Emmy Noether

Professor Einstein Writes in Appreciation of a Fellow-Mathematician.

To the Editor of The New York Times:

The efforts of most human-beings are consumed in the struggle for their daily bread, but most of those who are, either through fortune or some special gift, relieved of this struggle are largely absorbed in further improving their worldly lot. Beneath the effort directed toward the accumulation of worldly goods lies all too frequently the illusion that this is the most substantial and desirable end to be achieved; but there is, fortunately, a minority composed of those who recognize early in their lives that the most beautiful and satisfying experiences open to humankind are not derived from the outside, but are bound up with the development of the individual's own feeling, thinking and acting. The genuine artists, investigators and thinkers have always been persons of this kind. However inconspicuously the life of these individuals runs its course, none the less the fruits of their endeavors are the most valuable contributions which one generation can make to its successors.

Within the past few days a distinguished mathematician, Professor Emmy Noether, formerly connected with the University of Göttingen and for the past two years at Bryn Mawr College, died in her fifty-third year. In the judgment of the most competent living mathematicians, Fräulein Noether was the most significant creative mathematical genius thus far produced since the higher education of women began. In the realm of algebra, in which the most gifted mathematicians have been busy for centuries, she discovered methods which have proved of enormous importance in the development of the present-day younger generation of mathematicians. Pure mathematics is, in its way, the poetry of logical ideas. One seeks the most general ideas of operation which will bring together in simple, logical and unified form the largest possible circle of formal relationships. In this effort toward logical beauty spiritual formulas are discovered necessary for the deeper penetration into the laws of nature.

Born in a Jewish family distinguished for the love of learning, Emmy Noether, who, in spite of the efforts of the great Göttingen mathematician, Hilbert, never reached the academic standing due her in her own country, none the less surrounded herself with a group of students and investigators at Göttingen, who have already become distinguished as teachers and investigators. Her unselfish, significant work over a period of many years was rewarded by the new rulers of Germany with a dismissal, which cost her the means of maintaining her simple life and the opportunity to carry on her mathematical studies. Farsighted friends of science in this country were fortunately able to make such arrangements at Bryn Mawr College and at Princeton that she found in America up to the day of her death not only colleagues who esteemed her friendship but grateful pupils whose enthusiasm made her last years the happiest and perhaps the most fruitful of her entire career.

ALBERT EINSTEIN.
Princeton University, May 1, 1935.
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Quantum Field Theory

Notes on the Delta Function and Related Issues

In quantum field theory we often make use of the Dirac δ-function $\delta(x)$ and the θ-function $\theta(x)$ (also known as the Heaviside function, or step function). These are defined as follows.

The Delta Function

The δ-function, $\delta(x)$, is zero for all values of x except at $x = 0$, where it becomes infinite in such a way that

$$\int_{-\infty}^{\infty} dx\ \delta(x) = 1$$

It is therefore a single infinite spike at $x = 0$. We also have the important result,

$$\int_{-\infty}^{\infty} dx\ \delta(x - x_0) f(x) = f(x_0)$$

for any function $f(x)$.

One way of defining it more precisely is to consider the series of Gaussians

$$f_a(x) = \frac{1}{a\sqrt{\pi}} \exp\left(-\frac{x^2}{a^2}\right)$$

which clearly satisfy

$$\int_{-\infty}^{\infty} dx\ f_a(x) = 1$$

for all values of a. The functions $f_a(x)$ are peaked at $x = 0$ dropping off very rapidly as $|x|$ becomes large. If we let a become small, the Gaussian becomes progressively more peaked at $x = 0$, but in such a way that the total area underneath it remains 1. The δ-function is then defined by

$$\delta(x) = \lim_{a \to 0}\ f_a(x)$$

We are also interested in the Fourier transform of the δ-function. The Fourier transform $\tilde{f}(k)$ of a function $f(x)$ is defined by

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx\ e^{-ikx}\ f(x)$$

and its inverse is defined by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\ e^{ikx} \tilde{f}(k)$$

(We can equally use e^{ikx} in the definition of the Fourier transform as long as we use e^{-ikx} in the definition of the inverse. Similarly the factor of 2π may be moved around. Also, often the tilde is dropped.)

The Fourier transform of the δ-function therefore is

$$\tilde{\delta}(k) = \int_{-\infty}^{\infty} dx\ e^{-ikx} \delta(x) = 1$$

so the inverse relation is the important relation

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\ e^{ikx}$$

This relation also allows us to prove the simple property

$$\delta(cx) = \frac{1}{|c|}\delta(x)$$

by simply scaling k in the Fourier transform. In particular $\delta(-x) = \delta(x)$.

All of this easily generalizes to higher dimensions. We have, for example, in three dimensions

$$\delta^{(3)}(\mathbf{x}) = \frac{1}{(2\pi)^3}\int d^3k\ e^{i\mathbf{k}\cdot\mathbf{x}}$$

The δ-function strictly speaking only has meaning when it is sitting inside an integral. It is practically useful to remember this when interpreting combinations like $x\delta(x)$. This is obviously only non-zero at $x = 0$ but then it has the form $0\times\infty$. To see how to interpret this, we consider the quantity

$$\int dx\ x\delta(x)\ f(x)$$

This is clearly zero for arbitrary $f(x)$. So we interpret $x\delta(x)$ as zero.

One can also prove that

$$\delta(x^2 - a^2) = \frac{1}{2|a|}[\delta(x + a) + \delta(x - a)]$$

or more generally

$$\delta(g(x)) = \Sigma_j \frac{\delta(x - x_j)}{|g'(x_j)|}$$

where x_j are the roots of $g(x)$ and we have assumed that they are all single roots.

The Theta Function (a.k.a. the Heaviside or step function)

The θ-function is a discontinuous fucntion defined by

$$\theta(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Its value at $x = 0$ may be taken to be $1/2$ but it doesn't really affect much and it can also be taken to be defined as other values.

The θ-function has zero slope everywhere except at $x = 0$ where it appears to be infinitely steep. This suggests that the derivative of $\theta(x)$ is in fact $\delta(x)$. We can prove this more precisely as follows. Consider the integral

$$\int_{-\infty}^{x} dy\ \delta(y)$$

From the properties of the δ-function, this is clearly 0 if $x < 0$ and 1 if $x > 0$. That is,

$$\int_{-\infty}^{x} dy\ \delta(y) = \theta(x)$$

So if we differentiate with respect to x we get

$$\delta(x) = \theta'(x)$$

Notice also that

$$\frac{d}{dx}\left(\theta(-x)\right) = -\delta(x)$$

An Example

These properties are illustrated in the following example. Consider the differential equation

$$\ddot{g} + \omega^2 g = -i\delta(t)$$

This is the equation of a harmonic oscillator that receives a kick at $t = 0$. Clearly for $t > 0$ or $t < 0$, the solutions are linear combinations of $e^{\pm i\omega t}$. Then one possible solution with the δ-function source is

$$g(t) = N\left[\theta(t)e^{-i\omega t} + \theta(-t)e^{i\omega t}\right]$$

where N is a normalization factor, to be fixed. To see that this is a solution, we differentiate, using the known properties of $\theta(t)$. We have

$$\dot{g} = -i\omega N\left[\theta(t)e^{-i\omega t} - \theta(-t)e^{i\omega t}\right] + N\delta(t)\left[e^{-i\omega t} - e^{i\omega t}\right]$$

The second term in brackets is $\delta(t)$ times a function which vanishes at $t = 0$, so

$$\dot{g} = -i\omega N\left[\theta(t)e^{-i\omega t} - \theta(-t)e^{i\omega t}\right]$$

Differentiating a second time we get

$$\ddot{g} = -\omega^2 N\left[\theta(t)e^{-i\omega t} + \theta(-t)e^{i\omega t}\right] - i\omega N\delta(t)\left[e^{-i\omega t} + e^{i\omega t}\right]$$

In the second bracket, the $\delta(t)$ forces us to set $t = 0$ in the exponentials, and we get

$$\ddot{g} = -\omega^2 g - 2i\omega N\delta(t)$$

We thus obtain a solution to the differential equation as long as we choose

$$N = \frac{1}{2\omega}$$

Later in the course, you can consider this example as a simplified and alternative proof that the Feynman propagator D_F obeys the equation

$$(\partial^2 + m^2)D_F(x) = -i\delta^{(4)}(x)$$

Further comments on u's and v's

In constructing the general plane wave solutions to the Dirac equation we need spinors $u(p)$ and $v(p)$ that satisfy

$$(\not{p} - m)u(p) = 0, \qquad (\not{p} + m)v(p) = 0,$$

Here we complement the discussion given in the lectures and the problem sheets concerning the solutions to these equations. We will focus on $u(p)$, noting that there is an analogous discussion for $v(p)$.

We saw that in the rest frame with $p^\mu = (m, \mathbf{0})$ we have

$$u(m, \mathbf{0}) = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

where ξ is a two component spinor and the factor of $\sqrt{m}$ is for convenience. We obtain the general $u(p)$ by considering a boost in a direction $\hat{\mathbf{p}}$ (with $\hat{\mathbf{p}}^2 = 1$), that takes the rest frame four-momentum $(m, \mathbf{0})$ to $(p^0, \mathbf{p})$ with

$$p^0 = m \cosh \eta, \qquad |\mathbf{p}| = m \sinh \eta$$

This is a Lorentz transformation with $\omega_{0i} = \eta \hat{\mathbf{p}}^i$ and $\omega_{ij} = 0$. Thus the boost takes the spinor $u(m, \mathbf{0})$ to $u(p)$ with

$$\begin{aligned} u(p) &= exp[-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}]u(m, \mathbf{0}) \\ &= exp[-i\omega_{0i}S^{0i}]u(m, \mathbf{0}) \\ &= \exp[-\frac{\eta}{2}\begin{pmatrix} \hat{\mathbf{p}} \cdot \boldsymbol{\sigma} & 0 \\ 0 & -\hat{\mathbf{p}} \cdot \boldsymbol{\sigma} \end{pmatrix}]\sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{m}(\cosh \frac{\eta}{2} - \sinh \frac{\eta}{2} \hat{\mathbf{p}} \cdot \boldsymbol{\sigma})\xi \\ \sqrt{m}(\cosh \frac{\eta}{2} + \sinh \frac{\eta}{2} \hat{\mathbf{p}} \cdot \boldsymbol{\sigma})\xi \end{pmatrix} \end{aligned}$$

where we used $(\hat{\mathbf{p}} \cdot \boldsymbol{\sigma})^2 = 1$. Observing that

$$\cosh \frac{\eta}{2} = \frac{1}{\sqrt{2m}}(p^0 + m)^{1/2}, \qquad \sinh \frac{\eta}{2} = \frac{1}{\sqrt{2m}}(p^0 - m)^{1/2}$$

we can obtain the general expression

$$u(p) = \begin{pmatrix} \frac{1}{\sqrt{2(p^0+m))}}(p^0 + m - \mathbf{p} \cdot \boldsymbol{\sigma})\xi \\ \frac{1}{\sqrt{2(p^0+m))}}(p^0 + m + \mathbf{p} \cdot \boldsymbol{\sigma})\xi \end{pmatrix} . \tag{1}$$

It is straightforward to directly check that this expression for $u(p)$ satisfies $(\not{p} - m)u(p) = 0$ (which was guaranteed from the construction). One can also check that if we set $\mathbf{p} = (0, 0, p^3)$ then we obtain

$$u(p) = \begin{pmatrix} \begin{pmatrix} \sqrt{p^0 - p^3} & 0 \\ 0 & \sqrt{p^0 + p^3} \end{pmatrix} \xi \\ \begin{pmatrix} \sqrt{p^0 + p^3} & 0 \\ 0 & \sqrt{p^0 - p^3} \end{pmatrix} \xi \end{pmatrix}$$

as we had in the lectures.

Next we observe that

$$\left[\frac{1}{\sqrt{2(p^0+m))}}(p^0+m-\mathbf{p}\cdot\boldsymbol{\sigma})\right]^2 = p^0-\mathbf{p}\cdot\boldsymbol{\sigma} = p\cdot\sigma$$
$$\left[\frac{1}{\sqrt{2(p^0+m))}}(p^0+m+\mathbf{p}\cdot\boldsymbol{\sigma})\right]^2 = p^0+\mathbf{p}\cdot\boldsymbol{\sigma} = p\cdot\bar{\sigma}$$

which allows us to write the equivalent general expression

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\bar{\sigma}}\xi \end{pmatrix} \tag{2}$$

as we also had in the lectures.

Another perspective on the compact expressions given in (2) is the following. Observe that $p\cdot\sigma$ is an Hermitian matrix with positive eigenvalues given by $p^0 \pm |\mathbf{p}|$. Thus if we diagonalise $p\cdot\sigma$ using a non-singular matrix M i.e. $p\cdot\sigma = MDiag(p^0+|\mathbf{p}|, p^0-|\mathbf{p}|)M^{-1}$, then we can write

$$\sqrt{p\cdot\sigma} = MDiag(\sqrt{p^0+|\mathbf{p}|}, \sqrt{p^0-|\mathbf{p}|})M^{-1}$$

where we note that, by definition, the square root notation means take the positive roots of the eigenvalues. It is straightforward to check that this is consistent:

$$\begin{aligned}
[\sqrt{p\cdot\sigma}]^2 &= MDiag(\sqrt{p^0+|\mathbf{p}|}, \sqrt{p^0-|\mathbf{p}|})M^{-1}MDiag(\sqrt{p^0+|\mathbf{p}|}, \sqrt{p^0-|\mathbf{p}|})M^{-1} \\
&= MDiag(\sqrt{p^0+|\mathbf{p}|}, \sqrt{p^0-|\mathbf{p}|})Diag(\sqrt{p^0+|\mathbf{p}|}, \sqrt{p^0-|\mathbf{p}|})M^{-1} \\
&= MDiag(p^0+|\mathbf{p}|, p^0-|\mathbf{p}|)M^{-1} \\
&= p\cdot\sigma
\end{aligned}$$