Fourier Analysis (Prof. Warren)

Fourier Series:

$$f(x) = f(x+2L) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

Where, $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$e(x) = \frac{f(x)+f(-x)}{2}$$

$$o(x) = \frac{f(x)-f(-x)}{2}$$

$$\left\{1, \sum_{n=1}^{\infty}\cos\left(\frac{n\pi x}{L}\right), \sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\right\}$$ is a CONS.

- Use 'sin (cos D'
  'cos (cos D'
  'sin (sin D' identities!

Dirichlet Conditions (sufficient but not necessary)
- $f(x)$ must be periodic
- $f(x)$ must be single-valued with a finite number of discontinuities (within a period)
- $f(x)$ must have a finite number of maxima and minima (within a period)
- $\int_{-L}^{L}|f(x)|\,dx$ must converge

Parseval's Theorem (over a period $2L$):

$$\langle f^2(x)\rangle = \frac{1}{2L}\int_{-L}^{L} f^2(x)\,dx = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}a_n^2 + \frac{1}{2}\sum_{n=1}^{\infty}b_n^2 = \sum_{n=-\infty}^{\infty}|c_n|^2$$

Complex Fourier Series:

$$f(x) = f(x+2L) = \frac{a_0}{2} + \frac{1}{2}\sum_{n=1}^{\infty}a_n\left(e^{in\pi x/L} + e^{-in\pi x/L}\right) + \frac{1}{2i}\sum_{n=1}^{\infty}b_n\left(e^{in\pi x/L} - e^{-in\pi x/L}\right)$$

$$= \frac{a_0}{2} + \frac{1}{2}\sum_{n=1}^{\infty}\left[(a_n - ib_n)e^{in\pi x/L} + (a_n+ib_n)e^{-in\pi x/L}\right]$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$, where $c_0 = a_0/2$, $c_n = \frac{1}{2}(a_n - ib_n)$, $c_{-n} = \frac{1}{2}(a_n+ib_n)$

Where, $c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-in\pi x/L}\, dx$ $\qquad (\omega_n = \frac{n\pi}{L})$

Fourier Transform: $g(\omega) = \mathcal{F}[f(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$

Inverse Fourier Transform: $f(x) = \mathcal{F}^{-1}[g(\omega)] = \int_{-\infty}^{\infty} g(\omega)\, e^{i\omega x}\, d\omega$

$A(\omega) = g(\omega) + g(-\omega)$ $\rightarrow$ cosine spectrum

$B(\omega) = i[g(\omega) - g(-\omega)]$ $\rightarrow$ sine spectrum

$g(\omega) = \frac{1}{2}[A(\omega) - iB(\omega)]$

$g(-\omega) = \frac{1}{2}[A(\omega) + iB(\omega)]$

$$\delta\left(\frac{x}{a} - x_0\right) = a\,\delta(x - a x_0)$$
$$\delta(x - x_0) = \delta(x_0 - x)$$
$$\int \delta(x_1 - x)\,\delta(x_0 - x)\, f(x)\, dx = \delta(x_1 - x_0)\, f(x_0) = \delta(x_1 - x_0)\, f(x_1)$$
$$\int \delta[(x - x_1)(x - x_0)]\, f(x)\, dx = \frac{f(x_0) + f(x_1)}{|x_0 - x_1|}$$

Rectangle $\xrightarrow{FT}$ Sinc

Gaussian $\xrightarrow{FT}$ Gaussian

Delta $\xrightarrow{FT}$ Constant

Sifting Property: $\int_a^b f(x)\,\delta(x - x_0)\, dx = f(x_0)$, where $a < x_0 < b$; 0, otherwise

Convolution: $h(x) = f(x) * g(x) = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$

OR... $h(x) = g(x) * f(x) = \int_{-\infty}^{\infty} g(u)\, f(x-u)\, du$

Convolution Theorem: $\mathcal{F}[h(x)] = 2\pi\, \mathcal{F}[f(x)]\, \mathcal{F}[g(x)]$

- change order of integration
- change variable to $v = x - u$.

i) **Shift Theorem** $\mathcal{F}[f(x+a)] = e^{ia\omega}\, g(\omega)$

ii) **Exponential Multiplication** $\mathcal{F}[e^{ax} f(x)] = g(\omega + ia)$

iii) **Scaling** $\mathcal{F}[f(ax)] = \frac{1}{|a|}\, g\left(\frac{\omega}{a}\right)$