvector(s) belonging to the eigenvalue λ_1 and x_2, y_2 for the eigenvector(s) belonging to the eigenvalue λ_2.

<u>$\lambda_1 = \frac{1+\sqrt{5}}{2}$:</u>
Inserting the value λ_1 into Eq. (17.23), we find

$$\begin{cases} \lambda_2 x_1 + y_1 = 0 \\ x_1 - \lambda_1 y_1 = 0 \end{cases} \Leftrightarrow x_1 = \lambda_1 y_1. \tag{17.24}$$

The two equations are proportional because multiplying the second equation with λ_2 and using $\lambda_1 \cdot \lambda_2 = -1$ we recover the first equation. Therefore, we find that

$$\mathbf{x}_1 = \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix} \tag{17.25}$$

is an eigenvector associated with the eigenvalue $\lambda_1 = \frac{1+\sqrt{5}}{2}$.

<u>$\lambda_2 = \frac{1-\sqrt{5}}{2}$:</u>
Inserting the value λ_2 into Eq. (17.23), we find

$$\begin{cases} \lambda_1 x_2 + y_2 = 0 \\ x_2 - \lambda_2 y_2 = 0 \end{cases} \Leftrightarrow x_2 = \lambda_2 y_2. \tag{17.26}$$

The two equations are proportional because multiplying the second equation with λ_1 and using $\lambda_1 \cdot \lambda_2 = -1$ we recover the first equation. Therefore, we find that

$$\mathbf{x}_2 = \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix} \tag{17.27}$$

is an eigenvector associated with the eigenvalue $\lambda_1 = \frac{1-\sqrt{5}}{2}$.

Solving the problem

We could now construct the matrix of eigenvectors $\mathbf{S} = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix}$ and then use Eq. (17.8). This is left as an exercise to the reader. We take a slightly different approach, namely to write the initial vector $\mathbf{u}_0$ as a linear combination of the eigenvectors of $\mathbf{A}$. By inspection we see that

$$\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\mathbf{u}_0} = \frac{1}{\sqrt{5}} \underbrace{\begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix}}_{\mathbf{x}_1} - \frac{1}{\sqrt{5}} \underbrace{\begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}}_{\mathbf{x}_2} \Leftrightarrow \mathbf{u}_0 = \frac{1}{\sqrt{5}}\mathbf{x}_1 - \frac{1}{\sqrt{5}}\mathbf{x}_2. \tag{17.28}$$

Substituting this expression for $\mathbf{u}_0$ into Eq.(17.19), we find that

$$\begin{aligned} \mathbf{u}_k &= \mathbf{A}^k \mathbf{u}_0 \\ &= \mathbf{A}^k \left(\frac{1}{\sqrt{5}}\mathbf{x}_1 - \frac{1}{\sqrt{5}}\mathbf{x}_2 \right) \\ &= \frac{1}{\sqrt{5}}\mathbf{A}^k\mathbf{x}_1 - \frac{1}{\sqrt{5}}\mathbf{A}^k\mathbf{x}_2 \quad \text{linearity of } \mathbf{A}^k \\ &= \frac{1}{\sqrt{5}}\lambda_1^k\mathbf{x}_1 - \frac{1}{\sqrt{5}}\lambda_2^k\mathbf{x}_2 \quad \text{because } \mathbf{A}\mathbf{x}_i = \lambda_i\mathbf{x}_i, i = 1,2 \\ &= \frac{1}{\sqrt{5}}\left(\lambda_1^k\mathbf{x}_1 - \lambda_2^k\mathbf{x}_2\right). \end{aligned} \tag{17.29}$$

Writing out the entries explicitly, we find

$$\begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} \lambda_1^{k+1} - \lambda_2^{k+1} \\ \lambda_1^k - \lambda_2^k \end{pmatrix}. \tag{17.30}$$

We are now is a position to answer the questions raised initially. What is entry number 100? We set $k = 100$ and find

$$F_{100} = \frac{1}{\sqrt{5}}\left(\lambda_1^{100} - \lambda_2^{100}\right) = 354,224,848,179,261,915,075. \tag{17.31}$$

How fast do the entries in the sequence grow? For large k, the largest eigenvalue λ_1 controls the growth. We find

$$F_k = \frac{1}{\sqrt{5}}\left(\lambda_1^k - \lambda_2^k\right) = \frac{1}{\sqrt{5}}\lambda_1^k\left(1 - \left(\frac{\lambda_2}{\lambda_1}\right)^k\right) \approx \frac{1}{\sqrt{5}}\lambda_1^k \tag{17.32}$$

for $k \gg 1$ because $|\lambda_2/\lambda_1| < 1$, Therefore, the numbers grow by the factor $\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618$, the Golden ratio.

Finally, why does the ratio of two consecutive numbers in the sequence tend to the Golden ratio as k becomes large? We can calculate the ratio:

$$\frac{F_{k+1}}{F_k} = \frac{\lambda_1^{k+1} - \lambda_2^{k+1}}{\lambda_1^k - \lambda_2^k} \approx \frac{\lambda_1^{k+1}}{\lambda_1^k} = \lambda_1 = \frac{1+\sqrt{5}}{2} = \phi. \tag{17.33}$$

using that λ_2^k tends to zero for $k \gg 1$ because $|\lambda_2| < 1$.

17.3 System of linear differential equations

Let $\mathbf{x}(t) \in \mathbb{R}^n$ denote a vector with n components and let $\mathbf{x}'(t)$ denote the derivative of the vector. Let $\mathbf{A}$ denote an $n \times n$ matrix with constant elements. Then the general form of a system of coupled linear differential equations is

$$\begin{cases} x_1'(t) = a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) \\ x_2'(t) = a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ x_n'(t) = a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t) \end{cases} \Leftrightarrow \tag{17.34a}$$

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \tag{17.34b}$$

which can be written elegantly on matrix form:

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t). \tag{17.35}$$

What are the solutions $\mathbf{x}(t)$ to this system? Well, if it was a simple first-order differential equation

$$x' = ax \Leftrightarrow x(t) = ce^{at}. \tag{17.36}$$

Inspired by this, left us take a trial solution

$$\mathbf{x}(t) = \mathbf{x}_i e^{\lambda_i t}, \tag{17.37}$$

where $\mathbf{x}_i \in \mathbb{R}^n$ is an n-dimensional vector whose entries are time-independent and λ_i is a constant. We find that

$$\mathbf{x}'(t) = \lambda_i \mathbf{x}_i e^{\lambda_i t}, \tag{17.38}$$

so substituting into Eq. (17.35) we find

$$\begin{aligned} \mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) &\Leftrightarrow \lambda_i\mathbf{x}_i e^{\lambda_i t} = \mathbf{A}\mathbf{x}_i e^{\lambda_i t} \\ &\Leftrightarrow \mathbf{A}\mathbf{x}_i e^{\lambda_i t} - \lambda_i\mathbf{x}_i e^{\lambda_i t} = \mathbf{0} \\ &\Leftrightarrow \left(\mathbf{A}\mathbf{x}_i - \lambda_i\mathbf{x}_i\right) e^{\lambda_i t} = \mathbf{0} \\ &\Leftrightarrow \left(\mathbf{A}\mathbf{x}_i - \lambda_i\mathbf{x}_i\right) = \mathbf{0} \qquad \text{because } e^{\lambda_i t} \neq 0 \\ &\Leftrightarrow \mathbf{A}\mathbf{x}_i = \lambda_i\mathbf{x}_i. \end{aligned} \tag{17.39}$$

Hence, the trial solution proposed in Eq. (17.37) is a solution if λ_i is an eigenvalue for $\mathbf{A}$ with an associated eigenvector $\mathbf{x}_i$. Therefore, finding the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and the associated eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ then the general solution is given by the sum:

$$\mathbf{x}(t) = c_1\mathbf{x}_1 e^{\lambda_1 t} + c_2\mathbf{x}_2 e^{\lambda_2 t} + \cdots + c_n\mathbf{x}_n e^{\lambda_n t}. \tag{17.40}$$

Example 17.2. Consider the system of coupled differential equations

$$\begin{cases} x_1'(t) = x_1(t) + 2x_2(t) \\ x_2'(t) = 3x_1(t) + 2x_2(t) \end{cases} \Leftrightarrow \underbrace{\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix}}_{\mathbf{x}'(t)} = \underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}}_{\mathbf{A}} = \underbrace{\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}}_{\mathbf{x}(t)} \tag{17.41}$$

that is,

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t). \tag{17.42}$$

What is the solution to this equation? We have to determine the eigenvalues λ_1 and λ_2 and the associated eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ and then the general solution to this system of coupled linear equations is

$$\mathbf{x}(t) = c_1\mathbf{x}_1 e^{\lambda_1 t} + c_2\mathbf{x}_2 e^{\lambda_2 t}. \tag{17.43}$$

In order to determine the eigenvalues for $\mathbf{A}$, we must solve the characteristic equation

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) = 0 &\Leftrightarrow \begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = 0 \\ &\Leftrightarrow \lambda^2 - 3\lambda - 4 = 0 \\ &\Leftrightarrow \lambda = \frac{3 \pm \sqrt{9 - 4\cdot 1\cdot(-4)}}{2} \\ &\Leftrightarrow \lambda = \frac{3 \pm 5}{2}. \end{aligned} \tag{17.44}$$

Hence, the 2×2 matrix $\mathbf{A}$ has 2 eigenvalues, namely

$$\lambda_1 = 4, \tag{17.45a}$$
$$\lambda_2 = -1. \tag{17.45b}$$

Note that, as usual

$$\lambda_1 \cdot \lambda_2 = \det\mathbf{A} = -4, \tag{17.46a}$$
$$\lambda_1 + \lambda_2 = \operatorname{Trace}\mathbf{A} = 3. \tag{17.46b}$$

Finding the eigenvectors

We have to solve the homogeneous equation $(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{x} = \mathbf{0}$ with respect to the unknown vectors $\mathbf{x}$. We rewrite the homogeneous equation on component form

$$\begin{aligned} (\mathbf{A} - \lambda\mathbf{I})\,\mathbf{x} = \mathbf{0} &\Leftrightarrow \begin{pmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &\Leftrightarrow \begin{cases} (1-\lambda)x + 2y = 0 \\ 3x + (2-\lambda)y = 0. \end{cases} \end{aligned} \tag{17.47}$$

Now we have to consider each of the two eigenvalues separately. For clarity, we will use the notation x_1, y_1 for the eigenvector(s) belonging to the eigenvalue λ_1 and x_2, y_2 for the eigenvector(s) belonging to the eigenvalue λ_2.

$\underline{\lambda_1 = 4}$:
Inserting the value λ_1 into Eq. (17.47), we find

$$\begin{cases} -3x_1 + 2y_1 = 0 \\ 3x_1 - 2y_1 = 0 \end{cases} \Leftrightarrow x_1 = \frac{2}{3}y_1. \tag{17.48}$$

Therefore, we find that

$$\mathbf{x}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{17.49}$$

is an eigenvector associated with the eigenvalue $\lambda_1 = 4$.

$\underline{\lambda_2 = -1}$:
Inserting the value λ_2 into Eq. (17.47), we find

$$\begin{cases} 2x_2 + 2y_2 = 0 \\ 3x_2 + 3y_2 = 0 \end{cases} \Leftrightarrow x_2 = -y_2. \tag{17.50}$$

Therefore, we find that

$$\mathbf{x}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{17.51}$$

The general solution is therefore

$$\begin{aligned} \mathbf{x}(t) &= c_1\mathbf{x}_1 e^{\lambda_1 t} + c_2\mathbf{x}_2 e^{\lambda_2 t} \\ &= c_1 e^{4t}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}. \end{aligned} \tag{17.52}$$

or on component form

$$x_1(t) = 2c_1e^{4t} - c_2e^{-t}, \tag{17.53}$$
$$x_2(t) = 3c_1e^{4t} + c_2e^{-t}. \tag{17.54}$$

The constants c_1 and c_2 would be determined by initial conditions, say, $\mathbf{x}(0) = \mathbf{x}_0$. Let us check that this is really a solution to the system of differential equations we started out with. By differentiating we find

$$x_1'(t) = 8c_1e^{4t} + c_2e^{-t} = \left(2c_1e^{4t} - c_2e^{-t}\right) + 2\left(3c_1e^{4t} + c_2e^{-t}\right) = x_1(t) + 2x_2(t) \tag{17.55a}$$
$$x_2'(t) = 12c_1e^{4t} - c_2e^{-t} = 3\left(2c_1e^{4t} - c_2e^{-t}\right) + 2\left(3c_1e^{4t} + c_2e^{-t}\right) = 3x_1(t) + 2x_2(t). \tag{17.55b}$$

so indeed this is the solution we were looking for.

17.4 Summary

We considered a linear function $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ with its associated matrix $\mathbf{A}$ and discussed some applications of eigenvalues and eigenvectors. After studying Sec. 16, you should

- be able to calculate powers $\mathbf{A}^k$ of a diagonalisable matrix $\mathbf{A}$
- be able to calculate the trace of $\mathbf{A}^k$ of a diagonalisable matrix $\mathbf{A}$
- know that difference equations can be solved using linear algebra. The solution is formulated as powers of a matrix.
- know that a system of n coupled linear differential equations in n unknowns can be solved using the notion of eigenvalues and eigenvectors.

Appendix A

Logic

Let A and B denote two statements that are either true or false.

In mathematics, one often encounter the situation that A implies B or, equivalently, "if A then B", and we write

$$A \Rightarrow B. \tag{A.1}$$

Given that you can prove that $A \Rightarrow B$ you have not proven that B is true. However, if in addition you know or can prove that A is true, then for sure B is also true.

Using the negation, if you can prove that $A \Rightarrow B$ and that B is false, then you know that A is false.

If A implies B ($A \Rightarrow B$) AND B implies A ($B \Rightarrow A$) we say "A if and only if B" and we write

$$A \Leftrightarrow B. \tag{A.2}$$

The statement A if and only if B is only true if both statements A and B true or both statements are false. Given that you can prove $A \Leftrightarrow B$ (using the rules of mathematics) you have not proven that B is true. However, if in addition you know or can prove that A is true then for sure B is true. Similarly, if in addition you know or can prove that B is true then for sure A is true.

Using the negation, if you can prove that $A \Leftrightarrow B$ and that B is false, then you know that A is false. Similarly, if you can prove that $A \Leftrightarrow B$ and that A is false, then you know that B is false.

Example 1: $x = 3 \Rightarrow x^2 = 9$. However, $x^2 = 9$ does not imply that $x = 3$ because $x = -3$ is also a solution. Using the "if and only if", you have $x = \pm 3 \Leftrightarrow x^2 = 9$.

Example 2: If $\mathbf{x} = (3, 4)$ then $|\mathbf{x}| = 5$, that is, $\mathbf{x} = (3, 4) \Rightarrow |\mathbf{x}| = 5$ but clearly, $|\mathbf{x}| = 5$ does not imply that $\mathbf{x} = (3, 4)$, for example, the vector $\mathbf{x} = (0, 5)$ has also a magnitude equal to 5.

Appendix B

Greek alphabet

α, A	alpha
β, B	beta
γ, Γ	gamma
δ, Δ	delta
ϵ, E	epsilon
ζ, Z	zeta
η, H	eta
θ, Θ	theta
ι, I	iota
κ, K	kappa
λ, Λ	lambda
μ, M	mu
ν, N	nu
ξ, Ξ	xi
π, Π	pi
o, O	omicron
ρ, P	rho
σ, Σ	sigma
τ, T	tau
υ, Y	upsilon
ϕ (or φ), Φ	phi
χ, X	chi
ψ, Ψ	psi
ω, Ω	omega

Appendix C

Mathematical symbols

Notation	Meaning	Example
$\{\dots\}$	A set of objects	$A = \{a, b, c\}$ means a, b and c in no special order are the elements of the set A.
$\emptyset$	The empty set	The set with no elements. NB. *not* the same as set of one element, the number zero $\{0\} \neq \emptyset$.
$\in$	Is an element of *or* belongs to	If $A = \{a, b, c\}$ then $a \in A$
$\notin$	Is *not* an element of	If $A = \{a, b, c\}$ then $d \notin A$.
$\subset$	Subset of but *not* equal to	$\{a, b\} \subset \{a, b, c\}$.
$\subseteq$	Subset of *or* possibly equal to	$\{a, b, c\} \subseteq \{a, b, c\}$.
$\cup$	Union	$\{a, b, c\} \cup \{b, d, e\} = \{a, b, c, d, e\}$.
$\cap$	Intersection	$\{a, b, c\} \cap \{b, d, e\} = \{b\}$.
$\vert$	Subject to (in the context of sets)	$A = \{x \vert x \in \mathbb{R}, x > 0, x < 1\}$ means $A = (0, 1)$.
$\mathbb{N}$	Set of all Natural numbers	$\mathbb{N} = \{0, 1, 2, 3, \dots\}$ NB. sometimes 0 is not included.
$\mathbb{Z}$	Set of all integers	$-2 \in \mathbb{Z}$, but $-2/3 \notin \mathbb{Z}$. German *Zahl* – number.
$\mathbb{Q}$	Set of all rationals $\mathbb{Q} = \{x \vert x = a/b, a, b \in \mathbb{Z}\}$	$-2/3 \in \mathbb{Q}$, but $\sqrt{2} \notin \mathbb{Q}$. *Quotient*
$\mathbb{R}$	Set of all Real Numbers	$-3.51 \in \mathbb{R}$, $\pi \in \mathbb{R}$ but $3 + 2i \notin \mathbb{R}$, $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$
$\mathbb{C}$	Set of all Complex Numbers	$-3.51 + \pi i \in \mathbb{C}$, $\mathbb{R} \subset \mathbb{C}$
$\mathbb{R}^n$	Set of all n-tuples $(x_1, \dots, x_n), x_i \in \mathbb{R}$	$(2, 3.2, -1) \in \mathbb{R}^3$
$\mathbb{A}^+$	Positive elements of set A	$+3.51 \in \mathbb{R}^+$, $0 \not\subset \mathbb{Z}^+$
$\circ$	Function composition	$(f \circ g)(x) := f(g(x))$
Domain of f	Set on which function f is defined	$f : \mathbb{R} \to \mathbb{R}^2$, $\mathbb{R} =$ Domain of$f$.
Range of f	Set of values assumed by function f	$f : \mathbb{R} \to \mathbb{R}^2$, Range of $f \subseteq \mathbb{R}^2$.
$\mathbf{x}, \vec{x}, \underline{x}$	Vector, element in $\mathbb{R}^n$	$\mathbf{x} = (2, -3, 5) \in \mathbb{R}^3$.
$\vert\mathbf{x}\vert$	Magnitude of vector $\sqrt{x_1^2 + \cdots + x_n^2}$	$\mathbf{x} = (2, -3, 5) \Rightarrow \vert\mathbf{x}\vert = \sqrt{38}$.

Appendix D

Quadratic equation

The quadratic equation

$$ax^2 + bx + c = 0 \tag{D.1}$$

has two roots, namely

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \tag{D.2a}$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{D.2b}$$

We find that the product of the roots and that the sum of the roots

$$x_1 \cdot x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \cdot \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{b^2 - \left(\sqrt{b^2 - 4ac}\right)^2}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a} \tag{D.3a}$$

$$x_1 + x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-b - b}{2a} = \frac{-b}{a}. \tag{D.3b}$$

Hence, if the quadratic equation is on a form where the coefficient of x^2 is unity, for example,

$$\lambda^2 - \mathrm{Trace}\mathbf{A}\ \lambda + \det\mathbf{A} = 0, \tag{D.4}$$

then we have (because $a = 1, b = -\mathrm{Trace}\mathbf{A}$ and $c = \det\mathbf{A}$) that

$$\lambda_1 \cdot \lambda_2 = \det\mathbf{A}, \tag{D.5a}$$

$$\lambda_1 + \lambda_2 = \mathrm{Trace}\mathbf{A}. \tag{D.5b}$$