Fourier

Problem Sheet 1 October 17, 2011

Fourier series

1. Express the following as sums of even and odd functions:
   (a) $f(x) = (x+3)(x-2)$
   (b) $f(x) = (1 - \sin x)^3$
   (c) $f(x) = xe^{-x}\sin x$
   (d) $f(x) = e^{ix}$
   (e) $f(x) = xe^{ix(1-x)}$

2. Show that the $\cos(n\pi x/L)$ are orthogonal on the interval $-L \le x \le L$ for different integer values of n. [*Hint:* Use $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$.]

3. (a) Determine the Fourier series for the repeating sawtooth function, $f(x) = x$, for $-L \le x \le L$.
   (b) By considering the average value of $[f(x)]^2$, use Parseval's theorem to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

4. Sketch and calculate the Fourier series for the function

$$f(x) = \begin{cases} 0 & -L < x < 0 \\ \sin(\pi x/L) & 0 < x < L \end{cases}$$

5. Redo the previous two questions as complex exponential Fourier Series.

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Problem Sheet 2 October 31, 2011

Fourier transforms

1. Evaluate the following integrals (all integrals are from $-\infty$ to $+\infty$):

   $(a) \int dx\ \delta(x_0 - x)\ x$ $\quad$ $(b) \int dt\ \delta(t-2)\ \cos t$

   $(c) \int dx\ \delta(x_1 - x)\delta(x_2 - x)\ f(x)$ $\quad$ $(d) \int dx\ \delta(x/3 - 1)\ \sin(\pi x/2)$

   $(e) \int dx\ \delta\left[(x - x_0)(x - x_1)\right]\ f(x)$ $\quad$ $(f) \int dt\ \delta(t^2 - 3t + 2)\ \sin t$

2. A triangular function $f(x)$, of unit area, has the value $f(0) = 1/a$ and falls linearly to zero at $x = -a, a$. Express the function as two linear segments, to show that the Fourier transform of $f(x)$ is given by

   $$g(\omega) = \frac{1}{2\pi}\frac{\sin^2(\omega a/2)}{(\omega a/2)^2} = \frac{\mathrm{sinc}^2(\omega a/2)}{2\pi}\ .$$

   [You will ease the algebra by firstly calculating the indefinite integrals of $e^{-i\omega x}$, and $xe^{-i\omega x}$.]

3. Denoting the Fourier transform of $f(x)$ by $\mathcal{F}[f(x)]$, prove the following Fourier relationships:

   (a) *Phase shifts*:
   $$\mathcal{F}[f(x+a)] = e^{ia\omega}g(\omega)$$

   (b) *Exponential multiplication*:
   $$\mathcal{F}[e^{ax}f(x)] = g(\omega + ia)$$

   (c) *Scaling*:
   $$\mathcal{F}[f(ax)] = \frac{1}{a}g\left(\frac{\omega}{a}\right)$$

4. (a) Express the following function in terms of the rectangle function, Π:

   $$f(x) = \begin{cases} A & -(a+b)/2 < x < -(a-b)/2 \\ A & (a-b)/2 < x < (a+b)/2 \\ 0 & \text{elsewhere} \end{cases}$$

   where $a > b > 0$. *Hint:* $\Pi(x)$ is 1 when $-1/2 < x < 1/2$, so what is $\Pi[(x+c)/d]$?

   (b) Using the scaling and shift theorems show that its Fourier Transform is

   $$\frac{Ab}{\pi}\mathrm{sinc}(b\omega/2)\cos(a\omega/2)$$

   (c) Calculate the Fourier Transform of

   $$\delta(x - a/2) + \delta(x + a/2)\ .$$

   (d) Show that the convolution of $A\Pi(x/b)$ with $[\delta(x-a/2) + \delta(x+a/2)]$ is the same as the answer to the first part of this question.

   (e) Use the convolution theorem to verify Part (b).

Fourier

Answers to Problem Sheet 1

Fourier series

1. Using $e(x) = [f(x)+f(-x)]/2$ and $o(x) = [f(x)-f(-x)]/2$ leads to the following solutions:

$$\begin{array}{lll} (a) & e(x) = x^2 - 6 & o(x) = x \\ (b) & e(x) = 1 + 3\sin^2 x & o(x) = -3\sin x - \sin^3 x \\ (c) & e(x) = x\sin x\cosh x & o(x) = -x\sin x\sinh x \\ (d) & e(x) = \cos x & o(x) = i\sin x \\ (e) & e(x) = ie^{-ix^2} x\sin x & o(x) = e^{-x^2} x\cos x \end{array}$$

2. We wish to integrate

$$I = \int_{-L}^{L} \cos(n\pi x/L)\cos(m\pi x/L)dx \; ,$$

for integer $n \neq m$. Writing $a = n\pi/L$ and $b = m\pi/L$ and subsituting $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$, leads to

$$I = \frac{1}{4}\int_{-L}^{L} (e^{iax}+e^{-iax})(e^{ibx}+e^{-ibx})dx = \frac{1}{4}\int_{-L}^{L} e^{i(a+b)x}+e^{-i(a+b)x}+e^{i(a-b)x}+e^{-i(a-b)x}dx \; ,$$

$$I = \frac{1}{2}\int_{-L}^{L} \cos[(a+b)x]+\cos[(a-b)x]\,dx = \frac{1}{2}\int_{-L}^{L} \cos[(n+m)\pi x/L]+\cos[(n-m)\pi x/L]\,dx = 0 \; ,$$

because each $\cos$ terms oscillates through an integer number of periods in going from $-L$ to L, so the contribution of each term to the integral is zero. [Note that it doesn't matter if $n - m$ is negative as it can be replaced by $m - n$ since $\cos(-\theta) = \cos(\theta)$.]

3. (a) The function is odd, so $a_0 = 0$ and all $a_n = 0$. We can get the b_n using

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\pi x/L)dx = \frac{1}{L}\int_{-L}^{L} x\sin(n\pi x/L)dx = \frac{2}{L}\int_{0}^{L} x\sin(n\pi x/L)dx \; ,$$

because $x\sin(n\pi x/L)$ is even. Integrating by parts:

$$b_n = \frac{2}{L}\left(\left[-\frac{Lx}{n\pi}\cos(n\pi x/L)\right]_0^L + \int_0^L \frac{L}{n\pi}\cos(n\pi x/L)dx\right) \; ,$$

$$b_n = -\frac{2L}{n\pi}\cos(n\pi) + 0 \; .$$

So the Fourier series for $f(x) = x$ is

$$f(x) = \sum_{n=1}^{\infty} -\frac{2L}{n\pi}\cos(n\pi)\sin(n\pi x/L) \; .$$

(b) Parseval's theorem is that the average value of the square of the function is equal to the sum of the average value of the square of the Fourier terms. Now for a sin (or cos) wave, $y = A\sin(\omega x)$ the average value of y^2 over an integer number of periods is $A^2/2$. For the Fourier series above the amplitude of each term is $\frac{2L}{n\pi}\cos(n\pi)$, so Parseval's theorem in this case states

$$\frac{1}{2L}\int_{-L}^{L}[f(x)]^2 dx = \sum_{n=1}^{\infty}\frac{2L^2}{n^2\pi^2} \, .$$

Because $[f(x)]^2$ is even, we can calculate the average over half the period, i.e.

$$\frac{1}{L}\int_{0}^{L}[f(x)]^2 dx = \frac{1}{L}\int_{-0}^{L} x^2 dx = \frac{1}{L}\frac{L^3}{3} = \frac{L^2}{3} \, .$$

Equating the two values gives

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} \, .$$

4. The function is neither even nor odd. We have stressed in the course that even and odd functions are simpler to analyse. These are given by $e(x) = [f(x) + f(-x)]/2$ and $o(x) = [f(x) - f(-x)]/2$. So

$$e(x) = \begin{cases} -\frac{1}{2}\sin(\pi x/L) & -L < x < 0 \\ \frac{1}{2}\sin(\pi x/L) & 0 < x < L \end{cases} \, ,$$

and $o(x) = \frac{1}{2}\sin(\pi x/L)$. The function $o(x)$ is already in the correct form, while for the function $e(x)$, because it is even we only need to compute a_0 and a_n. Proceeding as usual

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x) dx = \frac{1}{L}\int_{0}^{L}\sin(\pi x/L) = -\frac{1}{\pi}[\cos(\pi x/L)]_0^L = \frac{2}{\pi} \, .$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\pi x/L) dx = \frac{1}{L}\int_{0}^{L}\sin(\pi x/L)\cos(n\pi x/L) dx \, .$$

Using the identity $\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$ gives

$$a_n = \frac{1}{2L}\int_{0}^{L}\sin((1+n)\pi x/L) + \sin((1-n)\pi x/L) dx \, .$$

This integral is clearly 0 for $n = 1, 3, 5 ...$ since sin goes through an even number of periods. For n even:

$$a_n = \frac{1}{2L}\left[-\frac{L\cos((1+n)\pi x/L)}{(1+n)\pi} - \frac{L\cos((1-n)\pi x/L)}{(1-n)\pi}\right]_0^L \, .$$

$$a_n = \frac{1}{\pi}\left[\frac{1}{(1+n)} + \frac{1}{(1-n)}\right] = \frac{2}{\pi(1-n^2)} \, .$$

So finally

$$f(x) = \frac{1}{\pi} + \sum_{n=2,4,6...}^{\infty}\frac{2}{\pi(1-n^2)}\cos(n\pi x/L) + \frac{1}{2}\sin(\pi x/L) \, .$$

5. Since the two formulations of a Fourier series on an interval $-L \le x \le L$

$$f(x) = a_0/2 + \sum_{n=1}^{\infty}[a_n\cos(n\pi x/L) + b_n\sin(n\pi x/L)] = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

are equivalent, if we know a_n and b_n we could derive c_n, and *vice versa*. Equating terms in $e^{in\pi x/L}$ and $e^{-in\pi x/L}$ we have that $c_n = \frac{1}{2}(a_n - ib_n)$, $c_{-n} = \frac{1}{2}(a_n + ib_n)$, and $c_0 = a_0/2$. So if a_n and b_n are real (which is true if $f(x)$ is real), then $c_{-n} = c_n*$ (where $*$ denotes the complex conjugate). Similarly starting from the c_n we could determine the a_n and b_n from $a_n = c_n + c_{-n}$ and $b_n = i(c_n - c_{-n})$. As a rule the integrations required to compute c_n are easier, which is why the complex exponential form is often preferred.

We can check these relations in this case by determining the c_n using

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx \ .$$

So for $f(x) = x$ we have $a_0 = 0$, $a_n = 0$ and $b_n = -\frac{2L}{n\pi}\cos(n\pi)$, so we expect $c_n = \frac{iL}{n\pi}\cos(n\pi)$. We compute

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx = \frac{1}{2L}\int_{-L}^{L} xe^{-in\pi x/L}dx \ .$$

Integrating by parts

$$c_n = \frac{1}{2L}\left(\left[\frac{iLx}{n\pi}e^{-in\pi x/L}\right]_{-L}^{L} - \frac{iL}{n\pi}\int_{-L}^{L} e^{-in\pi x/L}dx\right) = \frac{iL}{2n\pi}(e^{in\pi} + e^{-in\pi}) - 0$$

$$c_n = \frac{iL}{n\pi}\cos(n\pi) \ ,$$

as expected. So the complex exponential Fourier series representation of $f(x) = x$ on the interval $-L \le x \le L$ is

$$f(x) = x = \sum_{n=-\infty}^{\infty} \frac{iL}{n\pi}\cos(n\pi)e^{in\pi x/L} \ .$$

6. This time we will check the problem the other way, by computing the c_n, and from them computing the a_n and b_n and comparing against the previous answer. We have

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx = \frac{1}{2L}\int_{0}^{L} \sin(\pi x/L)e^{-in\pi x/L}dx \ .$$

For c_n, convert $\sin(\pi x/L)$ to exponential form:

$$c_n = \frac{1}{2L}\int_0^L \frac{1}{2i}(e^{i\pi x/L} - e^{-i\pi x/L})e^{-in\pi x/L}dx = \frac{1}{4iL}\int_0^L e^{i(1-n)\pi x/L} - e^{-i(1+n)\pi x/L}dx \ .$$

Note, at this point, that $n = 1$ is a special case, and that the first term integrates to $1/(4i)$, and $n = -1$ is also a special case, where the second term integrates to $-1/(4i)$.

$$c_n = \left[-\frac{e^{i(1-n)\pi x/L}}{4\pi(1-n)} - \frac{e^{-i(1+n)\pi x/L}}{4\pi(1+n)}\right]_0^L = -\frac{(e^{i(1-n)\pi} - 1)}{4\pi(1-n)} - \frac{(e^{-i(1+n)\pi} - 1)}{4\pi(1+n)} \ .$$

For $n = 0$ we recover $c_0 = 1/\pi = a_0/2$ (answer to Q4). For $n = 1$ the second term is 0, so $c_1 = 1/(4i)$ and for $n = -1$ the first term is zero, so $c_{-1} = -1/4i$ (see above). For all other odd values of n, $c_n = 0$. For even values of n

$$c_n = -\frac{-2}{4\pi(1-n)} - \frac{-2}{4\pi(1+n)} = -\frac{1}{\pi(1-n^2)} \ .$$

On this basis we have $b_1 = i(c_1 - c_{-1}) = 1/2$ and all other b are zero. Then $a_1 = 0$ as well as all other a_n, n odd. Finally for n even, except $n = 0$, $a_n = \frac{2}{\pi(1-n^2)}$. These values agree with those computed in Q4.

Fourier

Answers to Problem Sheet 2

Fourier transforms

1. The answers are provided below. As an aid to understanding the more difficult questions, consider what $\delta(2x)$ means, as follows. We define $\Pi(x)$ as a function having unit area, of height1, and extending over $-1/2 < x < 1/2$. Then $\frac{1}{L}\Pi(x/L)$ has height $\frac{1}{L}$ and extends over the interval $-L/2 < x < L/2$ (i.e. reaches 0 when the argument $x/L = \pm 1/2$). The function $\frac{1}{L}\Pi(x/L)$ has area 1. We can consider $\delta(x)$ as the limit of $\frac{1}{L}\Pi(x/L)$, as $L \to 0$. By this means we can see that $\delta(2x)$ is the limit of $\frac{1}{L}\Pi(2x/L)$, as $L \to 0$. But $2x/L = 1/2$ when $x = L/4$, meaning that the function $\delta(2x)$ is half as wide as $\delta(x)$, and so has area $1/2$. So it is clear that $\delta(ax)$ has area $1/a$, and that $\int dx\, \delta(ax) f(x) = f(0)/a$.

   (a) $\int dx\ \delta(x_0 - x)\ x = x_0$
   (b) $\int dt\ \delta(t-2)\ \cos t = \cos 2$
   (c) $\int dx\ \delta(x_1 - x)\delta(x_2 - x)\ f(x) = \delta(x_2 - x_1) f(x_1) = \delta(x_2 - x_1) f(x_2)$
   (d) $\int dx\ \delta(x/3 - 1)\ \sin(\pi x/2) = \int dx\ 3\delta(x-3)\ \sin(\pi x/2) = 3\sin(3\pi/2) = -3$
   (e) $\int dx\ \delta\left[(x - x_0)(x - x_1)\right]\ f(x) = \frac{f(x_0)}{|x_0 - x_1|} + \frac{f(x_1)}{|x_1 - x_0|} = \frac{f(x_0)+f(x_1)}{|x_0 - x_1|}$
   (f) $\int dt\ \delta(t^2 - 3t + 2)\ \sin t = \int dt\ \delta[(t-2)(t-1)]\ \sin t = \frac{\sin(2)+\sin(1)}{2-1}$

2. The triangular function is described by two linear segments:

$$f(x) = \begin{cases} (a+x)/a^2 & -a < x < 0 \\ (a-x)/a^2 & 0 < x < a \end{cases}$$

First evaluate the indefinite integrals for $e^{-i\omega x}$ and $xe^{-i\omega x}$

$$\int e^{-i\omega x} dx = \frac{1}{-i\omega} e^{-i\omega x} = \frac{i}{\omega} e^{-i\omega x}\ .$$

$$\int x e^{-i\omega x} dx = \frac{ix}{\omega} e^{-i\omega x} - \int \frac{i}{\omega} e^{-i\omega x} dx = \frac{ix}{\omega} e^{-i\omega x} + \frac{1}{\omega^2} e^{-i\omega x}\ .$$

Then from the definition of the Fourier transform, we have, splitting the integral into two parts:

$$g(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = \frac{1}{2\pi a^2}\int_{-a}^{0} (a+x) e^{-i\omega x} dx + \frac{1}{2\pi a^2}\int_{0}^{a} (a-x) e^{-i\omega x} dx$$

$$2\pi a^2 g(\omega) = \left[\frac{ia}{\omega} e^{-i\omega x} + \frac{ix}{\omega} e^{-i\omega x} + \frac{1}{\omega^2} e^{-i\omega x}\right]_{-a}^{0} + \left[\frac{ia}{\omega} e^{-i\omega x} - \frac{ix}{\omega} e^{-i\omega x} - \frac{1}{\omega^2} e^{-i\omega x}\right]_{0}^{a}$$

$$= \left(\frac{ia}{\omega} + \frac{1}{\omega^2}\right) - \left(\frac{ia}{\omega} e^{i\omega a} - \frac{ia}{\omega} e^{i\omega a} + \frac{1}{\omega^2} e^{i\omega a}\right) + \left(\frac{ia}{\omega} e^{-i\omega a} - \frac{ia}{\omega} e^{-i\omega a} - \frac{1}{\omega^2} e^{-i\omega a}\right) - \left(\frac{ia}{\omega} - \frac{1}{\omega^2}\right)$$

$$= \frac{2}{\omega^2} - \frac{1}{\omega^2}e^{i\omega a} - \frac{1}{\omega^2}e^{-i\omega a} = \frac{2}{\omega^2}(1-\cos(\omega a)) = \frac{4}{\omega^2}\sin^2(\omega a/2)$$

$$g(\omega) = \frac{4}{2\pi\omega^2 a^2}\sin^2(\omega a/2) = \frac{1}{2\pi}\frac{\sin^2(\omega a/2)}{(\omega a/2)^2} = \frac{1}{2\pi}\text{sinc}^2(\omega a/2)\ ,$$

as required.

3. (a) *Phase shifts*:

$$\mathcal{F}[f(x+a)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\ f(x+a)e^{-i\omega x} = \frac{1}{2\pi}\int du\ f(u)e^{-i\omega(u-a)} = \frac{e^{ia\omega}}{2\pi}\int du\ f(u)e^{-i\omega u} = e^{ia\omega}g(\omega)$$

(b) *Exponential multiplication*:

$$\mathcal{F}[e^{ax}f(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\ e^{ax}f(x)e^{-i\omega x} = \frac{1}{2\pi}\int dx\ f(x)e^{-i(\omega+ia)x} = g(\omega+ia)$$

(c) *Scaling*:

$$\mathcal{F}[f(ax)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\ f(ax)e^{-i\omega x} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d(ax)}{|a|}\ f(ax)e^{-i(\omega/a)(ax)} = \frac{1}{|a|}g\left(\frac{\omega}{a}\right)$$

Note that $|a|$ is needed because otherwise if a is -ve, the limits on the integral would change sign.

4. (a) The function has two parts. The first is A over a distance $L = -(a-b)/2 + (a+b)/2 = b$ centered at $x_0 = [-(a+b)/2 - (a-b)/2]/2 = -a/2$, which is represented by $\Pi\left[(x+a/2)/b\right]$. The second part is A over a distance $L = (a+b)/2 - (a-b)/2 = b$ centered at $x_0 = [(a-b)/2 + (a+b)/2]/2 = a/2$, which is represented by $\Pi\left[(x-a/2)/b\right]$. Hence

$$f(x) = A\Pi\left(\frac{x-a/2}{b}\right) + A\Pi\left(\frac{x+a/2}{b}\right)$$

(b) By linearity, we can do the two parts separately. Starting with

$$\mathcal{F}[A\Pi(x)] = \frac{A}{2\pi}\text{sinc}(\omega/2)\ ,$$

using the phase shift theorem we have

$$\mathcal{F}[A\Pi(x \pm a/2)] = \frac{A}{2\pi}e^{\pm ia\omega/2}\text{sinc}(\omega/2)\ .$$

Then using the scaling theorem we have

$$\mathcal{F}\left[A\Pi\left(\frac{(x\pm a/2)}{b}\right)\right] = \frac{Ab}{2\pi}e^{\pm ia\omega/2}\text{sinc}(b\omega/2)\ ,$$

and so we finally have

$$\mathcal{F}[f(x)] = \frac{Ab}{2\pi}\left(e^{ia\omega/2} + e^{-ia\omega/2}\right)\text{sinc}(b\omega/2) = \frac{Ab}{\pi}\cos(a\omega/2)\text{sinc}(b\omega/2)\ ,$$

as required.

(c)

$$g(\omega) = \frac{1}{2\pi}\int dx\ \left[\delta(x-a/2)+\delta(x+a/2)\right] e^{-i\omega x} = \frac{1}{2\pi}\left(e^{i\omega a/2}+e^{-\omega a/2}\right) = \frac{1}{\pi}\cos(a\omega/2)$$

(d) The convolution may be thought of as smearing out the δ functions by the rectangular function. And since the areas of the delta functions are each 1, the result is two versions of the rectangular function, width b, shifted to the positions $x = \pm a/2$, as expected. Alternatively we can use the formula for the convolution of $[\delta(x-a/2)+\delta(x+a/2)]$ with $A\Pi(x/b)$, which is

$$h(x) = \int du\ \left[\delta(u-a/2)+\delta(u+a/2)\right] A\Pi\left(\frac{x-u}{b}\right) = A\Pi\left(\frac{x-a/2}{b}\right) + A\Pi\left(\frac{x+a/2}{b}\right)$$

(e) The convolution theorem states that if $h(x) = f(x)*g(x)$ then $\mathcal{F}(h(x)) = 2\pi\mathcal{F}(f(x))\mathcal{F}(g(x))$. Therefore

$$\mathcal{F}\left[A\Pi\left(\frac{x-a/2}{b}\right) + A\Pi\left(\frac{x+a/2}{b}\right)\right] = 2\pi\mathcal{F}[\delta(x-a/2)+\delta(x+a/2)]\mathcal{F}A\Pi(x/b)$$

$$= 2\pi\frac{1}{\pi}\cos(a\omega/2)\frac{Ab}{2\pi}\mathrm{sinc}(b\omega/2) = \frac{Ab}{\pi}\cos(a\omega/2)\mathrm{sinc}(b\omega/2)\ ,$$

as before.