Vibrations and Waves
Physics Year 1
Handout 1: Course Details

Dr Carl Paterson
(Blackett 621, carl.paterson@imperial.ac.uk)

Office Hours

Office hours are **Mondays 3pm-4pm** and **Wednesdays 12pm-1pm** (starting Week 2). Alternatively, you can call by my office at any other time (if I can't see you then, we can arrange a time for you to come back) or you can drop me an e-mail.

Suggested Books

Vibrations and Waves in Physics
I.G. Main
3rd edition (Cambridge, 1994)

University Physics
H.D. Young & R.A. Freedman
12th edition (Addison-Wesley, 2004)
Chapters 13, 15, 16, 35, 36

The Physics of Vibrations and Waves
H.J. Pain
6th edition (Wiley, 2005)

Vibrations and Waves (The M.I.T. Introductory Physics Series)
A. P. French
(Norton, 1971)

Assessment

- Assessed Problem Sheets
- Written exam: "Structure of Matter, Vibrations and Waves, Quantum Physics" (one third on Vibrations and Waves)

Course materials and note-taking

I will post learning materials on WebCT(Blackboard) (http://learn.imperial.ac.uk/) as the course progresses. These will include classworks, problem sheets, handouts, and dataprojector slides (where used). You will need to make your own notes during the lectures. I will post very brief summaries of key formulae used in the lectures on Blackboard but these will not be a substitute for your own notes.

Course Overview

Lectures 1-2 (Introduction, SHM)
Introduction to vibrations and waves. Simple harmonic motion (SHM): general solution to ideal SHM, mass on a spring, circular motion, using complex notation, SHM in other systems (LC circuit, pendulum), phasors, velocity and acceleration, potential, kinetic and total energy

Lectures 3-4 (Damped SHM)
Damped SHM: resistive forces, general solution for damped oscillations, light(under) damping, Q-factor, decay and energy loss, LCR circuit, heavy(over) damping, critical damping.

Lectures 5-6 (Forced SHM)
Forced SHM: steady state solutions, amplitude and phase response, resonance, Q-factor, LCR circuit, mechanical resonances, power dissipation, resonance absorption, transients and beating.

Lecture 7 (Coupled Oscillators)
Coupled oscillators: two coupled masses, general solution and normal modes, resonances, superposition and beating, energy transfer; N-coupled oscillators, normal-mode solutions, superposition, wave motion.

Lectures 8-9 (Travelling Waves)
Travelling waves waves on a string, the wave equation, superposition, reflection, transmission, refraction, transverse and longitudinal waves, acoustic waves.

Lecture 10 (Standing Waves)
Standing waves, nodes, anti-nodes, boundary conditions, normal modes, harmonics, 2-D wave equation and normal modes.

Lecture 11 (Dispersion)
Wave-packets, dispersion, phase velocity and group velocity.

Lecture 12 (Doppler, Diffraction and Interference)
Doppler effect; diffraction and interference, two source interference.

Vibrations and Waves
Handout 2: Complex Notation and Phasors

It is often convenient to solve oscillator and wave problems in the complex plane. We represent the displacement of an oscillator (such as a mass on a spring) as

$$\begin{aligned}\tilde{x}(t) &= A\cos(\omega t+\phi) + \mathrm{i}A\sin(\omega t+\phi)\\ &= A\exp[\,\mathrm{i}(\omega t+\phi)]\\ &= \tilde{A}\exp(\,\mathrm{i}\omega t)\end{aligned}$$

where in the last of these forms the phase factor is incorporated into a complex coefficient $\tilde{A} \equiv A\exp(\,\mathrm{i}\phi)$. The physical quantity is given by the real part of $\tilde{x}(t)$. Here we use the tilde (˜) to remind ourselves that we are dealing with the complex quantity, though when we get more proficient we may drop this.

The quantity $\tilde{x}(t)$ is a **phasor**, i.e., a vector that is rotating in the complex plane. We find that manipulation is simplified by using this complex representation. For example, differential operators can be replaced by algebraic expressions

$$\frac{\mathrm{d}\tilde{x}}{\mathrm{d}t} = \mathrm{i}\omega\tilde{x} \qquad \frac{\mathrm{d}^n\tilde{x}}{\mathrm{d}t^n} = (\,\mathrm{i}\omega)^n\tilde{x}$$

This is because the complex exponential is unchanged by differentiation except for a constant multiplier. (We say that the complex exponential is an *eigenfunction* of the differentiation operator.) This is not the case for sine and cosine, which do change on differentiation.

Phasors help us analyse and visualise simple harmonic motion. In an Argand diagram, we see that the complex displacement $\tilde{x}$ is a vector that is rotating around the origin with an angular velocity equal to the angular frequency of the oscillator ω. The complex velocity is obtained by differentiating the complex displacement with respect to time,

$$\tilde{v} = \frac{\mathrm{d}\tilde{x}}{\mathrm{d}t} = \mathrm{i}\omega\tilde{x} = \exp(\,\mathrm{i}\pi/2)\omega\tilde{x} = \omega\tilde{A}\exp[i(\omega t+\pi/2)]$$

We can see from this that the complex velocity is itself a phasor rotating with the same angular velocity that is $\pi/2$ ahead of the displacement—we say it *leads* the displacement by $\pi/2$. Similarly, the acceleration leads the velocity by $\pi/2$.

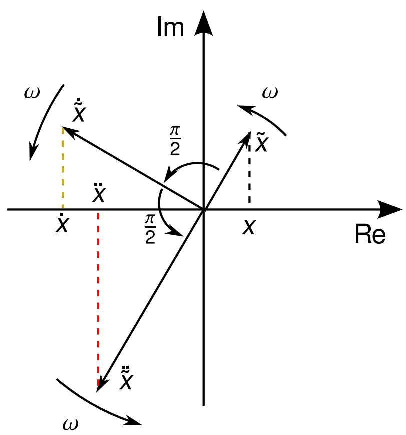


Figure 1: Differentiation of phasors.

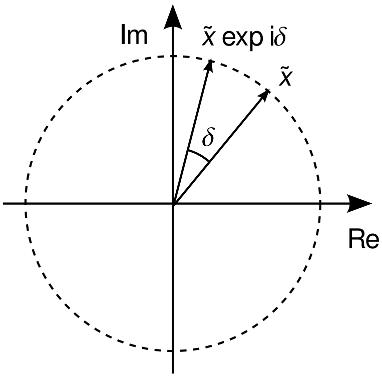


Figure 2: Multiplication of a phasor by a phase factor $\exp(\,\mathrm{i}\delta)$.

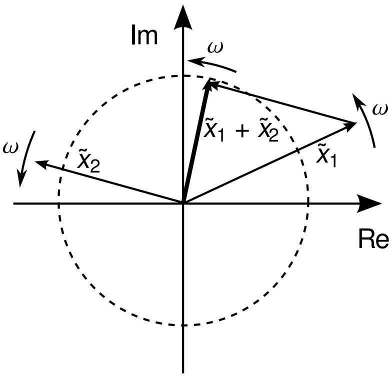


Figure 3: Linear superposition — adding phasors.

More generally, we can see that to add an extra phase $\delta\phi$ to an oscillating quantity we just multiply the phasor by $\exp(\,\mathrm{i}\delta\phi)$ which just adds $\delta\phi$ to the phasor angle.

The superposition (adding together) of two oscillators with the same frequency can be visualised as the vector addition of two phasors. This results in a phasor that also rotates around the complex plane with angular frequency ω

$$\tilde{x} = \tilde{x}_1 + \tilde{x}_2 = (\tilde{A}_1 + \tilde{A}_2)\exp(\,\mathrm{i}\omega t)$$

When we are dealing with systems that have a single angular frequency ω, we find the same factor $\exp(\,\mathrm{i}\omega t)$ in all the phasor quantities. It is sometimes a shorthand convenience to drop this factor, then all the phasor quantities become fixed in the Argand diagram.

We can perform any manipulation on the complex quantity $\tilde{x}$ providing we restrict ourselves to linear combinations of the phasor quantities. [Note: multiplying by a phase factor is perfectly acceptable, since we are not multiplying a phasor by a phasor — the phase factor $\exp(\,\mathrm{i}\delta\phi)$ is not itself a phasor, it is just a coefficient that happens to be complex.] However, a final caution is in order. We must always remember that the physical quantity is the real part. This is particularly important when we want a function that involves non-linear quantities of the oscillator variables, for example, when we are calculating energy which contains squared quantities. We must take the real part of the phasor BEFORE taking the square, otherwise we will get incorrect results. The usefulness of the complex representation is such that it is sometimes too easy to forget this.

Useful conversions

The following are useful identities for converting between the various representations of complex numbers

$$z = x + \mathrm{i}y = r\cos\theta + \mathrm{i}r\sin\theta = r\exp(\,\mathrm{i}\theta)$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$r^2 = |z|^2 = z^* z = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad \cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r}$$

$$z^n = r^n\exp(\,\mathrm{i}n\theta) = r^n\cos(n\theta) + \mathrm{i}r^n\sin(n\theta) \qquad z_1 z_2 = r_1 r_2\exp[\,\mathrm{i}(\theta_1 + \theta_2)]$$

$$\log z = \log r + \mathrm{i}\theta \qquad \mathrm{i} = \exp[\,\mathrm{i}\pi/2]$$

Vibrations and Waves
Handout 3: Non-Linear Restoring Forces

In the lectures we saw that a linear restoring force results in simple harmonic motion. We also saw that this is indeed the case about any stable equilibrium point, provided that we restrict ourselves to small oscillations. This is one of the reasons why simple harmonic motion is so important. It is extremely common in a wide range of physical situations.

However, what happens when we do not have small oscillations? A mechanical spring is only linear within a certain range beyond which Hooke's law starts to break down. The approximation $\sin\theta \approx \theta$ in the derivation of SHM for a pendulum is only valid for small θ. In these cases, the motion is no longer simple harmonic; it becomes *anharmonic.* For a one-dimensional system, the resulting oscillations are still periodic, but rather than having a single harmonic component $\cos\omega t$, we also have higher harmonics present $\cos n\omega t$, where $n = 2, 3, 4, ...$ However, using Fourier series, even this can be treated as a superposition of harmonic oscillations — so we don't have to abandon our harmonic oscillator description completely.

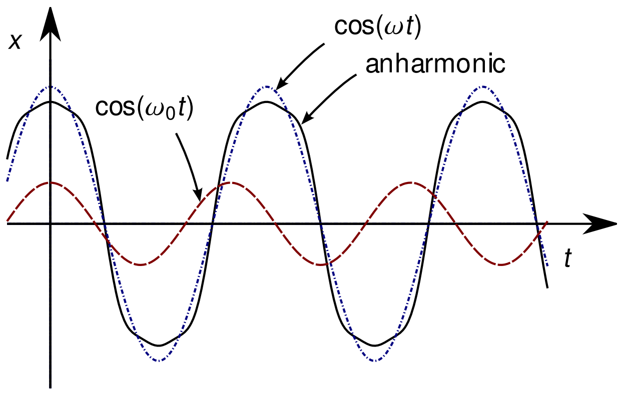


Figure 1: Anharmonic oscillator. The anharmonic oscillator displacement has higher harmonics, $\cos(n\omega t)$ in addition to the fundamental $\cos(\omega t)$. The period of the anharmonic oscillator in general is dependent on the amplitude and is different to the small displacement (harmonic) period $2\pi/\omega_0$.

A second consequence of the non-linearity is that the period of the oscillation is no-longer independent of the amplitude of the oscillations. For example, in the case of the pendulum the period increases for larger oscillations.

For small amplitudes the pendulum period is given by

$$T_0 = 2\pi\sqrt{\frac{\ell}{g}}$$

For large amplitudes, a more accurate (series) approximation for the pendulum period is given by

$$T = T_0\left(1 + \frac{1}{16}\theta_0^2 + \frac{11}{3072}\theta_0^4 + ...\right)$$

where θ_0 is the amplitude of the oscillation in radians.

The effect is still quite small, and it can be surprising how large an angle θ is required to give a significant alteration to the period. Even at $\theta_0 = 20^\circ$, which is definitely not a small angle, the error in the period is still less than 1%.

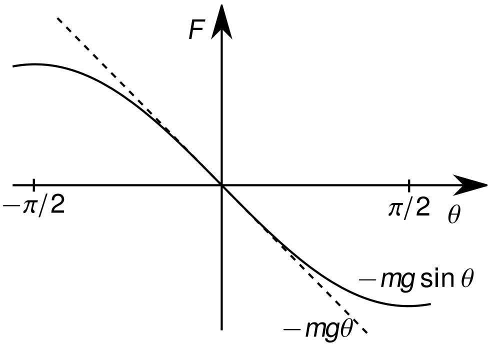


Figure 2: Non-linear restoring force for a pendulum

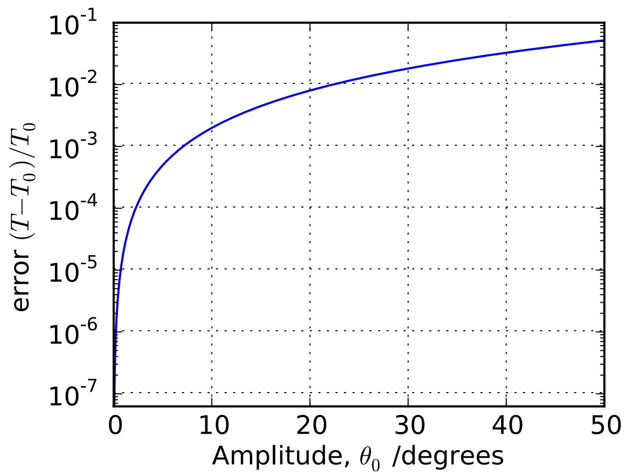


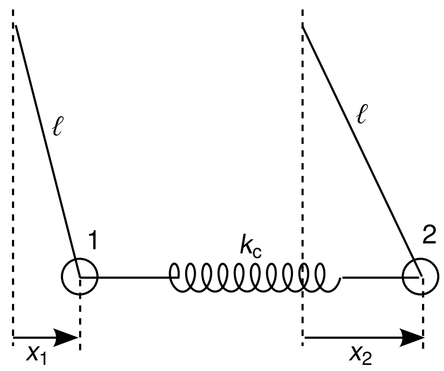
Figure 3: Error of the linear approximation for the period of the pendulum.

Such non-linear behaviour can can be quite useful, for example in laser frequency doubling we can exploit material non-linearities to produce an output with double the frequency (half the wavelength) of the input light. However, we normally have to work quite hard to access the non-linear regime—in many optical systems, we can assume linearity up to quite high electric field amplitudes.

Vibrations and Waves
Handout 4: Coupled Oscillators
(for Lecture 7)

Two coupled oscillators

Consider a system of coupled oscillators consisting of two identical pendulums coupled together with a spring.



Finding the Normal Mode Solutions

STEP 1: Displace each oscillator from its equilibrium position and calculate the forces on it. (In this case, by equilibrium we mean the position of the oscillator when the whole system is at rest.)

For example, on mass 1, the forces are the restoring force of the pendulum itself $F = -(mg/\ell)x_1$ (we have assumed small angles such that $\sin\theta \approx \theta \approx x/\ell$) and the force from the coupling spring, $F = k_c(x_2 - x_1)$.

Then the force on each mass is

$$F_1 = -\frac{mg}{\ell}x_1 + k_c(x_2 - x_1) \quad (1)$$
$$F_2 = -\frac{mg}{\ell}x_2 - k_c(x_2 - x_1) \quad (2)$$

STEP 2: Apply Newton's second law,

$$m\ddot{x}_1 + \frac{mg}{\ell}x_1 - k_c(x_2 - x_1) = 0 \tag{3}$$

$$m\ddot{x}_2 + \frac{mg}{\ell}x_2 + k_c(x_2 - x_1) = 0 \tag{4}$$

We write

$$\omega_0^2 = g/\ell \qquad \omega_c^2 = k_c/m$$

to get the equations into a standard form (ω_0 is the natural angular frequency of the uncoupled oscillator.)

This gives,

$$\ddot{x}_1 + \omega_0^2 x_1 - \omega_c^2(x_2 - x_1) = 0 \tag{5}$$

$$\ddot{x}_2 + \omega_0^2 x_2 + \omega_c^2(x_2 - x_1) = 0 \tag{6}$$

The equations of motion are a system of two coupled linear differential equations.

STEP 3: We shall look for **normal mode solutions** of the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \cos(\omega t) \tag{7}$$

Substituting this form as a trial solution into the coupled equations of motion gives

$$-A_1\omega^2 + \omega_0^2 A_1 - \omega_c^2 A_2 + \omega_c^2 A_1 = 0 \tag{8}$$

$$-A_2\omega^2 + \omega_0^2 A_2 + \omega_c^2 A_2 - \omega_c^2 A_1 = 0 \tag{9}$$

Note that this is a system of two simultaneous equations with three unknowns.

STEP 4: Solve for ω and the ratio A_2/A_1. We first divide through by A_1, which gives

$$-\omega^2 + \omega_0^2 + \omega_{\text{c}}^2 - \omega_{\text{c}}^2 \frac{A_2}{A_1} = 0 \qquad (10)$$

$$(-\omega^2 + \omega_0^2 + \omega_{\text{c}}^2)\frac{A_2}{A_1} - \omega_{\text{c}}^2 = 0 \qquad (11)$$

Eliminating A_2/A_1 from these equations gives

$$(-\omega^2 + \omega_0^2 + \omega_{\text{c}}^2)^2 = \omega_{\text{c}}^4 \quad \Rightarrow \quad -\omega^2 + \omega_0^2 + \omega_{\text{c}}^2 = \pm\omega_{\text{c}}^2 \qquad (12)$$

which (by inspection in this case) has the possible solutions

$$\omega^2 = \omega_0^2 \qquad \text{and} \qquad \omega^2 = \omega_0^2 + 2\omega_{\text{c}}^2 \qquad (13)$$

Substituting back into (8) and (9) gives

$$\begin{array}{lll} A_2 = A_1 & \text{for} & \omega^2 = \omega_0^2 \\ A_2 = -A_1 & \text{for} & \omega^2 = \omega_0^2 + 2\omega_{\text{c}}^2 \end{array} \qquad (14)$$

Thus, we have found the two normal mode solutions (which we label 1 and 2)

$$\begin{array}{lll} \text{Mode 1:} & \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_1 t) & \omega_1 = \omega_0 \\ \text{Mode 2:} & \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_2 t) & \omega_2 = \sqrt{\omega_0^2 + 2\omega_c^2} \end{array} \qquad (15)$$

All motion of the coupled system can be described as a linear superposition of these normal modes.

Summary

To find the normal mode solutions:

1. Displace each oscillator from its equilibrium position and calculate the forces on it.
2. Apply Newton's second law to each oscillator to get a system of coupled differential equations
3. Use as a trial function the normal mode solutions of the form $\mathbf{x} = \mathbf{A}\cos\omega t$, where $\mathbf{x}$ is the vector of the oscillator displacements, and $\mathbf{A}$ is the vector of the displacements for the normal mode.
4. Solve for the ω and the ratios A_2/A_1, A_3/A_1, etc.

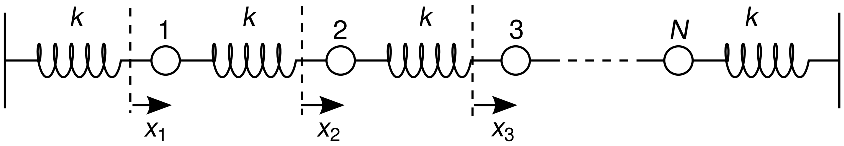
In this case, two oscillators gave two normal modes.

In general there will be as many normal modes as there are independent oscillations.

In practice, matrix methods and numerical matrix inversion are often used for systems with more than a small number of oscillators.

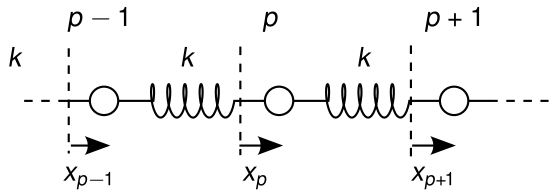
N-coupled oscillators

Consider the following system of N identical masses, joined together by identical springs.



We use the same method to find normal mode solutions as for the two coupled oscillators:

STEP 1: Displace each oscillator from its equilibrium position and calculate the forces on it.



The force on the p^{th} mass is

$$F_p = -k(x_p - x_{p-1}) + k(x_{p+1} - x_p) \quad \text{for } p = 1 \dots N$$
$$\text{(with boundary conditions } x_0 = 0, \;\; x_{N+1} = 0) \qquad (16)$$

STEP 2: Apply Newton's second law to each oscillator to get N coupled differential equations.

$$\ddot{x}_p = \omega_0^2(x_{p+1} + x_{p-1} - 2x_p) \qquad p = 1 \dots N \qquad (17)$$

where $\omega_0^2 = k/m$.

STEP 3: Look for normal mode solutions

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{pmatrix} \cos(\omega t) \qquad (18)$$

Substitute the normal mode solution into equations (17) to get

$$\omega^2 A_p + \omega_0^2 \left(A_{p+1} + A_{p-1} - 2A_p\right) = 0 \qquad p = 1 \ldots N$$
$$\text{(with boundary conditions } A_0 = 0,\ \ A_{N+1} = 0) \quad (19)$$

STEP 4: We could try to solve these N simultaneous equations using brute force, (perhaps using matrix methods.) However, we will guess a trial form of solution—if our guess satisfies the equations, then it must be a valid solution.

The solutions we will try are

$$A_p = \sin\left(\frac{np\pi}{N+1}\right) \qquad n = 1, 2, \ldots, N \qquad (20)$$

(We expect N normal mode solutions since we have N oscillators.)

Substituting into equations (19),

$$\omega^2 \sin\left(\frac{np\pi}{N+1}\right) + \omega_0^2 \left[\underbrace{\sin\left(\frac{n(p+1)\pi}{N+1}\right) + \sin\left(\frac{n(p-1)\pi}{N+1}\right)}_{2\sin\frac{n\pi p}{N+1}\cos\frac{n\pi}{N+1}} - 2\sin\left(\frac{np\pi}{N+1}\right)\right] = 0 \qquad (21)$$

where we have used the trigonometric identity $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$ to simplify the braced terms (spot the similarity with how we derived beating.) This gives

$$\left\{\omega^2 + \omega_0^2 \left[2\cos\left(\frac{n\pi}{N+1}\right) - 2\right]\right\} \sin\left(\frac{np\pi}{N+1}\right) = 0 \qquad (22)$$

from which we obtain the equation for ω for the trial solution to satisfy the equations of motion,

$$\omega^2 = 2\omega_0^2 \left[1 - \cos\left(\frac{n\pi}{N+1}\right)\right] \tag{23}$$

Using $\cos 2A = 1 - 2\sin^2 A$, allows us to write

$$\omega^2 = 4\omega_0^2 \sin^2 \left[\frac{n\pi}{2(N+1)}\right]$$

$$\Rightarrow \qquad \omega = 2\omega_0 \sin\left[\frac{n\pi}{2(N+1)}\right] \tag{24}$$

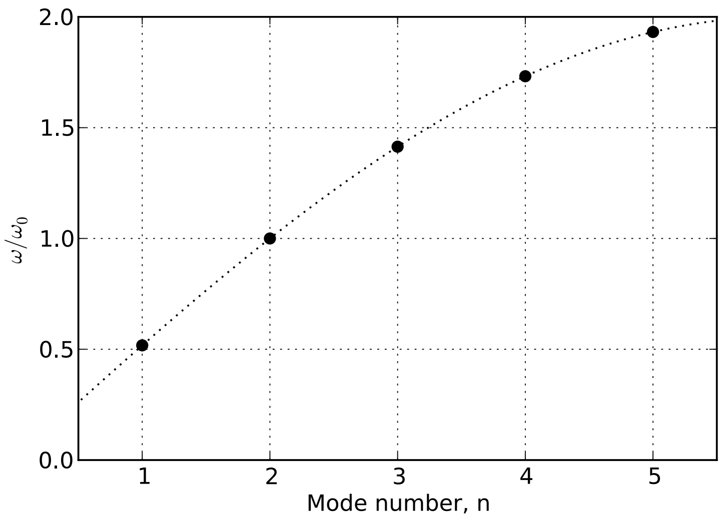


Figure 1: Angular frequencies of the normal modes for $N = 5$ identical coupled oscillators.

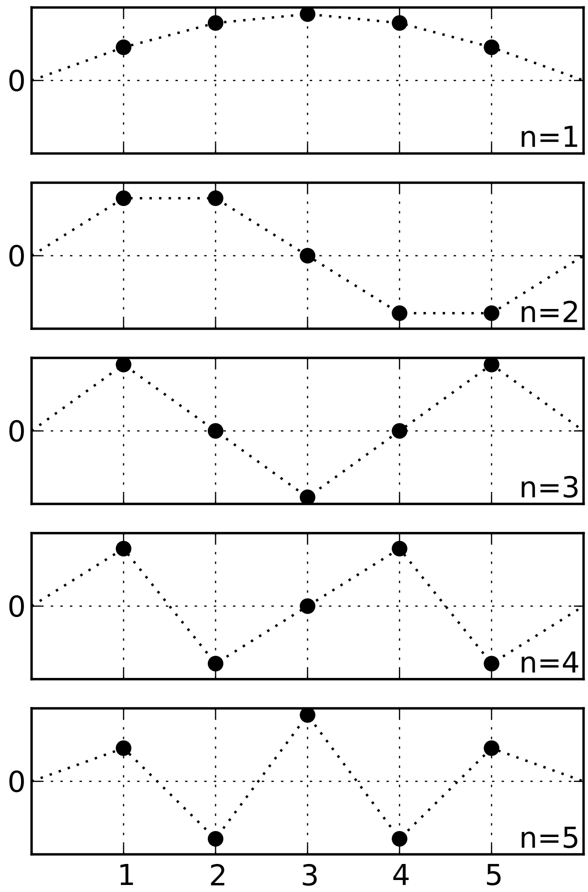


Figure 2: Oscillator displacements in the normal modes for $N = 5$ identical coupled oscillators.