1 Vectors

See Chapter 7 in Riley, Hobson, & Bence.

Definition		length, direction, coordinate representation, transformation properties
Dot product	$\mathbf{A}\cdot\mathbf{B}$	$= A_xB_x + A_yB_y + A_zB_z$
		$= \lvert\mathbf{A}\rvert\,\lvert\mathbf{B}\rvert\cos\theta_{AB}$
Projection	$\mathbf{A}$ onto $\mathbf{B}$	$= \mathbf{A}\cdot\mathbf{B}/\lvert\mathbf{B}\rvert$
Area of triangle		$\lvert\mathbf{A}\times\mathbf{B}\rvert/2$
Volume of parallelpiped		$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}$
Triple vector product	$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$	$= \mathbf{B}\,(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}\,(\mathbf{A}\cdot\mathbf{B})$

2 Lines and Planes

See Chapter 7 in Riley, Hobson, & Bence.

In the following $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ is a position vector, with $\mathbf{r}_o$ a specific point on the line or plane in question. $\hat{\mathbf{n}}$ is a unit vector normal to the plane: $\hat{\mathbf{n}} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$ in these cases with $a^2 + b^2 + c^2 = 1$.

Line in 2D	y	$= mx + b$
Line in 3D	$\dfrac{x - x_o}{a}$	$= \dfrac{y = y_o}{b} = \dfrac{z - z_o}{c} = \lambda$
Line in 3D	$\mathbf{r}$	$= \mathbf{r}_o + \lambda\mathbf{u}$ where $\mathbf{u}$ is in the direction of the line.
Plane distance d from O	$\mathbf{r}\cdot\hat{\mathbf{n}}$	$= d$
	$ax + by + cz$	$= d$
	$\mathbf{r}$	$= \mathbf{r}_o + \lambda\mathbf{A} + \beta\mathbf{B}$
	$\hat{\mathbf{n}}$	$= \mathbf{A}\times\mathbf{B}/\lvert\mathbf{A}\times\mathbf{B}\rvert$

where $\mathbf{A}$ and $\mathbf{B}$ are non-parallel vectors that are both parallel to the plane

3 Derivatives

Definition	$\dfrac{df}{dx}$	$= \lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x) - f(x)}{\Delta x}$
Chain rule	$\dfrac{df(g(x))}{dx}$	$= \dfrac{df}{dg}\dfrac{dg}{dx}$
Product rule	$\dfrac{d(fg)}{dx}$	$= f\dfrac{dg}{dx} + \dfrac{df}{dx}g$
Partial derivative	$\dfrac{\partial f(x,y)}{\partial x}$	$= \lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$
Chain rule	$\dfrac{\partial f}{\partial u}$	$= \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial u}$
Notation	∂_x	$\equiv \dfrac{\partial}{\partial x}$

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6 January 2011

Orthogonal Curvilinear Coordinates

Contents

1 Introduction

Many problems in physics have a central point or axis. For these, cartesian (x, y, z) coordinates can be tedious, and it is natural to introduce a coordinate system that reflects the shapes and symmetries of the problem. Examples include cylindrical and spherical polar coordinates, which we shall explore further here, parabolic or hyperbolic coordinates, and others. Many are constructed so that the corresponding unit vectors $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$, $(\hat{\rho}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{k}})$, etc., are orthogonal (i.e., perpendicular to one another). Since in these systems lines of constant components (e.g., constant r) are curved, we refer to such coordinate systems as "orthogonal curvilinear coordinates." Below is a summary of the main aspects of two of the most important systems, cylindrical and spherical polar coordinates. Many of the steps presented take subtle advantage of the orthogonal nature of these systems.

You can find complementary material in both Rile et al., *Mathematical Methods for Physics and Engineering*, Sections 10.9 and 10.10, and in Boas, *Mathematical Methods in the Physics Sciences*, Chapter 10, Sections 6–9. These approaches tend to be more mathematical and general than the one given here.

You will not necessarily be expected to reproduce the calculations given below. However, much of it is quite instructive in terms of the way cylindrical and spherical polar coordinates work. Additionally, seeing how the forms of grad, div, and curl, together with line, surface and volume elements, are derived in different systems helps provide some insight into their interpretation.

2 Cylindrical Polar Coordinates

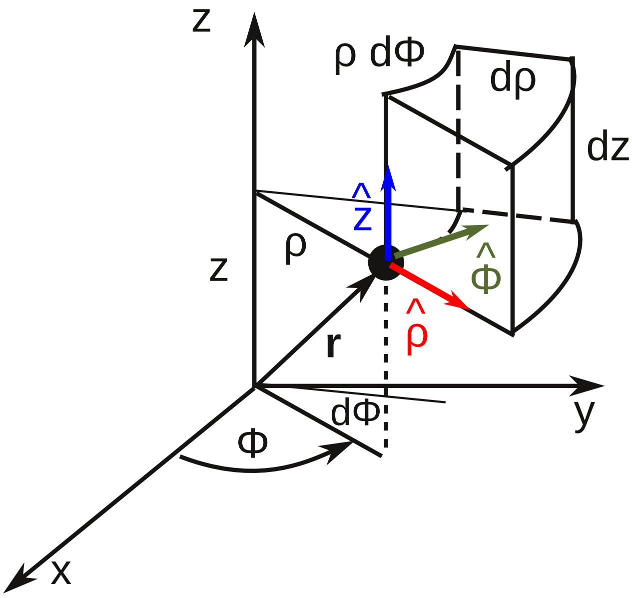

Figure 1: Cylindrical polar coordinates and their volume element.

To begin, let us recall some basics about cylindrical polar coordinates (see Figure 1). From this figure it is apparent that

$$x = \rho \cos \phi \tag{1}$$
$$y = \rho \sin \phi \tag{2}$$
$$z = z \tag{3}$$

It is quite common to use the symbol r instead of ρ, and you will often encounter the notation (r, ϕ, z) for cylindrical polar coordinates. Here, to avoid all possible confusion with r in spherical polars, we will use ρ for the distance from the z-axis.

2.1 Position Vector

The point P indicated by the black dot in Figure 1 has position vector

$$\mathbf{r} = \rho\,\hat{\boldsymbol{\rho}} + z\,\hat{\mathbf{k}} \tag{4}$$

At first sight this might seem strange; where is the $\hat{\boldsymbol{\phi}}$ component? But if you look at Figure 1 you can see that you do need to know three things to locate P: ρ, z and $\hat{\rho} = \hat{\rho}(\phi)$. So the information about ϕ is contained in $\hat{\rho}$; different ϕ values give you $\hat{\rho}$ unit vectors that point in different directions. A general vector field $\mathbf{B}$ will have three components $\mathbf{B}(\rho, \phi, z) = B_\rho\,\hat{\boldsymbol{\rho}} + B_\phi\,\hat{\boldsymbol{\phi}} + B_z\,\hat{\mathbf{k}}$; the position vector is special in this regard.

From Figure 1 or from (1)–(3) we see that

$$\begin{aligned}
\mathbf{r} &= \rho\cos\phi\,\hat{\mathbf{i}} + \rho\sin\phi\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}} &(5)\\
&= \rho\left(\cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}}\right) + z\,\hat{\mathbf{k}} &(6)\\
&= \rho\,\hat{\boldsymbol{\rho}} + z\,\hat{\mathbf{k}} &(7)
\end{aligned}$$

from which we deduce that

$$\hat{\boldsymbol{\rho}} = \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}} \tag{8}$$

From Figure 1 we can find the third unit vector

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}} \tag{9}$$

You can see this either by constructing the necessary trigonometry or simply note that $\hat{\boldsymbol{\phi}}$ must lie in the $x - y$ plane and be perpendicular to $\hat{\rho}$. Indeed, (ρ, ϕ) are just the plane polar coordinates (r, θ) in disguise.

Finally, from (8) we see that

$$\frac{d\hat{\rho}}{d\phi} = \hat{\boldsymbol{\phi}} \tag{10}$$

2.2 Line Element

Now that we have established the representation of a position $\mathbf{r}$ we can proceed to consider a displacement $\mathbf{dr}$ from that position. Graphically with reference to Figure 1, if we increment ρ by an amount $d\rho$ then the vector displacement would be $d\rho\,\hat{\boldsymbol{\rho}}$. Incrementing ϕ by an amount $d\phi$ would be a vector displacement $\rho\,d\phi\,\hat{\boldsymbol{\phi}}$. And incrementing z by dz would be a vector displacement $dz\,\hat{\mathbf{k}}$. An arbitrary displacement would be the vector sum of these, i.e.,

$$\mathbf{dr} = d\rho\,\hat{\boldsymbol{\rho}} + \rho\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{k}} \tag{11}$$

Interestingly, and perhaps reassuringly, you can get to (11) by taking the differential of $\mathbf{r}$ directly from (7):

$$\begin{aligned}
\mathbf{dr} &= \frac{\partial \mathbf{r}}{\partial \rho}\,d\rho + \frac{\partial \mathbf{r}}{\partial \phi}\,d\phi + \frac{\partial \mathbf{r}}{\partial z}\,dz &(12)\\
&= d\rho\,\hat{\boldsymbol{\rho}} + \rho\frac{d\hat{\rho}}{d\phi}\,d\phi + dz\,\hat{\mathbf{k}} &(13)\\
&= d\rho\,\hat{\boldsymbol{\rho}} + \rho\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{k}} &(14)
\end{aligned}$$

where we have made use of (10).

2.3 Surface Elements

To find a surface element $\mathbf{dS}$ we would need to use the formal machinery we derived in lecture, namely if we have a surface S parametrised by two variables (u, v) so that a point on the surface is $\mathbf{r} = \mathbf{r}(u, v)$, then

$$\mathbf{dS} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\,du\,dv \tag{15}$$

Since $\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{k}}$ are mutually orthogonal, you could perform this cross product in cylindrical polars if that was convenient, i.e.,

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \hat{\boldsymbol{\rho}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{k}} \\ \left(\dfrac{\partial \mathbf{r}}{\partial u}\right)_\rho & \left(\dfrac{\partial \mathbf{r}}{\partial u}\right)_\phi & \left(\dfrac{\partial \mathbf{r}}{\partial u}\right)_z \\ \left(\dfrac{\partial \mathbf{r}}{\partial v}\right)_\rho & \left(\dfrac{\partial \mathbf{r}}{\partial v}\right)_\phi & \left(\dfrac{\partial \mathbf{r}}{\partial v}\right)_z \end{vmatrix} \tag{16}$$

Let's look at the surfaces of the elemental volume shown in Figure 1. If we start with the bottom surface, defined by z = constant = z_o say, then we can parametrise this surface by ρ and ϕ, and a point on this surface is described simply by (7) with constant z, i.e., $\mathbf{r}(\rho, \phi) = \rho\,\hat{\rho}(\phi) + z_o\,\hat{\mathbf{k}}$. So we have

$$\frac{\partial \mathbf{r}}{\partial \rho} = \hat{\boldsymbol{\rho}} + 0\,\hat{\boldsymbol{\phi}} + 0\,\hat{\mathbf{k}}; \qquad \frac{\partial \mathbf{r}}{\partial \phi} = 0\,\hat{\boldsymbol{\rho}} + \rho\,\hat{\boldsymbol{\phi}} + 0\,\hat{\mathbf{k}}$$

again making use of (10). Thus

$$\mathbf{dS} = \begin{vmatrix} \hat{\boldsymbol{\rho}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{k}} \\ 1 & 0 & 0 \\ 0 & \rho & 0 \end{vmatrix} d\rho\,d\phi = \rho\,d\rho\,d\phi\,\hat{\mathbf{k}} \tag{17}$$

Actually, you should be able to look at Figure 1, see that this face is roughly rectangular and has an area $\rho\, d\phi \times d\rho$ and that its normal is $\hat{\mathbf{k}}$ and write down immediately (17).

In a similar way it is now easy to see (I hope) that the surface element for the flat-sided side face nearest the viewer is

$$\mathbf{dS} = d\rho\, dz\, \hat{\boldsymbol{\phi}} \tag{18}$$

while that of the inner curved face is

$$\mathbf{dS} = \rho\, d\phi\, dz\, \hat{\boldsymbol{\rho}} \tag{19}$$

Note that for all of these pieces of surface, there is an ambiguity of a ± sign depending on the application at hand. For example, if we were interested in the surface elements with normals pointing *out* of the volume element, we would need to take the negative of all the above **dS** expressions for some faces (see Figure 2 below).

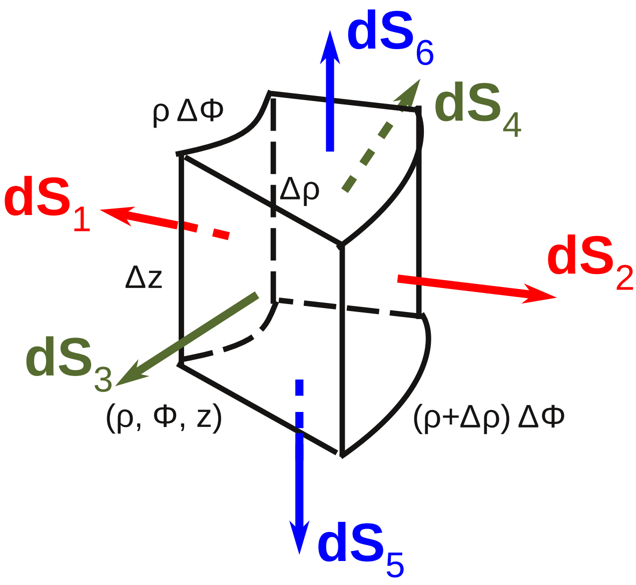

Figure 2: Volume element in cylindrical polars with surface elements marked for application of the Divergence Theorem

2.4 Volume Elements

Again taking advantage of the orthogonality of the unit vectors and looking at Figure 1 we can immediately write down the volume of the differential volume depicted there. This volume is simply the product of the three sides of the (pseudo-)rectangular volume, i.e.,

$$dV = \rho\, d\phi\, d\rho\, dz \tag{20}$$

You can also deduce this by calculating the Jacobian $\dfrac{\partial(x, y, z)}{\partial(\rho, \phi, z)}$ as we did in lecture.

2.5 Gradient Operator

Let's now turn our attention to the vector calculus operators, beginning with the gradient. We will do this by constructing an exact differential of a scalar function f by writing

$$\begin{aligned} df &= \frac{\partial f}{\partial \rho} d\rho + \frac{\partial f}{\partial \phi} d\phi + \frac{\partial f}{\partial z} dz && (21) \\ &= \nabla f \cdot \mathbf{dr} && (22) \end{aligned}$$

If we now express ∇f in components, i.e., $\nabla f = (\nabla f)_\rho \hat{\boldsymbol{\rho}} + (\nabla f)_\phi \hat{\boldsymbol{\phi}} + (\nabla f)_z \hat{\mathbf{k}}$, and make use of (7) for **dr** this gives

$$\begin{aligned} df &= \left((\nabla f)_\rho \hat{\boldsymbol{\rho}} + (\nabla f)_\phi \hat{\boldsymbol{\phi}} + (\nabla f)_z \hat{\mathbf{k}}\right) \cdot \\ &\quad \cdot \left(d\rho\, \hat{\boldsymbol{\rho}} + \rho\, d\phi\, \hat{\boldsymbol{\phi}} + dz\, \hat{\mathbf{k}}\right) \\ &= (\nabla f)_\rho\, d\rho + (\nabla f)_\phi \rho\, d\phi + (\nabla f)_z\, dz && (23) \end{aligned}$$

Now $d\rho$, $d\phi$, and dz are all independent, so their coefficients in (21) and (23) must be equal. Thus $(\nabla f)_\rho = \dfrac{\partial f}{\partial \rho}$, $(\nabla f)_\phi \rho = \dfrac{\partial f}{\partial \phi}$, and $(\nabla f)_z = \dfrac{\partial f}{\partial z}$ so we reach

$$\nabla f = \frac{\partial f}{\partial \rho} \hat{\boldsymbol{\rho}} + \frac{1}{\rho} \frac{\partial f}{\partial \phi} \hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z} \hat{\mathbf{k}} \tag{24}$$

2.6 Divergence Operator

We can derive an expression for the divergence operator of a vector field $\mathbf{B}(\rho, \phi, z)$ by applying the Divergence Theorem to the elemental volume shown in Figure 1 and expanded in Figure 2. The steps below are essentially the reverse of those used to derive the Divergence Theorem in the first place in cartesian coordinates. The underlying concepts are identical, although the algebra here is a bit more involved, essentially because the lengths of some of the sides of the volume element depend on the value of ρ.

The Divergence Theorem states that

$$\iiint_V \nabla \cdot \mathbf{B}\, dV = \oiint_S \mathbf{B} \cdot \mathbf{dS} \tag{25}$$

When applied to an elemental volume, we can remove the integration signs to reveal that

$$\nabla \cdot \mathbf{B} = \sum_{i=1}^{6} \mathbf{B} \cdot \mathbf{dS}_i / dV \tag{26}$$

If we consider $\mathbf{dS}_1$ and $\mathbf{dS}_2$ to begin with, we can see from Figure 2 that

$$\begin{aligned}\mathbf{dS}_1 &= -\rho\,\Delta\phi\,\Delta z\,\hat{\boldsymbol{\rho}}\\ \mathbf{dS}_2 &= +(\rho+\Delta\rho)\,\Delta\phi\,\Delta z\,\hat{\boldsymbol{\rho}}\end{aligned}$$

so that

$$\mathbf{B}\cdot\mathbf{dS}_1 = -B_\rho(\rho,\phi^*,z^*)\rho\,\Delta\phi\,\Delta z \tag{27}$$

$$\mathbf{B}\cdot\mathbf{dS}_2 = +B_\rho(\rho+\Delta\rho,\phi^{**},z^{**})(\rho+\Delta\rho)\,\Delta\phi\,\Delta z \tag{28}$$

where ϕ^* and z^* denote the values of ϕ and z for which B_ρ takes on its average value over $\mathbf{dS}_1$, and similarly for $\mathbf{dS}_2$. By virtue of the mean value theorem, $\phi \le \phi^* \le (\phi+\Delta\phi)$, etc. Combining these two expressions then yields

$$\mathbf{B}\cdot\mathbf{dS}_1 + \mathbf{B}\cdot\mathbf{dS}_2 = \Big[(\rho+\Delta\rho)B_\rho(\rho+\Delta\rho,,) - \rho B_\rho(\rho,,)\Big]\,\Delta\phi\,\Delta z \tag{29}$$

where we have suppressed ϕ^*, etc., quantities for the sake of brevity. Now as we let the volume shrink to differential proportions,

$$\begin{aligned}\Delta\phi &\to d\phi\\ \Delta z &\to dz\\ \phi^* &\to \phi\\ \phi^{**} &\to \phi\\ z^* &\to z\\ z^{**} &\to z\end{aligned}$$

It remains only to let $\Delta\rho$ shrink to its limiting differential form:

$$\begin{aligned}\Delta\rho &\to d\rho\\ (\rho+\Delta\rho)B_\rho(\rho+\Delta\rho) - \rho B_\rho(\rho) &\to \frac{\partial(\rho B_\rho)}{\partial\rho}\,d\rho\end{aligned}$$

so that (29) becomes

$$\mathbf{B}\cdot\mathbf{dS}_1 + \mathbf{B}\cdot\mathbf{dS}_2 = \left(\frac{\partial(\rho B_\rho)}{\partial\rho}\,d\rho\right)d\phi\,dz \tag{30}$$

$$= \frac{1}{\rho}\frac{\partial(\rho B_\rho)}{\partial\rho}\,dV \tag{31}$$

where we have multiplied and divided by ρ in the first line to form the volume element $\rho\,d\rho\,d\phi\,dz \equiv dV$ as in (20).

In a similar way, $\mathbf{dS}_3$ and $\mathbf{dS}_4$ are in the $\mp\hat{\boldsymbol{\phi}}$ direction and in this case both have area $\Delta\rho\,\Delta z$. $\mathbf{dS}_3$ is at ϕ while $\mathbf{dS}_4$ is at $\phi+\Delta\phi$. So the analog to (29) is

$$\mathbf{B}\cdot\mathbf{dS}_3 + \mathbf{B}\cdot\mathbf{dS}_4 = \Big[B_\phi(,\phi+\Delta\phi,) - B_\phi(,\phi,)\Big]\,\Delta\rho\,\Delta z \tag{32}$$

which leads to

$$\mathbf{B}\cdot\mathbf{dS}_3 + \mathbf{B}\cdot\mathbf{dS}_4 = \left(\frac{\partial B_\phi}{\partial\phi}\,d\phi\right)d\rho\,dz \tag{33}$$

$$= \frac{1}{\rho}\frac{\partial B_\phi}{\partial\phi}\,dV \tag{34}$$

Finally, $\mathbf{dS}_{5,6} = \mp\Delta\rho\,\rho\,\Delta\phi\,\hat{\mathbf{k}}$ so that

$$\mathbf{B}\cdot\mathbf{dS}_5 + \mathbf{B}\cdot\mathbf{dS}_6 = [B_z(,,z+\Delta z) - B_z(,,z)]\,\Delta\rho\,\rho\,\Delta\phi \tag{35}$$

which leads to

$$\mathbf{B}\cdot\mathbf{dS}_5 + \mathbf{B}\cdot\mathbf{dS}_6 = \left(\frac{\partial B_z}{\partial z}dz\right)\rho\,d\rho\,d\phi \tag{36}$$

$$= \frac{\partial B_z}{\partial z}\,dV \tag{37}$$

Putting (31), (34), and (37) into the summation in (26) yields the desired result, namely an expression for $\nabla\cdot\mathbf{B}$ in cylindrical polar coordinates:

$$\nabla\cdot\mathbf{B} = \frac{1}{\rho}\frac{\partial(\rho B_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial B_\phi}{\partial\phi} + \frac{\partial B_z}{\partial z} \tag{38}$$

2.7 Curl Operator

A similar approach to that in Section 2.6 can be applied to derive an expression for the curl of a vector field in cylindrical polar coordinates, this time starting from Stoke's Theorem:

$$\oint_C \mathbf{B}\cdot\mathbf{dr} = \iint_S \nabla\times\mathbf{B}\cdot\mathbf{dS} \tag{39}$$

which for a differential surface element reduces to

$$\sum_{i=1}^{n_{edges}} \mathbf{B}\cdot\mathbf{dr}_i = \nabla\times\mathbf{B}\cdot\mathbf{dS} \tag{40}$$

We will pick our surface elements from the faces of the volume element at ρ, ϕ, and z shown in Figure 2.

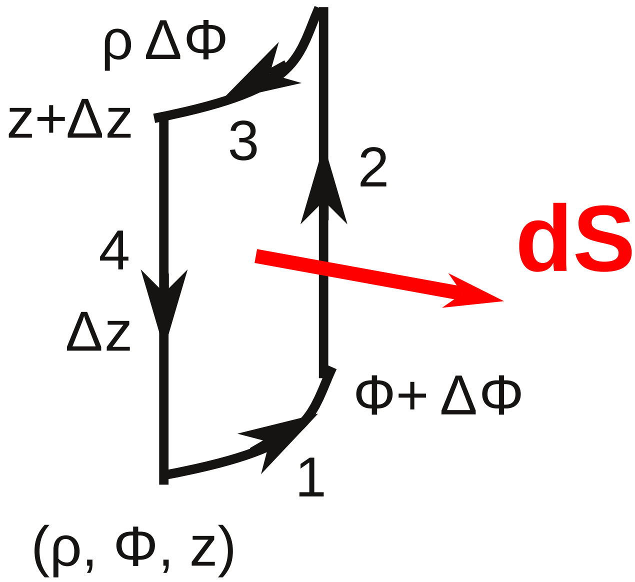

Figure 3: Line integration to determine the $\hat{\rho}$ component of $\nabla \times \mathbf{B}$.

2.7.1 $\hat{\rho}$ Component

The face with normal $\hat{\rho}$ is shown in Figure 3. The four edges have $\mathbf{dr}_1 = +\rho\,\Delta\phi\,\hat{\boldsymbol{\phi}}$, $\mathbf{dr}_2 = +\Delta z\,\hat{\mathbf{k}}$, $\mathbf{dr}_3 = -\rho\,\Delta\phi\,\hat{\boldsymbol{\phi}}$, and $\mathbf{dr}_4 = -\Delta z\,\hat{\mathbf{k}}$ while $\mathbf{dS}_\rho = \rho\,\Delta\phi\,\Delta z\,\hat{\rho}$. Thus (40) becomes

$$\begin{aligned}
&\mathbf{B}\cdot\mathbf{dr}_1 + \mathbf{B}\cdot\mathbf{dr}_2 + \mathbf{B}\cdot\mathbf{dr}_3 + \mathbf{B}\cdot\mathbf{dr}_4 = \\
&\quad B_\phi(\rho,\phi^*,z)\,\rho\,\Delta\phi + B_z(\rho,\phi+\Delta\phi,z^*)\,\Delta z - \\
&\quad -B_\phi(\rho,\phi^{**},z+\Delta z)\,\rho\Delta\phi - B_z(\rho,\phi,z^{**})\,\Delta z \\
&= \left[B_z(\rho,\phi+\Delta\phi,z^*) - B_z(\rho,\phi,z^{**})\right]\,\Delta z - \\
&\qquad -\left[B_\phi(\rho,\phi^{**},z+\Delta z) - B_\phi(\rho,\phi^*,z)\right]\,\rho\Delta\phi \\
&= (\nabla\times\mathbf{B})_\rho\,\rho\,\Delta\phi\,\Delta z \qquad (41)
\end{aligned}$$

with ϕ^*, z^* again denoting the value within the interval for which the relevant component of $\mathbf{B}$ takes on its mean value. As we let this surface shrink to differential proportions, this reduces to:

$$\begin{aligned}
(\nabla\times\mathbf{B})_\rho\,\rho\,d\phi\,dz &= \left(\frac{\partial B_z}{\partial\phi}\,d\phi\right)dz - \left(\frac{\partial B_\phi}{\partial z}\,dz\right)\rho d\phi \\
&= \left(\frac{1}{\rho}\frac{\partial B_z}{\partial\phi} - \frac{\partial B_\phi}{\partial z}\right)\rho\,d\phi\,dz \qquad (42)
\end{aligned}$$

from which we deduce that the ρ component of $\nabla\times\mathbf{B}$ is

$$(\nabla\times\mathbf{B})_\rho = \frac{1}{\rho}\frac{\partial B_z}{\partial\phi} - \frac{\partial B_\phi}{\partial z} \qquad (43)$$

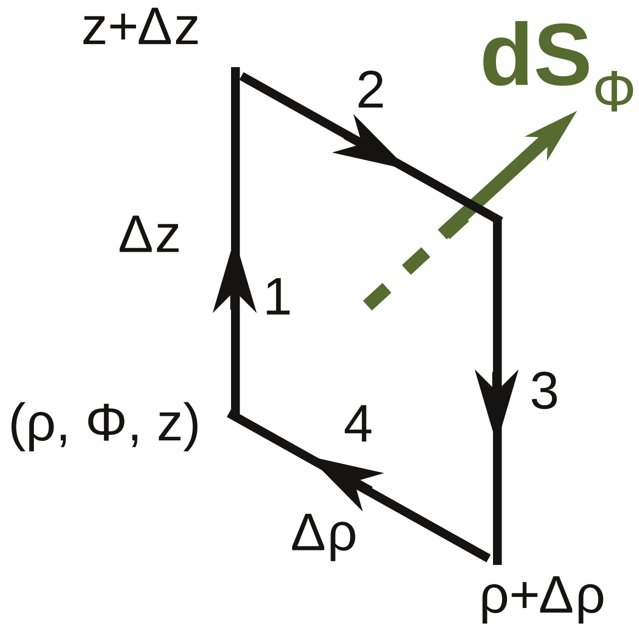

Figure 4: Line integration to determine the $\hat{\phi}$ component of $\nabla \times \mathbf{B}$.

2.7.2 $\hat{\phi}$ Component

The other components of $\nabla\times\mathbf{B}$ follow similarly. Figure 4 shows the $\hat{\phi}$ face. Be careful to ensure that you go around the edges in a right-handed sense with respect to $\mathbf{dS}$, which we have chosen here to be in the $+\hat{\phi}$ direction, so that $\mathbf{dS} = \Delta\rho\,\Delta z\,\hat{\phi}$. Evaluating (40) for this face yields

$$\begin{aligned}
&\mathbf{B}\cdot\mathbf{dr}_1 + \mathbf{B}\cdot\mathbf{dr}_2 + \mathbf{B}\cdot\mathbf{dr}_3 + \mathbf{B}\cdot\mathbf{dr}_4 = \\
&\quad = B_z(\rho,,)\,\Delta z + B_\rho(,,z+\Delta z)\,\Delta\rho - \\
&\quad\quad -B_z(\rho+\Delta\rho,,)\,\Delta z - B_\rho(,,z)\,\Delta\rho \\
&\quad = \left[B_\rho(,,z+\Delta z) - B_\rho(,,z)\right]\,\Delta\rho - \\
&\quad\quad -\left[B_z(\rho+\Delta\rho,,) - B_z(\rho,,)\right]\,\Delta z \\
&\quad = (\nabla\times\mathbf{B})_\phi\,\Delta\rho\,\Delta z \qquad (44)
\end{aligned}$$

Here for brevity and clarity I have omitted the dependencies which are either constant along a particular edge or evaluated at some *'ed value along them while retaining the dependency that characterises which edge we are following.

Letting the surface shrink to its differential form yields

$$\begin{aligned}
&(\nabla\times\mathbf{B})_\phi\,d\rho\,dz = \\
&\quad = \left(\frac{\partial B_\rho}{\partial z}\,dz\right)d\rho - \left(\frac{\partial B_z}{\partial\rho}\,d\rho\right)dz \qquad (45)
\end{aligned}$$

from which we can see that

$$(\nabla\times\mathbf{B})_\phi = \frac{\partial B_\rho}{\partial z} - \frac{\partial B_z}{\partial\rho} \qquad (46)$$

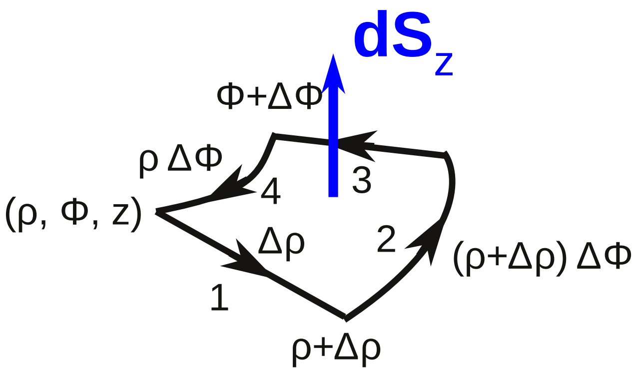

Figure 5: Line integration to determine the $\hat{\mathbf{k}}$ component of $\nabla\times\mathbf{B}$.

2.7.3 $\hat{\mathbf{k}}$ Component

There remains only the $\hat{\mathbf{k}}$-component of $\nabla\times\mathbf{B}$ to calculate, by reference to Figure 5. Following the now-familiar pattern, $\mathbf{dS}_z = \Delta\rho\,\rho\,\Delta\phi\,\hat{\mathbf{k}}$ and

$$\begin{aligned}
&\mathbf{B}\cdot\mathbf{dr}_1 + \mathbf{B}\cdot\mathbf{dr}_2 + \mathbf{B}\cdot\mathbf{dr}_3 + \mathbf{B}\cdot\mathbf{dr}_4 = \\
&\quad = B_\rho(,\phi,)\,\Delta\rho + B_\phi(\rho+\Delta\rho,,)\,(\rho+\Delta\rho)\Delta\phi - \\
&\qquad - B_\rho(,\phi+\Delta\phi,)\,\Delta\rho - B_\phi(\rho,,)\,\rho\Delta\phi \\
&\quad = \left[(\rho+\Delta\rho)B_\phi(\rho+\Delta\rho,,) - \rho B_\phi(\rho,,)\right]\,\Delta\phi \\
&\qquad - \left[B_\rho(,\phi+\Delta\phi,) - B_\rho(,\phi,)\right]\,\Delta\rho \\
&\quad = (\nabla\times\mathbf{B})_z\;\rho\Delta\rho\,\Delta\phi
\end{aligned} \tag{47}$$

Again shrinking to differential size yields

$$\begin{aligned}
&(\nabla\times\mathbf{B})_z\;\rho\,d\rho\,d\phi = \\
&\quad = \left(\frac{\partial(\rho B_\phi)}{\partial\rho}\,d\rho\right)\,d\phi - \left(\frac{\partial B_\rho}{\partial\phi}\,d\phi\right)\,d\rho
\end{aligned} \tag{48}$$

from which we deduce

$$(\nabla\times\mathbf{B})_z = \frac{1}{\rho}\frac{\partial(\rho B_\phi)}{\partial\rho} - \frac{1}{\rho}\frac{\partial B_\rho}{\partial\phi} \tag{49}$$

2.7.4 $\nabla\times\mathbf{B}$

If we now collect together (43), (46), and (49) we see that in cylindrical polar coordinates the curl takes the form

$$\begin{aligned}
\nabla\times\mathbf{B} \;=\; &\left(\frac{1}{\rho}\frac{\partial B_z}{\partial\phi} - \frac{\partial B_\phi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \\
&+\left(\frac{\partial B_\rho}{\partial z} - \frac{\partial B_z}{\partial\rho}\right)\hat{\boldsymbol{\phi}} + \\
&+\left(\frac{1}{\rho}\frac{\partial(\rho B_\phi)}{\partial\rho} - \frac{1}{\rho}\frac{\partial B_\rho}{\partial\phi}\right)\hat{\mathbf{k}}
\end{aligned} \tag{50}$$

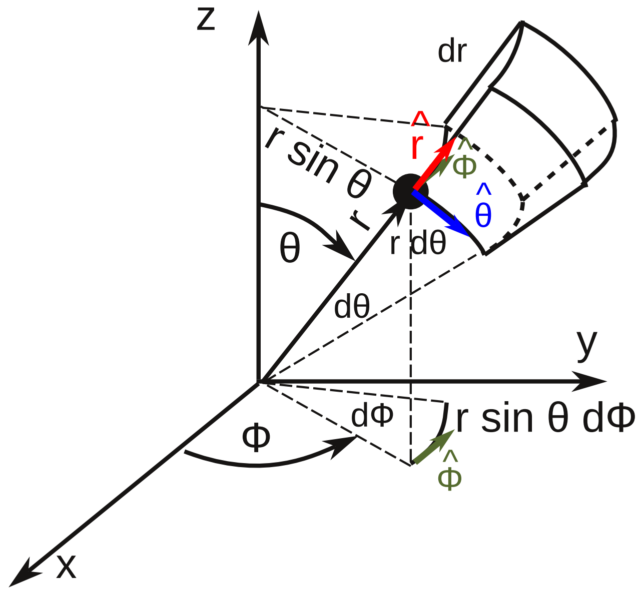

Figure 6: Spherical polar coordinates and their volume element

3 Spherical Polar Coordinates

All the methods we applied in the previous section for cylindrical polar coordinates can be applied in the same way to spherical polar coordinates.

To begin, let us recall some basics about spherical polar coordinates (see Figure 6). From this figure it is apparent that

$$x = r\sin\theta\cos\phi \tag{51}$$

$$y = r\sin\theta\sin\phi \tag{52}$$

$$z = r\cos\theta \tag{53}$$

Note carefully that the two angles, θ and ϕ, are intrinsically different. θ is a polar angle, that measures inclination with respect to an axis. ϕ is an azimuthal angle, that measures a rotation about an axis.

3.1 Position Vector

The point P indicated by the black dot in Figure 6 has position vector

$$\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}} = r\,\hat{\mathbf{r}} \tag{54}$$

This might look even stranger than (4), but you should now expect position vectors in curvilinear coordinates to have information contained within the (non-constant) unit vectors. Here $\hat{\mathbf{r}} = \hat{\mathbf{r}}\,(\theta,\phi)$ contains all the direction information about $\mathbf{r}$. That is, to get to a point P, you travel a distance $r = |\mathbf{r}|$ in the $\hat{\mathbf{r}}$ direction.

From (54) we can write down

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{i}} + \sin\theta\sin\phi\,\hat{\mathbf{j}} + \cos\theta\,\hat{\mathbf{k}} \quad (55)$$

Differentiating this with respect to θ and ϕ will lead us to unit vectors in the direction of increasing θ and increasing ϕ respectively:

$$\begin{aligned}\frac{\partial\hat{\mathbf{r}}}{\partial\theta} &= \cos\theta\cos\phi\,\hat{\mathbf{i}} + \cos\theta\sin\phi\,\hat{\mathbf{j}} - \sin\theta\,\hat{\mathbf{k}} \\ &= \hat{\boldsymbol{\theta}} \quad (56)\\ \frac{\partial\hat{\mathbf{r}}}{\partial\phi} &= -\sin\theta\sin\phi\,\hat{\mathbf{i}} + \sin\theta\cos\phi\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}} \\ &= \sin\theta\left(-\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}}\right) \\ &= \sin\theta\,\hat{\boldsymbol{\phi}} \quad (57)\end{aligned}$$

You should convince yourself that $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ are indeed unit vectors and that (56) and (57) give directions that agree with what trigonometry would tell you from Figure 6.

3.2 Line Element

Looking at Figure 6 to reveal the displacement vectors resulting from increments dr, $d\theta$, and $d\phi$, or taking the differential $\mathbf{dr}$ of the position vector (54) and making use of (56)–(57) leads to the expression for the line element in spherical polar coordinates:

$$\mathbf{dr} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}} \quad (58)$$

3.3 Surface Elements

With reference to Figure 6 we can write down the basic surface elements of the three faces that meet at the black dot (r, θ, ϕ):

$$\begin{aligned}\mathbf{dS}_r &= r^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}} \quad (59)\\ \mathbf{dS}_\theta &= r\sin\theta\,d\phi\,dr\,\hat{\boldsymbol{\theta}} \quad (60)\\ \mathbf{dS}_\phi &= r\,d\theta\,dr\,\hat{\boldsymbol{\phi}} \quad (61)\end{aligned}$$

We will see these again in Sections 3.6 and 3.7 when we work out the divergence and curl in spherical polar coordinates.

3.4 Volume Element

The volume of the element shown in Figure 6 is the product of the three orthogonal edges, i.e.,

$$dV = r^2\sin\theta\,dr\,d\theta\,d\phi \quad (62)$$

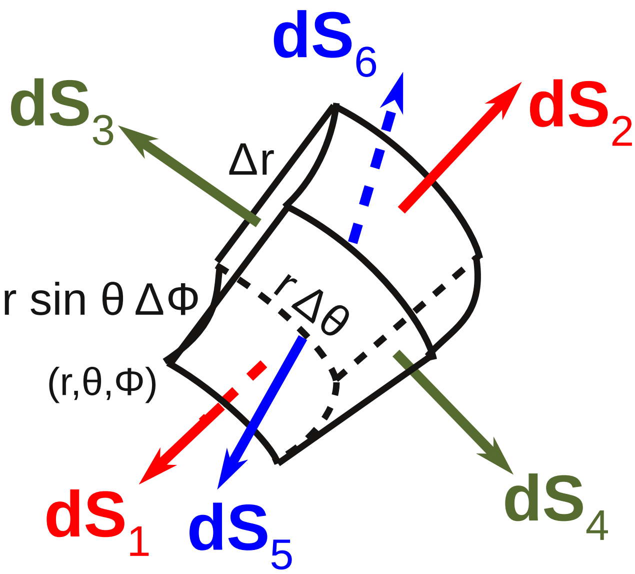

Figure 7: Volume element in spherical polars with surface elements marked for application of the Divergence Theorem

3.5 Gradient Operator

We can find the gradient through the exact differential as for cylindrical polars:

$$\begin{aligned}df &= \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial\theta}d\theta + \frac{\partial f}{\partial\phi}d\phi \quad (63)\\ &= \nabla\mathbf{f}\cdot\mathbf{dr} \quad (64)\\ &= (\nabla f)_r\,dr + (\nabla f)_\theta\,r\,d\theta + \\ &\quad (\nabla f)_\phi\,r\sin\theta\,d\phi \quad (65)\end{aligned}$$

Comparing (63) and (65) gives

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\hat{\boldsymbol{\phi}} \quad (66)$$

3.6 Divergence Operator

We derived the form of the divergence in spherical polar coordinates in lecture by a method that mimics closely that employed in Section 2.6. Here we will outline the key steps which follow from the divergence theorem applied to the volume depicted in Figure 7.

Recall (26)

$$\nabla\cdot\mathbf{B} = \sum_{i=1}^{6}\mathbf{B}\cdot\mathbf{dS}_i/dV$$

Retaining only the dependencies related to the face in question we have

$$\mathbf{B}\cdot\mathbf{dS}_1 = -B_r(r,,)\,r\Delta\theta\,r\sin\theta\,\Delta\phi \quad (67)$$

$$
\begin{aligned}
\mathbf{B}\cdot\mathbf{dS}_2 &= +B_r(r+\Delta r,\,,\,)\times \\
&\quad \times(r+\Delta r)\Delta\theta\,(r+\Delta r)\sin\theta\,\Delta\phi && (68)\\
\mathbf{B}\cdot\mathbf{dS}_3 &= -B_\theta(,\theta,)\,\Delta r\,r\sin\theta\,\Delta\phi && (69)\\
\mathbf{B}\cdot\mathbf{dS}_4 &= +B_\theta(,\theta+\Delta\theta,)\times \\
&\quad \Delta r\,r\sin(\theta+\Delta\theta)\,\Delta\phi && (70)\\
\mathbf{B}\cdot\mathbf{dS}_5 &= -B_\phi(,,\phi)\,\Delta r\,r\Delta\theta && (71)\\
\mathbf{B}\cdot\mathbf{dS}_6 &= +B_\phi(,,\phi+\Delta\phi)\,\Delta r\,r\Delta\theta && (72)\\
& && (73)
\end{aligned}
$$

in which you should recognise the forms of the different surface elements given in (59)-(61) with minus signs in some places to ensure that all the **dS**'s point out of the volume. Summing these pairwise and letting the volume shrink to differential proportions yields

$$
\begin{aligned}
\sum_{i=1}^{6}\mathbf{B}\cdot\mathbf{dS} &= \frac{\partial(r^2 B_r)}{\partial r}\,dr\,\sin\theta\,d\theta\,d\phi + \\
&\quad +\frac{\partial(\sin\theta\,B_\theta)}{\partial\theta}\,d\theta\,r\,dr\,d\phi + \\
&\quad +\frac{\partial B_\phi}{\partial\phi}\,d\phi\,r\,dr\,d\theta \qquad (74)
\end{aligned}
$$

Dividing this summation by $dV = r^2\sin\theta\,dr\,d\theta\,d\phi$ then produces the desired expression:

$$
\nabla\cdot\mathbf{B} = \frac{1}{r^2}\frac{\partial(r^2 B_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\,B_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial B_\phi}{\partial\phi} \qquad (75)
$$

3.7 Curl Operator

As in the calculation of $\nabla\times\mathbf{B}$ in Section 2.7, we shall apply Stoke's Theorem to path integrals around faces of the volume element (Figure 7) to find the components of $\nabla\times\mathbf{B}$ in spherical polar coordinates. The three faces, area elements **dS**, and corresponding right-handed paths are shown in Figure 8.

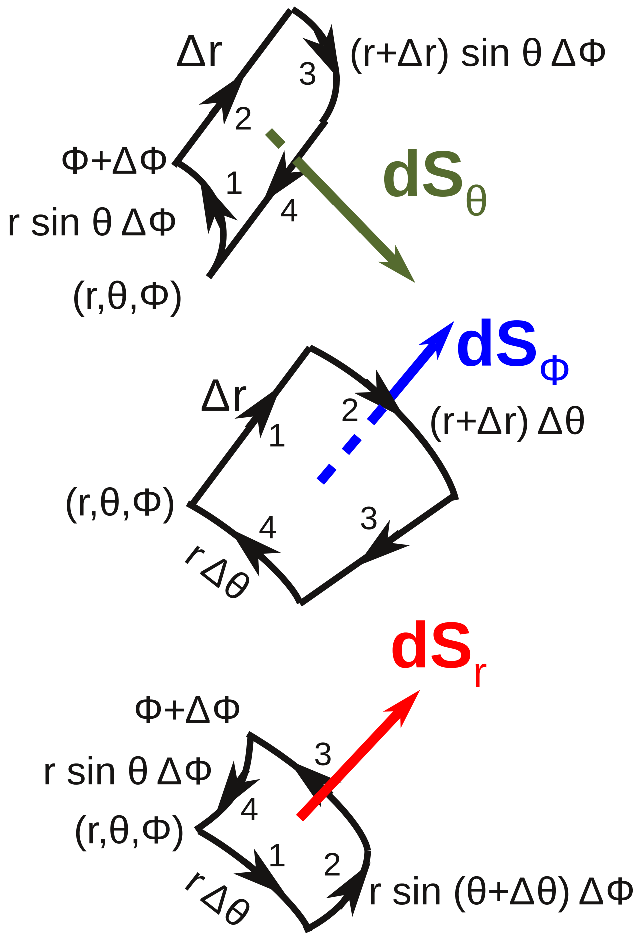

Figure 8: Three faces of the volume element in spherical polar coordinates with paths indicated for derivation of the $\hat{\boldsymbol{\theta}}$ (top), $\hat{\boldsymbol{\phi}}$ (middle), and $\hat{\mathbf{r}}$ (bottom) components of $\nabla\times\mathbf{B}$.

3.7.1 $\hat{\mathbf{r}}$ component

The bottom sketch in Figure 8 shows the face of the volume element at radial distance r and that therefore has its surface element directed in the $\hat{\mathbf{r}}$ direction. (We take it to be in the $+\hat{\mathbf{r}}$ direction here, which is opposite to what is sketched as $\mathbf{dS}_1$ in Figure 7). Applying Stoke's Theorem to this face gives

$$
\begin{aligned}
\sum_{i=1}^{4}\mathbf{B}\cdot\mathbf{dr}_i &= B_\theta(,,\phi)\,r\Delta\theta + \\
&\quad +B_\phi(,\theta+\Delta\theta,)\,r\sin(\theta+\Delta\theta)\,\Delta\phi - \\
&\quad -B_\theta(,,\phi+\Delta\phi)\,r\Delta\theta - \\
&\quad -B_\phi(,\theta,)\,r\sin\theta\Delta\phi \\
&= (\nabla\times\mathbf{B})_r\;r^2\sin\theta\,\Delta\theta\,\Delta\phi \qquad (76)
\end{aligned}
$$

Dividing by $r^2\sin\theta\,\Delta\theta\,\Delta\phi$, re-arranging and letting $\Delta\theta\to d\theta$ and $\Delta\phi\to d\phi$ gives

$$
(\nabla\times\mathbf{B})_r = \frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta\,B_\phi)}{\partial\theta} - \frac{\partial B_\theta}{\partial\phi}\right) \qquad (77)
$$

3.7.2 $\hat{\boldsymbol{\theta}}$ component

The top sketch in Figure 8 shows the face of the volume element at polar angle θ and that therefore has its surface element directed in the $\hat{\boldsymbol{\theta}}$ direction. (We take it to be in the $+\hat{\boldsymbol{\theta}}$ direction here, which is opposite to what is sketched as $\mathbf{dS}_3$ in Figure 7).

Applying Stoke's Theorem to this face gives

$$\begin{aligned}\sum_{i=1}^{4} \mathbf{B}\cdot\mathbf{dr}_i &= B_\phi(r,,)\, r\sin\theta\,\Delta f + \\ &\quad + B_r(,,\phi+\Delta\phi)\,\Delta r - \\ &\quad - B_\phi(r+\Delta r,,)\,(r+\Delta r)\sin\theta\,\Delta\phi - \\ &\quad - B_r(,,\phi)\,\Delta r \\ &= (\nabla\times\mathbf{B})_\theta\; r\sin\theta\,\Delta\phi\,\Delta r \end{aligned} \tag{78}$$

Dividing by $r\sin\theta\,\Delta\phi\,\Delta r$, re-arranging, and letting $\Delta\phi \to d\phi$ and $\Delta r \to dr$ yields

$$(\nabla\times\mathbf{B})_\theta = \left(\frac{1}{r\sin\theta}\frac{\partial B_r}{\partial\phi} - \frac{1}{r}\frac{\partial(rB_\phi)}{\partial r}\right) \tag{79}$$

3.7.3 $\hat{\boldsymbol{\phi}}$ component

The middle sketch in Figure 8 shows the face of the volume element at azimuthal angle ϕ and that therefore has its surface element directed in the $\hat{\boldsymbol{\phi}}$ direction. (We take it to be in the $+\hat{\boldsymbol{\phi}}$ direction here, which is opposite to what is sketched for the front face $\mathbf{dS}_5$ in Figure 7). Applying Stoke's Theorem to this face gives

$$\begin{aligned}\sum_{i=1}^{4} \mathbf{B}\cdot\mathbf{dr}_i &= B_r(,\theta,)\,\Delta r + \\ &\quad + B_\theta(r+\Delta r,,)\,(r+\Delta r)\,\Delta\theta - \\ &\quad - B_r(,\theta+\Delta\theta,)\,\Delta r - \\ &\quad - B_\theta(r,,)\, r\,\Delta\theta \\ &= (\nabla\times\mathbf{B})_\phi\; r\,\Delta\theta\,\Delta r \end{aligned} \tag{80}$$

Dividing by $r\Delta\theta\,\Delta r$, re-arranging, and letting $\Delta\theta \to d\theta$ and $\Delta r \to dr$ yields

$$(\nabla\times\mathbf{B})_\phi = \frac{1}{r}\left(\frac{\partial(rB_\theta)}{\partial r} - \frac{\partial B_r}{\partial\theta}\right) \tag{81}$$

3.7.4 $\nabla\times\mathbf{B}$

Collecting the components of $\nabla\times\mathbf{B}$ from (77), (79), and (81) gives our final result

$$\begin{aligned}(\nabla\times\mathbf{B}) &= \frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta\,B_\phi)}{\partial\theta} - \frac{\partial B_\theta}{\partial\phi}\right)\hat{\mathbf{r}} + \\ &\quad + \left(\frac{1}{r\sin\theta}\frac{\partial B_r}{\partial\phi} - \frac{1}{r}\frac{\partial(rB_\phi)}{\partial r}\right)\hat{\boldsymbol{\theta}} + \\ &\quad + \frac{1}{r}\left(\frac{\partial(rB_\theta)}{\partial r} - \frac{\partial B_r}{\partial\theta}\right)\hat{\boldsymbol{\phi}} \end{aligned} \tag{82}$$

4 General Orthogonal Coordinate Systems

The methodology used in this handout can be applied to other orthogonal curvilinear coordinate systems. You will also find this done in several books by pure mathematical manipulations. All start from the cornerstone of coordinate systems, namely a set of unit vectors $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_3$, and the corresponding scale factors h_1, h_1, and h_3 that convert differentials in the coordinates (u_1, u_2, u_3) into vector differential line elements, i.e.,

$$\mathbf{dr} = h_1\, du_1\,\hat{\mathbf{e}}_1 + h_2\, du_2\,\hat{\mathbf{e}}_2 + h_3\, du_3\,\hat{\mathbf{e}}_3 \tag{83}$$

You might try to replicate the approach here for such a general system, and then compare your answer with the results in Riley et al., or Boas.

Steve Schwartz
4 March 2011

Surfaces and Volumes of Revolution

1 Objects formed by revolution

Many objects have symmetry about an axis and can be considered as *Objects of Revolution*. Although many books will distinguish the axis of revolution, with different results depending on whether it is the x-axis or y-axis, the most natural way for us to do this is to always consider the z-axis as the axis of symmetry. This allows us to work in cylindrical polar coordinates. To avoid later confusion, we shall use ρ as the variable representing distance from the z-axis.

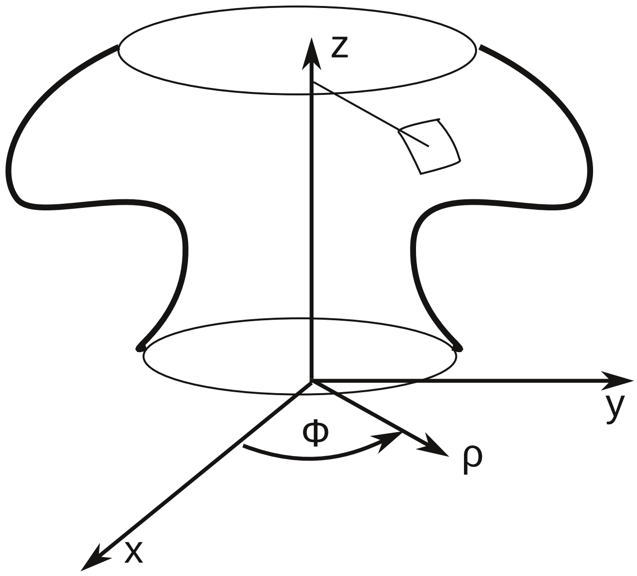

Figure 1: A general surface of revolution formed by revolving the curve $z(y)$ about the z-axis to leave a surface $z(\rho)$ that is independent of ϕ.

Consider a curve $z(y)$ and revolve it about the z-axis. This becomes a surface $z(\rho)$ as sketched in Figure 1. This surface is independent of the azimuthal angle ϕ.

1.1 Cylindrical Polar Coordinates

To begin, let us recall some basics about cylindrical polar coordinates (see Figure 2). From this figure it is apparent that

$$x = \rho\cos\phi \quad (1)$$
$$y = \rho\sin\phi \quad (2)$$
$$z = z \quad (3)$$

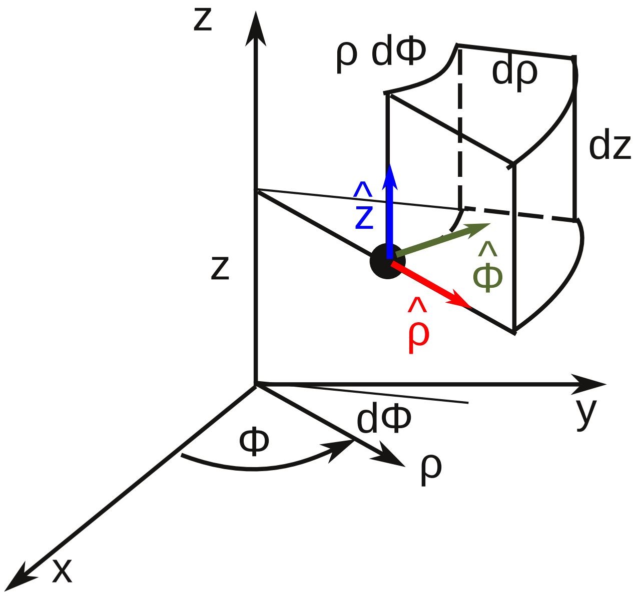

Figure 2: Cylindrical polar coordinates and their volume element

and that the volume element is

$$dV = \rho\, d\phi\, dr\, dz \quad (4)$$

You can also deduce this by calculating the Jacobian $\frac{\partial(x,y,z)}{\partial(\rho,\phi,z)}$ as we did in lecture. Here this volume is simply the product of the three sides of the (pseudo-)rectangular volume since the coordinate system is orthogonal.

Moreover, you can see from this representation of the coordinates that a general path element would be

$$\mathbf{dr} = d\rho\,\hat{\boldsymbol{\rho}} + \rho\, d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{k}} \quad (5)$$

where the three unit vectors are also shown in Figure 2.

1.2 Volumes of Revolution

We can now easily find the volume of the object formed by revolving $z(y)$ about the z-axis, making use of our expression for dV in cylindrical polar coordinates from (4).

$$V = \iiint_V \rho\, d\phi\, d\rho\, dz \quad (6)$$
$$= 2\pi \int_{z_1}^{z_2} \int_{\rho=0}^{\rho=\rho(z)} \rho\, d\rho\, dz \quad (7)$$
$$= \int_{z_1}^{z_2} \pi\rho^2(z)\, dz \quad (8)$$

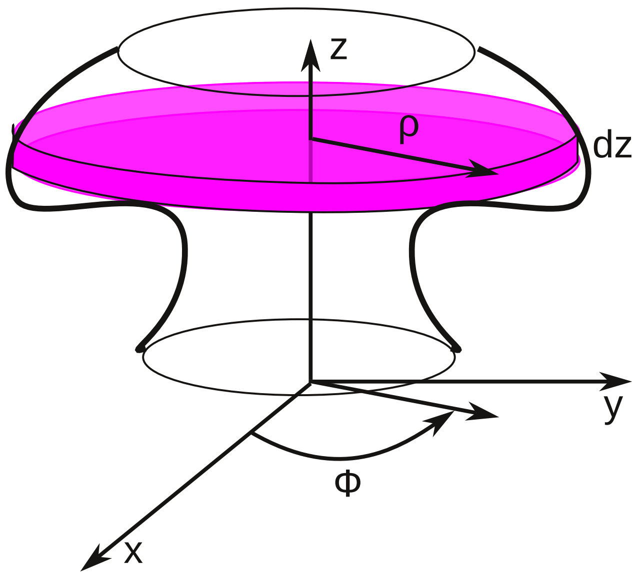

Figure 3: Final volume integral of a volume of revolution showing a disk of radius ρ and height dz. The total volume is a sum of such disks from the lowest z-value to the the top.

Note that here the 2π is the result of the ϕ integration. We chose to perform the ρ integration second, so this means we would need to invert mathematically the curve $z(\rho)$ to find $\rho(z)$. The result in (8) makes perfect sense. The quantity $\pi\rho^2$ is the area of a disk of radius ρ, while dz is the thickness of the disk. So the integrand is the volume of such a disk, and the entire volume is found by summing the volume of these disks between the z-limits. This is illustrated in Figure 3.

1.3 Surface of Revolution - the Easy Way

There are two ways to calculate the surface of a body such as that sketched in Figure 1. The easy way is to consider the surface area of a slice taken parallel to the $x - y$ plane, as sketched in Figure 4. Imagine cutting the resulting ribbon and unrolling it. The ribbon is $2\pi\rho$ in length and is ds wide, so its area is simply

$$dS = 2\pi\rho\, ds \tag{9}$$

This isn't 100% true, of course, because it won't be exactly rectangular in shape, but the errors involved will shrink to zero as we make the differentials smaller. Note, however, that we have to retain the arc length ds in this process, and that ds never reduces to simply dz or $d\rho$. Now as we've constructed it, the arc length is taken at some fixed

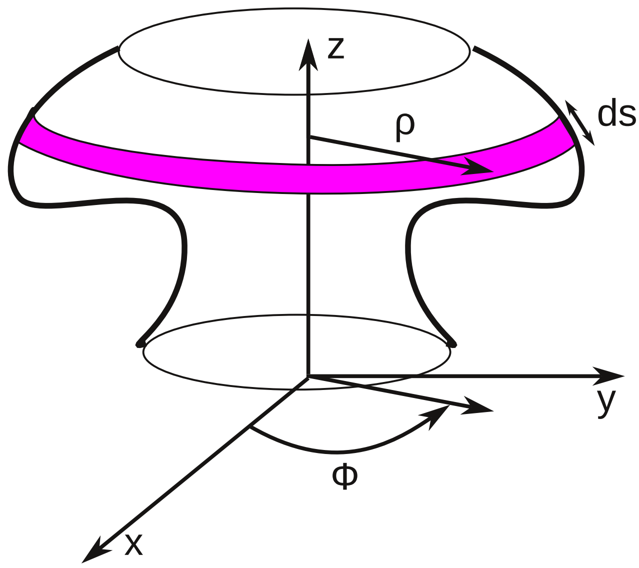

Figure 4: Elemental surface for a body of revolution.

value of ϕ, so noting that $d\phi = 0$ from (5) we can write

$$ds = \sqrt{(dz)^2 + (d\rho)^2} = \sqrt{1 + \left(\frac{d\rho}{dz}\right)^2}\quad dz \tag{10}$$

So we can find the surface area of this volume of revolution by summing all these ribbons:

$$S = \int_{z_1}^{z_2} 2\pi\rho(z)\sqrt{1 + \left(\frac{d\rho}{dz}\right)^2}\quad dz \tag{11}$$

1.4 Surface of Revolution - the Hard Way

It is possible to invoke all the machinery we have for evaluating surface integrals, namely starting from

$$\mathbf{dS} = \mathbf{N}\, du\, dv \tag{12}$$

where u and v are two variables that parameterise the 2D surface, $\mathbf{N} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$ and $\mathbf{r}$ is an arbitrary point on the surface.

We could write $\mathbf{r} = \rho\,\hat{\boldsymbol{\rho}} + z\,\hat{\mathbf{k}}$ but we would need to remember that $\hat{\boldsymbol{\rho}}$ is a function of ϕ, with $d\hat{\boldsymbol{\rho}}/d\phi = \hat{\boldsymbol{\phi}}$. You might like to try to tackle the problem this way, parameterising the surface by ϕ and z. Here, let's try to do this in cartesians making use of (1)-(3). So

$$\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z(x, y)\,\hat{\mathbf{k}} \tag{13}$$

We can make use of the fact here that the problem is cylindrically symmetric, so that z doesn't depend on

ϕ and so we shall write $z = z(\rho)$ with $\rho = \sqrt{x^2 + y^2}$. Now we can calculate

$$\frac{\partial \mathbf{r}}{\partial x} = \hat{\mathbf{i}} + z' \frac{\partial \rho}{\partial x} \hat{\mathbf{k}} \quad (14)$$

$$\frac{\partial \mathbf{r}}{\partial y} = \hat{\mathbf{j}} + z' \frac{\partial \rho}{\partial y} \hat{\mathbf{k}} \quad (15)$$

where $z' \equiv dz/d\rho$. Now $\frac{\partial \rho}{\partial x} = x/\rho$ and $\frac{\partial \rho}{\partial y} = y/\rho$. Thus

$$\mathbf{N} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & z'\frac{x}{\rho} \\ 0 & 1 & z'\frac{y}{\rho} \end{vmatrix} \quad (16)$$

$$= -z' \frac{x}{\rho} \hat{\mathbf{i}} - z' \frac{y}{\rho} \hat{\mathbf{j}} + \hat{\mathbf{k}} \quad (17)$$

so that

$$dS = |\mathbf{dS}| = |\mathbf{N}| \, dx \, dy \quad (18)$$

$$= \left[(z')^2 \left(\frac{x^2 + y^2}{\rho^2} \right) + 1 \right]^{1/2} dx \, dy \quad (19)$$

$$= \sqrt{1 + \left(\frac{dz}{d\rho} \right)^2} \; \rho \, d\phi \, d\rho \quad (20)$$

where we have transformed the integration from (x, y) to (ρ, ϕ) - essentially plane polar coordinates. So, finally

$$S = \int_{\rho(z_1)}^{\rho(z_2)} \int_{\phi=0}^{2\pi} \sqrt{1 + \left(\frac{dz}{d\rho} \right)^2} \; \rho \, d\phi \, d\rho \quad (21)$$

$$= \int_{\rho(z_1)}^{\rho(z_2)} 2\pi\rho \sqrt{1 + \left(\frac{dz}{d\rho} \right)^2} \; d\rho \quad (22)$$

$$= \int_{\rho(z_1)}^{\rho(z_2)} 2\pi\rho \, ds \quad (23)$$

using (10) to recognise the arc length ds found here in terms of $dz/d\rho$. Alternatively, change variables from ρ to z formally as $d\rho = (d\rho/dz) \, dz$, take the $d\rho/dz$ underneath the square root sign, and note that since these are total derivatives

$$\frac{d\rho}{dz} = \frac{1}{dz/d\rho}$$

This then recovers (11).

As an aside, you can also parameterise $\mathbf{r}$ in terms of ρ and ϕ but leave these expressed as cartesian components (i.e., with unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$). That is, $\mathbf{r} = \rho \cos\phi \, \hat{\mathbf{i}} + \rho \sin\phi \, \hat{\mathbf{j}} + z(\rho) \, \hat{\mathbf{k}}$. Then write $\mathbf{dS} = \left| \frac{\partial \mathbf{r}}{\partial \rho} \times \frac{\partial \mathbf{r}}{\partial \phi} \right| d\rho \, d\phi$. This avoids the change of variables $(x, y) \to (\rho, \phi)$ and gets you directly to (20).

2 And finally ...

One of your problems asked you to find the surface element of a body expressed in *spherical polar coordinates* that was symmetric about the polar (z) axis. The solution will follow closely what we've done here (and what was expected was probably the "easy way"), provided you note that $\rho = r \sin\phi$. It is possible to follow through a similar procedure to the "hard way" adopted here, noting that in spherical polars the curve corresponding to $z = z(\rho)$ will be of the form $r = r(\theta)$.

Steve Schwartz
14 February 2011

Differential Calculus	
Ordinary derivative	$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$
Chain rule	$\frac{df(u(x))}{dx} = \frac{df}{du}\frac{du}{dx}$
Mean Value Theorem	$\frac{df}{dx}(x=b) = \frac{f(c)-f(a)}{c-a}$ for some b in $a < b < c$
Partial derivative	$\frac{\partial f(x,y)}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$
	$\frac{\partial f(x,y)}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$
Total differential	$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$
Total derivative	$\frac{df(x,y(x))}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$
Exact differential	$A\,dx + B\,dy \equiv \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ if and only if $\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$
Chain rule (1 parameter)	$\frac{df(x(u),y(u))}{du} = \frac{\partial f}{\partial x}\frac{dx}{du} + \frac{\partial f}{\partial y}\frac{dy}{du}$
Change of variables $(x,y) \to (u,v)$ s.t. $x = x(u,v)$, $y = y(u,v)$	$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$
	$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$

Integration	
Riemann Sum	$S_n = \sum_{i=1}^{n} f(x_i)(\xi_i - \xi_{i-1})$
Integral	$F(b) - F(a) = \int_a^b f(x)\,dx = \lim_{n\to\infty} S_n$
Fundamental Theorem	$\int_a^b f(x)\,dx = F(b) - F(a)$ if $f(x)$ integrable and $f(x) = dF/dx$
Change of variables	$F(b) - F(a) = \int_{a(u)}^{b(u)} f(x(u))\frac{dx}{du}\,du$
Double Integral	limit of S_N where $S_N = \sum_{sub-regions}\sum_{p=1}^{N} f(x_p, y_p)\,\Delta A_p$
	$I = \iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy$
Jacobian	$J \equiv \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$
Change of variables	$I = \iint_R f(x,y)\,dx\,dy = \iint_R f(x(u,v), y(u,v))\,\lvert J\rvert\,du\,dv$

... continued

Plane polar coordinates	$x = r\cos\theta$, $y = r\sin\theta$, $\lvert J\rvert = r$, $dA = r\,d\theta\,dr$

Triple Integration	
Integral over volume	$\iiint_R f(x,y,z)\,dV = \iiint_R f\,dx\,dy\,dz$
Change of variables	$dV = \left\lvert\dfrac{\partial(x,y,z)}{\partial(u,v,w)}\right\rvert du\,dv\,dw$
Jacobian	$J = \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}$
Cylindrical polar coordinates	$x = r\cos\phi$, $y = r\sin\phi$, $z = z$ use $\rho = \sqrt{x^2+y^2}$ instead if danger of confusion with r $J = r$ so $dV = r\,dr\,d\phi\,dz$
Spherical polar coordinates	$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ $J = r^2\sin\theta$ so $J\,dV = r^2\sin\theta\,d\theta\,d\phi\,dr$
Surface integrals	weighted sum over **dS**
Surface element	If $\mathbf{r}(u,v)$ is on surface S then $\mathbf{dS} = \left(\dfrac{\partial\mathbf{r}}{\partial u}\times\dfrac{\partial\mathbf{r}}{\partial v}\right)du\,dv = \mathbf{N}\,du\,dv = \hat{\mathbf{n}}\,dS$
Projected area element	$dS = dA/\cos\alpha$ where α is angle between **dS** on surface and its projection dA onto a coordinate plane.
Arc length	$ds = \sqrt{(dy)^2+(dx)^2}$ so $s = \int\sqrt{\left(\dfrac{dy}{dx}\right)^2+1}\,dx$
Volume of revolution	Cast so z-axis is axis of revolution & use cylindrical polar coordinates $V = \iiint dV = \iiint_R r\,dr\,d\phi\,dz = \pi\int r^2(z)\,dz$
Surface of revolution	Use above properties of surface integrals $S = 2\pi\int r(z)\sqrt{\left(\dfrac{dr}{dz}\right)^2+1}\,dz$

Line Integrals	
Work done by force **F**	$dW = \mathbf{F}\cdot\mathbf{dr}$
Vector line element	$\mathbf{dr} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}}$

... continued

Cylindrical polar form	$\mathbf{dr} = d\rho\,\hat{\boldsymbol{\rho}} + \rho d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{k}}$
Spherical polar form	$\mathbf{dr} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$
Line integrals	$\int_C \phi\,\mathbf{dr}$, $\int_C \mathbf{F}\cdot\mathbf{dr}$, $\int_C \mathbf{B}\times\mathbf{dr}$, etc.
Parametrisation of C	$\displaystyle\int_C \mathbf{F}\cdot\mathbf{dr} = \int_C F_x\frac{dx}{du}du + F_y\frac{dy}{du}du + F_z\frac{dz}{du}du$
Path independence	$\mathbf{F}\cdot\mathbf{dr} = F_x\,dx + F_y\,dy + F_z\,dz$ is an exact differential
[or Conservative field]	$\mathbf{F}\cdot\mathbf{dr} = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi$ for some $\phi(x,y,z)$
	$\mathbf{F} = \nabla\phi$ for some $\phi(x,y,z)$
	$\displaystyle\int_A^B \mathbf{F}\cdot\mathbf{dr} = \phi(B) - \phi(A)$ independent of path $A\to B$
	$\oint_C \mathbf{F}\cdot\mathbf{dr} = 0$ for all closed paths C.
	$\nabla\times\mathbf{F} = \mathbf{0}$

Vector Fields	
Differentiation of vector	e.g., $\mathbf{r}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}} + z(t)\,\hat{\mathbf{k}}$
	$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}}$
Vector differential	$\mathbf{dF} = \frac{d\mathbf{F}}{du}\,du$
Gradient	$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{\mathbf{i}} + \frac{\partial\phi}{\partial y}\hat{\mathbf{j}} + \frac{\partial\phi}{\partial z}\hat{\mathbf{k}}$
Cylindrical polar form	$\nabla f = \frac{\partial f}{\partial\rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}$
Spherical polar form	$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\hat{\boldsymbol{\phi}}$
Differential	$d\phi = \nabla\phi\cdot\mathbf{dr}$
Directional derivative	$\frac{d\phi}{ds} = \nabla\phi\cdot\hat{\mathbf{a}}$
Normal to surface $\phi = const$	$\hat{\mathbf{n}} = \nabla\phi/\lvert\nabla\phi\rvert$

Divergence	
Divergence	$\nabla\cdot\mathbf{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}$
Laplacian	$\nabla^2\phi \equiv \nabla\cdot\nabla\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$
Divergence Theorem	$\displaystyle\oiint_S \mathbf{B}\cdot\mathbf{dS} = \iiint_V \nabla\cdot\mathbf{B}\,dV$
Spherical Polar Coordinates	$\nabla\cdot\mathbf{B} = \frac{1}{r^2}\frac{\partial(r^2B_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\,B_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial B_\phi}{\partial\phi}$

... continued

Cylindrical Polar Coordinates	$\nabla \cdot \mathbf{B} = \frac{1}{r}\frac{\partial(r\,B_r)}{\partial r} + \frac{1}{r}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z}$

Curl	
Curl	$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$ $= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{k}}$
Green's Theorem	$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \oint_C P\,dx + Q\,dy$
Stoke's Theorem	$\oint_C \mathbf{A}\cdot\mathbf{dr} = \iint_S \nabla\times\mathbf{A}\cdot\mathbf{dS}$
Cylindrical Polar form	$\nabla \times \mathbf{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\boldsymbol{\phi}} + \left(\frac{1}{r}\frac{\partial}{\partial r}\left(rA_\phi\right) - \frac{1}{r}\frac{\partial A_r}{\partial \phi}\right)\hat{\mathbf{k}}$
Spherical Polar form	$\nabla \times \mathbf{A} = \left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\phi\sin\theta\right) - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi}\right)\hat{\mathbf{r}} + \left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}\left(rA_\phi\right)\right)\hat{\boldsymbol{\theta}} + \left(\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol{\phi}}$

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9 March 2011